ROYAUME DU MAROC

كليـــــــة العلوم السملالية - مراكش

FACULTÉ DES SCIENCES SEMLALIA - MARRAKECH

MARRAKECH

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# The backward problem for time-fractional evolution equations


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# Dedication

Dedicated to the memory of Professor **HAMMADI BOUSLOUS** (——June 2, 2023)

Founder of **T**eam of **A**nalysis and **C**ontrol of **S**ystems and **I**nteractions (TACSI, Marrakesh)

# Introduction

Let $0 < \alpha \leq 1$ and $T > 0$. We consider

$$\begin{cases} \partial_t^\alpha u(t) = Au(t), & t \in (0, T), \\ u(0) = u_0, \end{cases} \tag{1}$$

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where $A : D(A) \subset H \to H$ is a densely defined s.t.

(i) $A$ is self-adjoint,

(ii) $A$ is bounded above: there exists $\kappa \geq 0$ such that $\langle Au, u \rangle \leq \kappa \|u\|^2$ for all $u \in D(A)$,

(iii) $A$ has compact resolvent.

# Introduction

The Caputo derivative $\partial_t^\alpha g$ is defined by

$$\partial_t^\alpha g(t) = \begin{cases} \dfrac{1}{\Gamma(1-\alpha)} \displaystyle\int_0^t (t-s)^{-\alpha} \frac{\mathrm{d}}{\mathrm{d}s} g(s)\,\mathrm{d}s, & 0 < \alpha < 1, \\ \dfrac{\mathrm{d}}{\mathrm{d}t} g(t), & \alpha = 1. \end{cases}$$

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**Backward problem:** Given $u(T)$, can we recover $u(t_0)$, $0 \leq t_0 < T$?

# Logarithmic convexity

**Theorem (C-Maniar-Yamamoto)**

*Let* $0 < \alpha \leq 1$. *Let* $u$ *be the solution to* (1). *Then there exists a constant* $M \geq 1$ *such that*

$$\|u(t)\| \leq M\|u(0)\|^{1-\frac{t}{T}}\|u(T)\|^{\frac{t}{T}}, \qquad 0 \leq t \leq T. \tag{2}$$

*Moreover, if* $\kappa = 0$, *then we can choose* $M = 1$.

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• A function $f(t)$ that is $C^2[0,\infty)$ is log-convex if and only if the differential inequality

$$f''(t)f(t) - (f'(t))^2 \geq 0 \tag{3}$$

holds for all $t \geq 0$.

**Proof for $\alpha = 1$ (Agmon-Nirenberg (1963))**

Since $D(A^2)$ is dense in $H$, it suffices to consider $u_0 \in D(A^2) \setminus \{0\}$.
We have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2 = 2\langle u'(t), u(t)\rangle = 2\langle Au(t), u(t)\rangle,$$

and since $A$ is self-adjoint,

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\|u(t)\|^2 = 4\|Au(t)\|^2.$$

It follows that

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2}\|u(t)\|^2\right)\|u(t)\|^2 - \left(\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2\right)^2 = 4(\|Au(t)\|^2\|u(t)\|^2 - \langle Au(t), u(t)\rangle^2).$$

By Cauchy-Schwarz inequality, we obtain

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2}\|u(t)\|^2\right)\|u(t)\|^2 - \left(\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2\right)^2 \geq 0, \qquad 0 \leq t \leq T. \qquad (4)$$

# Ideas of the proof for $0 < \alpha < 1$

- No similar formula is available for

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- The functions $t \mapsto \left(E_\alpha(-\lambda_n t^\alpha)\right)^2$ are completely monotone on $[0, T]$ for $\lambda_n \geq 0$ (Schneider, 1996), i.e.,

$$(-1)^k f^{(k)}(t) \geq 0 \qquad \text{for all } t > 0,\ k = 0, 1, 2, \dots$$

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- Any completely monotone function is log-convex.

# Proof for $0 < \alpha < 1$

S.E. C, L. Maniar & M. Yamamoto, The backward problem for time-fractional evolution equations, *Appl. Anal.*, **103** (2023), 2194-2212.

# Numerical test

We consider

$$\begin{cases} \partial_t^\alpha u(t,x) = u_{xx}(t,x), & (t,x) \in (0,0.02) \times (0,1), \\ u(t,0) = u(t,1) = 0, & t \in (0,0.02), \\ u(0,x) = \sin(\pi x), & x \in (0,1). \end{cases}$$

The solution is given by

$$u_\alpha(t,x) = E_\alpha\left(-\pi^2 t^\alpha\right)\sin(\pi x), \quad t \in (0,0.02),\ x \in (0,1).$$

# Numerical test

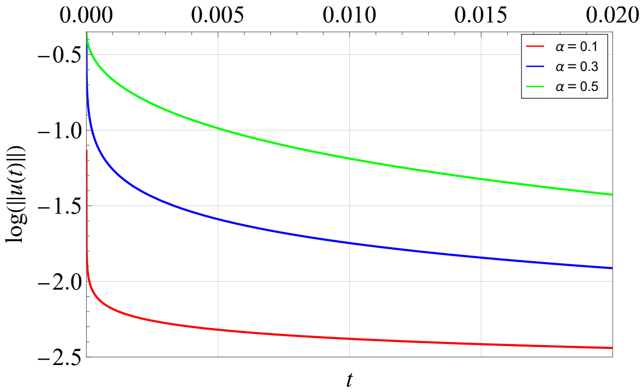


Figure: $\log \|u_\alpha(t,\cdot)\|_{L^2(0,1)}$ for $\alpha = 0.1,\ 0.3,\ 0.5$.

