X Partial differential equations, optimal design and numerics

Criterion of the global solvability and attracting sets for singular and abstract differential-algebraic equations and applications

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A system of differential and algebraic equations can be represented in the form of an abstract evolution equation which is often called a **differential-algebraic equation** (**DAE**), when it is considered in finite-dimensional spaces, and an **abstract DAE**, when it is considered in infinite-dimensional spaces.
Any type of a **PDE** can be represented as an abstract DAE in appropriate infinite-dimensional spaces, possibly, with a complementary boundary condition.

Types of DAEs

Nonlinear DAE: $F(t,x,\dot{x})=0$ such that it cannot be reduced to the explicit form $\dot{x}=f(t,x)$ (e.g., $F(t,x,p)$ has the continuous partial derivatives in p, x and $\partial_p F(t,x,p)$ is degenerate (noninvertible) for all (t,x,p) from the domain of definition of F)

Quasilinear DAE: $A(t,x)\frac{d}{dt}[D(t)x]=f(t,x)$ or $A(t,x)\dot{x}+B(t)x=f(t,x)$, where $A(t,x)$ is degenerate

Semilinear DAE: $\frac{d}{dt}[A(t)x]+B(t)x=f(t,x)$ or $\frac{d}{dt}[A(t)x]=f(t,x)$, where $A(t,x)$ is degenerate

Linear DAE: $\frac{d}{dt}[A(t)x]+B(t)x=f(t)$, where $A(t)$ is degenerate

Semi-implicit DAE: $f(t,x_1,x_2,\dot{x}_1)=0$, $g(t,x_1,x_2)=0$

Semi-explicit DAE: $\dot{x}_1=f(t,x_1,x_2)$, $g(t,x_1,x_2)=0$

Hessenberg DAE: $\dot{x}_1=f(t,x_1,x_2)$, $g(t,x_1)=0$

The classification is taken from [Lamour R., März R., Tischendorf C. Differential-Algebraic Equations: A Projector Based Analysis, 2013]

Applications

DAEs are used to describe mathematical models in **cybernetics, radioelectronics, mechanics, robotics technology, economics, ecology, chemical kinetics** and **gas industry**, e.g., in modelling

- dynamics of neural networks
- transient processes in electrical circuits
- dynamics of gas networks
- dynamics of complex mechanical and technical systems (e.g., robots)
- multi-sectoral economic models (e.g., the dynamics of corporate enterprises using investment)
- kinetics of chemical reactions

1. Rabier P.J., Rheinboldt W.C., Nonholonomic motion of mechanical systems from a DAE viewpoint, 2000.
2. Riaza R. Differential-algebraic systems. Analytical aspects and circuit applications, 2008.
3. Morishima M. Equilibrium, stability, and growth, 1964.
4. Benner P., Grundel S., Himpe C., Huck C., Streubel T., Tischendorf C. Gas Network Benchmark Models, 2018.

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DAEs are also referred to as degenerate DEs, descriptor systems, singular systems, operator-differential equations, DEs or dynamical systems on manifolds, abstract evolution equations, PDAEs and DEs of Sobolev type.

Consider a **semilinear DAE**

$$\frac{d}{dt}[Ax] + Bx = f(t,x), \tag{1}$$

where $f \in C(\mathscr{T} \times D, Y)$, $\mathscr{T} \subseteq [0,\infty)$ is an interval, A and B are closed linear operators from X into Y with domains D_A and D_B respectively, $D = D_A \cap D_B \neq \{0\}$ is a lineal (linear manifold), X and Y are Banach spaces, D_A and D_B are dense in X.

The operators A, B can be degenerate (noninvertible).

We consider the initial value problem (IVP) for the DAE (1) with the initial condition

$$x(t_0) = x_0. \tag{2}$$

A function $x \in C([t_0,t_1),X)$ is said to be a *solution of* (1) *on* $[t_0,t_1)$ $(t_1 \leq \infty)$ if the function Ax is continuously differentiable on (t_0,t_1) and $x(t)$ satisfies (1) on $[t_0,t_1)$. If the function $x(t)$ additionally satisfies the initial condition (2), then it is called a *solution of the initial value problem* (1), (2).

Denote by $\rho = \rho(A,B) := \{\lambda \in \mathbb{C} \mid \exists (\lambda A + B)^{-1} \in L(Y,X)\}$ the *set of the regular points* λ of the pencil $\lambda A + B$ ($\lambda \in \mathbb{C}$ is a parameter). The set $\rho(A,B)$ is open, and the resolvent as the operator function $R \colon \rho \to L(Y,X)$ is holomorphic on $\rho(A,B)$.

The pencil $\lambda A + B$ is called **regular** if $\rho(A,B) \neq \emptyset$ and **singular** if $\rho(A,B) = \emptyset$.

In general, here X, Y are complex Banach spaces (BSs). If X, Y are real BSs, then the pencil $\lambda A + B$ is called *regular* if $\rho = \rho(\tilde{A},\tilde{B}) = \{\lambda \in \mathbb{C} \mid \exists (\lambda \tilde{A} + \tilde{B})^{-1} \in L(\tilde{Y},\tilde{X})\} \neq \emptyset$, where the operators $\tilde{A}$, $\tilde{B}$ and the complex BSs $\tilde{X}$, $\tilde{Y}$ are the complex extensions of A, B and the complexifications of X, Y, respectively.

Let $X = \mathbb{R}^n$ and $Y = \mathbb{R}^m$, i.e., $A, B \in L(\mathbb{R}^n,\mathbb{R}^m)$.

