p-Wasserstein metrics for gas networks

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HELMHOLTZ IMAGING

DESY.

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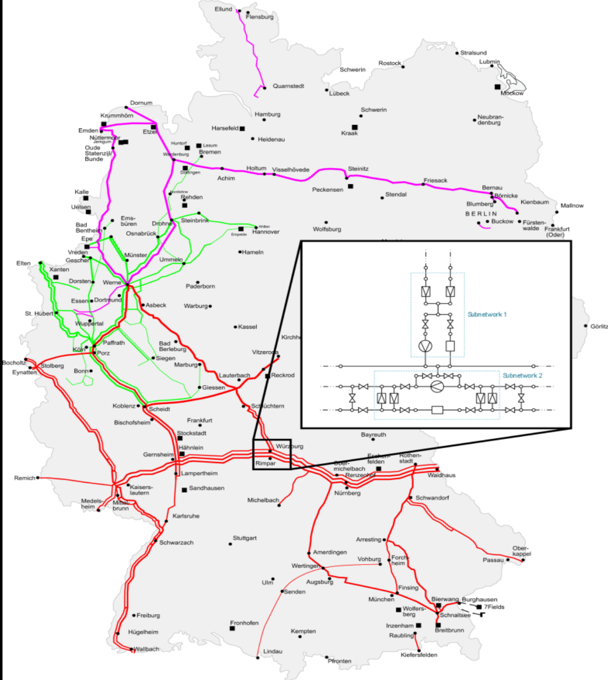

p-Wasserstein metric

For two probability measures $\mu_0, \mu_T \in \mathbb{P}(\Omega)$ and $p \in [1, \infty)$:

(STATIC) $$W_p(\mu_0, \mu_T) = \min_{\pi \in \Pi(\mu_0, \mu_T)} \left\{ \int_{\Omega \times \Omega} |x - y|^p \, \mathrm{d}\pi(x, y) \right\}^{\frac{1}{p}}$$

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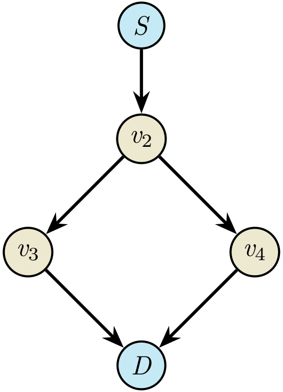
Based on the work of Benamou and Brenier [BB00]:

For a reference measure η, density μ and mass flux j:

(DYNAMIC)
$$W_p(\mu_0, \mu_T) = \min_{\substack{\mu \in \mathcal{M}_+(\Omega \times [0,T]), \\ j \in \mathcal{M}(\Omega \times [0,T])}} \left\{ T^{p-1} \int_0^T \int_\Omega \frac{|j(x,t)|^p}{\mu(x,t)^{p-1}} \, \mathrm{d}\eta \, \mathrm{d}t \;\middle|\; \begin{array}{l} \frac{\partial \mu}{\partial t} + \frac{\partial j}{\partial x} = 0, \\ \mu|_{t=0} = \mu_0, \\ \mu|_{t=T} = \mu_T \end{array} \right\}^{\frac{1}{p}}$$

Gas networks as metric graphs [FBP24]



1) Gas flow in individual pipes
2) Mass conservation at vertices
3) Storage of gas at interior vertices
4) Time-(in)dependent boundary conditions at boundary vertices

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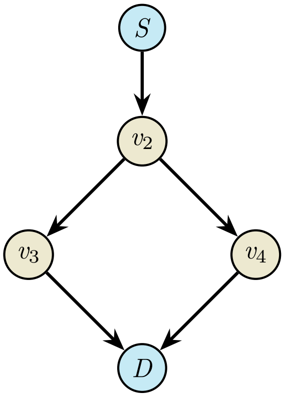

1) **Gas flow in individual pipes**
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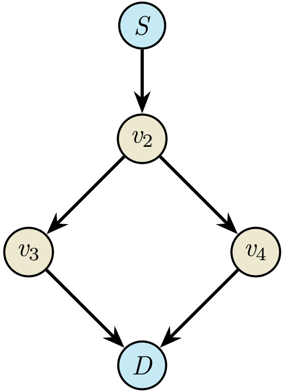
For small flows $|v| << c$ and friction-dominance:

(ISO3)

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) = 0$$

$$\frac{\partial P}{\partial x} = -\frac{\lambda}{2D}\rho v|v| - g\rho \sin(\alpha)$$