# Non-symmetrci case

We consider the following backward problem:

$$\begin{cases} \partial_t^\alpha u(t,x) = Lu(t,x), & \text{in } (0,T)\times\Omega, \\ u|_{\partial\Omega} = 0, & \text{on } (0,T)\times\partial\Omega, \\ u(0,x) = u_0(x) & \text{in } \Omega, \end{cases} \tag{5}$$

where

$$Lu(x) := \mathrm{div}(\mathcal{A}(x)\nabla u(x)) + \mathcal{B}(x)\cdot\nabla u(x) + p(x)u(x),$$

with symmetric and uniformly elliptic principal part.

The main assumption on the drift term:

**(H)** There exists a function $b \in W^{2,\infty}(\Omega)$ such that $\mathcal{B} = \mathcal{A}\nabla b$.

# Non-symmetric case

The main result reads as follows:

**Theorem**

*Assume that Assumption **(H)** is fulfilled. Then there exists a constant* $\kappa = \kappa(\mathcal{A}, b, p, \alpha, T) \geq 1$ *such that*

$$\|u(t,\cdot)\|_{L^2(\Omega)} \leq \kappa e^{\|b\|_\infty} \|u(0,\cdot)\|_{L^2(\Omega)}^{1-\frac{t}{T}} \|u(T,\cdot)\|_{L^2(\Omega)}^{\frac{t}{T}}, \qquad 0 \leq t \leq T. \tag{6}$$

# Ideas of the proof

By the change of variable $v(t,x) = \mathrm{e}^{\frac{b}{2}} u(t,x)$, we obtain a **symmetric equation**:

$$\begin{cases} \partial_t^\alpha v(t,x) = L_0 v(t,x), & \text{in } (0,T)\times\Omega, \\ v|_{\partial\Omega} = 0, & \text{on } (0,T)\times\partial\Omega, \\ v(0,x) = v_0(x) & \text{in } \Omega, \end{cases} \tag{7}$$

where $v_0 = \mathrm{e}^{\frac{b}{2}} u_0$ and the operator $L_0$ is given by

$$L_0 v(x) = \mathrm{div}(\mathcal{A}(x)\nabla v(x)) + q(x)v(x),$$

with

$$q(x) = p(x) - \frac{1}{2}\mathrm{div}(\mathcal{A}(x)\nabla b(x)) - \frac{1}{4}\mathcal{A}(x)\nabla b(x)\cdot\nabla b(x), \qquad x\in\Omega.$$

# Numerical test

$$\begin{cases} \partial_t^\alpha u(t,x) = u_{xx}(t,x) + u_x(t,x), & (t,x) \in (0,0.02) \times (0,1), \\ u(t,0) = u(t,1) = 0, & t \in (0,0.02), \\ u(0,x) = \sin(\pi x), & x \in (0,1). \end{cases} \tag{8}$$

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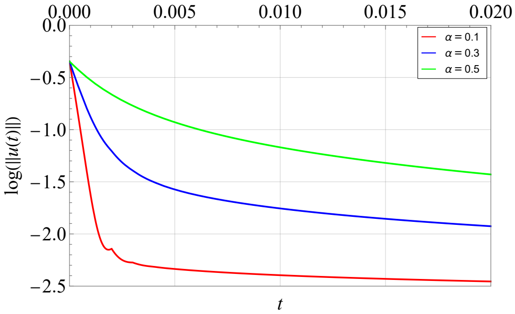


Figure: $\log \|u_\alpha(t,\cdot)\|_{L^2(0,1)}$ for $\alpha = 0.1,\ 0.3,\ 0.5$ in Example 2.

# Non-symmetric case

S.E. C, L. Maniar, and M. Yamamoto, Logarithmic convexity of non-symmetric time-fractional diffusion equations, *Math. Meth. Appl. Sci.*, (2024), 1–11, Doi: 10.1002/mma.10421.

# Open problems

- Similar results for **coupled** systems? e.g.,

$$\begin{cases} \partial_t^{\alpha_1} u_1 = \Delta u_1 + a_{11} u_1 + a_{12} u_2, \\ \partial_t^{\alpha_2} u_2 = \Delta u_2 + a_{12} u_1 + a_{22} u_2. \end{cases}$$

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- Logarithmic convexity without Assumption **(H)**.

- Backward uniqueness for analytic semigroups.

**Thank you for your attention**