Gas networks as metric graphs [FBP24]



1) Gas flow in individual pipes
2) **Mass conservation at vertices**
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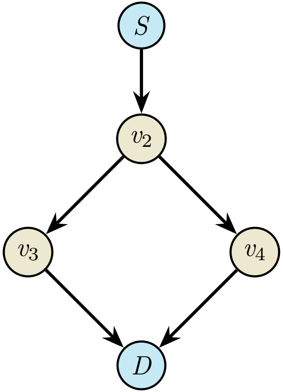
4) Time-(in)dependent boundary conditions at boundary vertices

Classical Kirchhoff's law at each vertex ν:

$$0 = \sum_{e=(\omega,\nu)} j_e|_{x=L_e} - \sum_{e=(\nu,\omega)} j_e|_{x=0}$$

Gas networks as metric graphs [FBP24]



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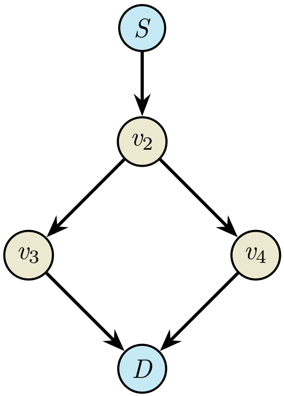
Generalized Kirchhoff's law at each vertex ν:

$$f_\nu = \sum_{e=(\omega,\nu)} j_e|_{x=L_e} - \sum_{e=(\nu,\omega)} j_e|_{x=0}$$

with vertex flux f_ν and vertex density $\gamma_\nu \geq 0$ such that $\frac{\partial \gamma_\nu}{\partial t} = f_\nu$

Gas networks as metric graphs [FBP24]



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Time-dependent boundary condition at boundary vertex ν:

Given supply $s_\nu : [0, T] \longrightarrow \mathbb{R}_{\leq 0}$ for source vertex ν:

$$s_\nu = \sum_{e=(\omega,\nu)} j_e|_{x=L_e} - \sum_{e=(\nu,\omega)} j_e|_{x=0}$$

Given demand $d_\nu : [0, T] \longrightarrow \mathbb{R}_{\geq 0}$ for sink vertex ν:

$$d_\nu = \sum_{e=(\omega,\nu)} j_e|_{x=L_e} - \sum_{e=(\nu,\omega)} j_e|_{x=0}$$

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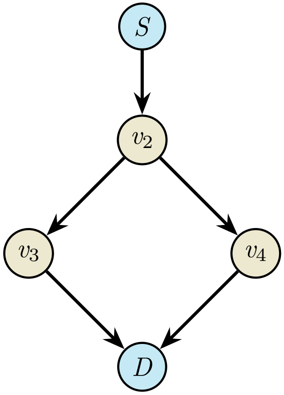

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Time-independent boundary condition at boundary vertex ν:

Given total supply $S_\nu \leq 0$ for source vertex ν:

$$s_\nu = \sum_{e=(\omega,\nu)} j_e|_{x=L_e} - \sum_{e=(\nu,\omega)} j_e|_{x=0} \quad \text{such that} \quad \int_0^T s_\nu(t)\,\mathrm{d}t = S_\nu$$

Given total demand $D_\nu \geq 0$ for sink vertex ν:

$$d_\nu = \sum_{e=(\omega,\nu)} j_e|_{x=L_e} - \sum_{e=(\nu,\omega)} j_e|_{x=0} \quad \text{such that} \quad \int_0^T d_\nu(t)\,\mathrm{d}t = D_\nu$$

p-Wasserstein metric on gas networks [FBP24]

For a collection of edge and vertex measures $\mu_t = (\rho(t), \gamma(t), S(t), D(t))$:

$$W_p(\mu_0, \mu_T) = \inf_{\substack{\rho, j, \gamma, f, \\ S, s, D, d}} \left\{ T^{p-1} \int_0^T \left(\sum_e \int_0^{L_e} \frac{|j_e|^p}{\rho_e^{p-1}} \,\mathrm{d}x + \sum_{\nu \text{ interior vertex}} \frac{|f_\nu|^p}{\gamma_\nu^{p-1}} + \sum_{\nu \text{ source vertex}} \frac{|s_\nu|^p}{S_\nu^{p-1}} + \sum_{\nu \text{ sink vertex}} \frac{|d_\nu|^p}{D_\nu^{p-1}} \right) \mathrm{d}t \,\middle|\, \dots \right\}^{\frac{1}{p}}$$

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Classical Kirchhoff's law and no boundary vertices, i.e. $\mu_t = (\rho(t))$:

$$W_p(\mu_0, \mu_T) = \inf_{\rho, j} \left\{ T^{p-1} \int_0^T \sum_e \int_0^{L_e} \frac{|j_e|^p}{\rho_e^{p-1}} \, \mathrm{d}x \, \mathrm{d}t \;\middle|\; \begin{array}{l} \forall e : \frac{\partial \rho_e}{\partial t} + \frac{\partial j_e}{\partial x} = 0, \\ \rho_e|_{t=0} = (\rho_0)_e, \; \rho_e|_{t=T} = (\rho_T)_e, \\ \forall \nu : 0 = \sum_{e=(\omega,\nu)} j_e|_{x=L_e} - \sum_{e=(\nu,\omega)} j_e|_{x=0} \end{array} \right\}^{\frac{1}{p}}$$

p-Wasserstein gradient flows [FBP24]

p-Wasserstein gradient flow derived in the limit $\tau \to 0$ for $\tau > 0$, from the minimizing movement scheme:

$$\rho^{\tau}_{(k+1)\tau} = \text{arg}\min_{\rho} \left\{ \frac{1}{p\tau^{p-1}} W_p\left(\rho, \rho^{\tau}_{k\tau}\right)^p + \sum_e E\left(\rho_e\right) \right\}$$

with energy E, entropy F and integration constants c_e such that $\sum_e c_e = 0$:

$$E\left(\rho_e\right) = \int_0^{L_e} \left(F_e\left(\rho_e\right) + \frac{2\mathcal{D}_e}{\lambda_e} g \sin\left(\alpha_e\right) x \rho_e + c_e \rho_e \right) dx \qquad F_e''\left(\rho_e\right) = \frac{2\mathcal{D}_e}{\lambda_e} \frac{P_e'\left(\rho_e\right)}{\rho_e}$$

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Result:

$$\forall e: \qquad \frac{\partial \rho_e}{\partial t} + \frac{\partial j_e}{\partial x} = 0, \qquad j_e \left|j_e\right|^{p-2} = -\rho_e^{p-1} \frac{\partial}{\partial x}\left(E'\left(\rho_e\right)\right)$$

$$\forall \nu: \qquad 0 = \sum_{e=(\omega,\nu)} j_e\big|_{x=L_e} - \sum_{e=(\nu,\omega)} j_e\big|_{x=0}$$

p-Wasserstein gradient flows [FBP24]

Corresponds to the (ISO3) formulation with classical Kirchhoff's law for $p = 3$:

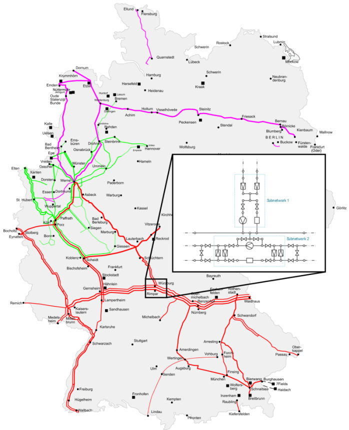
$$\forall e: \quad \frac{\partial \rho_e}{\partial t} + \frac{\partial j_e}{\partial x} = 0$$

$$\forall e: \quad j_e\,|j_e| = -\rho_e^2 \left(\frac{2\mathcal{D}_e}{\lambda_e \rho_e} \frac{\partial P_e}{\partial x} - \frac{2D_e}{\lambda_e} g \sin(\alpha_e) \right) \iff \frac{\lambda_e}{2D_e} \rho_e v_e |v_e| = -\frac{\partial P_e}{\partial x} - g\rho_e \sin(\alpha_e) \iff$$

$$\frac{\partial P_e}{\partial x} = -\frac{\lambda_e}{2\mathcal{D}_e} \rho_e v_e |v_e| - g\rho_e \sin(\alpha_e)$$

$$\forall \nu: \quad 0 = \sum_{e=(\omega,\nu)} j_e|_{x=L_e} - \sum_{e=(\nu,\omega)} j_e|_{x=0}$$

Current work and next steps...



- Properties of the generalized p-Wasserstein metric
- Port-Hamiltonian formulation of (ISO2) in Wasserstein spaces
- Gas mixture model in (ISO3) formulation

Thank you for your attention!

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