The pencil $\lambda A + B$ is called **regular** if $n = m = \mathrm{rk}(\lambda A + B)$. Otherwise, if $n \neq m$ or $n = m$ and $\mathrm{rk}(\lambda A + B) < n$, the pencil is called **singular** or **nonregular** (irregular).

The operator pencil $\lambda A + B$, associated with the linear part $\frac{d}{dt}[Ax] + Bx$ of the DAE (1), is called *characteristic*. If the characteristic pencil is singular (respectively, regular), then the DAE is called **singular** (respectively, **regular**), or *nonregular*, or *irregular*.

Notice that *the system of equations corresponding the DAE with the singular characteristic pencil may be underdetermined or overdetermined.*

Index of the regular pencil

Let the following conditions hold:

1. The pencil $P(\lambda) = \lambda A + B$ is regular for all λ from some neighborhood of the infinity, i.e., there exists a number $R > 0$ such that each λ with $|\lambda| > R$ is a regular point of $P(\lambda)$.
2. The point $\lambda = \infty$ is a pole of the resolvent $R(\lambda) = P^{-1}(\lambda) = (\lambda A + B)^{-1}$ of order r. This is equivalent to the fact that the resolvent $\widehat{R}(\mu) = (A + \mu B)^{-1}$ of the pencil $A + \mu B$ has a pole of order $\nu = r + 1$ at the point $\mu = 0$.

Then $P(\lambda)$ is called a **regular pencil of index** ν ($\nu \in \mathbb{N}$).

If there exists the inverse operator $A^{-1} \in L(Y,X)$ (or $\mu = 0$ is a regular point of the pencil $A + \mu B$) and $D_B \supseteq D_A$, then $P(\lambda)$ is a regular pencil of **index** 0.

The above definition can be reformulated in the following way.

Let condition 1 hold and $\nu \in \mathbb{N}$ be the least number such that for some constants $C, R > 0$ the estimate

$$\|R(\lambda)\| \le C|\lambda|^{\nu-1}, \quad |\lambda| \ge R, \tag{3}$$

or the equivalent estimate $\|\widehat{R}(\mu)\| \le C|\mu|^{-\nu}, \quad |\mu| \le R^{-1},$ holds, then $P(\lambda)$ is a **regular pencil of index** ν.

Notice that for a regular pencil $P(\lambda)$ acting in finite-dimensional spaces, there is always a number $\nu \in \mathbb{N}$ for which the condition (3) is satisfied.

Direct decompositions of spaces and the associated projectors

Let $P(\lambda)=\lambda A+B$ *be a regular pencil of index* ν.

Then there exists the pair of mutually complementary projectors $P_k : D \to D_k$ ($P_i P_j x = \delta_{ij} P_i x$, $(P_1+P_2)x = x$, $x \in D_A$) and the pair of mutually complementary projectors $Q_k : Y \to Y_k$ ($Q_i Q_j = \delta_{ij} Q_i$, $Q_1+Q_2 = I_Y$), $k=1,2$, which generate the decompositions of D and Y into the direct sums

$$D = D_1 \dot{+} D_2, \quad Y = Y_1 \dot{+} Y_2, \quad D_k := P_k D, \quad Y_k := Q_k Y, \quad k=1,2, \tag{4}$$

such that $AD_k \subset Y_k$ and $BD_k \subset Y_k$, $k=1,2$.
The restricted operators $A_k := A\big|_{D_k} : D_k \to Y_k$ and $B_k := B\big|_{D_k} : D_k \to Y_k$, $k=1,2$, are such that there exist $A_1^{-1} \in L(Y_1, \overline{D_1})$ and $B_2^{-1} \in L(Y_2, \overline{D_2})$.

Thus, A, B are the direct sums of the operators A_1, A_2 and B_1, B_2:

$$A = A_1 \dot{+} A_2,\ B = B_1 \dot{+} B_2 : D_1 \dot{+} D_2 \to Y_1 \dot{+} Y_2 \tag{5}$$

If $P(\lambda)$ *is a regular pencil of index not higher than 1*, then $A_2 = 0$.

[Rutkas A.G., Vlasenko L.A. Existence of solutions of degenerate nonlinear differential operator equations, *Nonlinear Oscillations*, 2001]
[Vlasenko L.A. Evolution Models with Implicit and Degenerate Differential Equations. 2006 (in Russian)].

The projectors can be constructively determined by using contour integration

$$\begin{aligned}
&\mathrm{P}_1 = \frac{1}{2\pi \mathrm{i}} \oint\limits_{|\lambda|=\mathrm{R}} (\lambda \mathrm{A} + \mathrm{B})^{-1} \mathrm{A}\, d\lambda, \quad \mathrm{Q}_1 = \frac{1}{2\pi \mathrm{i}} \oint\limits_{|\lambda|=\mathrm{R}} \mathrm{A}\, (\lambda \mathrm{A} + \mathrm{B})^{-1} d\lambda, \\
&\mathrm{P}_2 = \mathrm{I}_{\mathrm{X}} - \mathrm{P}_1, \qquad \mathrm{Q}_2 = \mathrm{I}_{\mathrm{Y}} - \mathrm{Q}_1.
\end{aligned} \tag{6}$$

[Rutkas A.G., Vlasenko L.A. *Nonlinear Oscillations*, 2001] (as well as other works by Rutkas, Vlasenko and co-authors)

or by using residues

$$\begin{aligned}
&\mathrm{P}_1 = \operatorname*{Res}_{\mu=0} \left(\frac{(\mathrm{A} + \mu \mathrm{B})^{-1} \mathrm{A}}{\mu} \right), \quad \mathrm{Q}_1 = \operatorname*{Res}_{\mu=0} \left(\frac{\mathrm{A}(\mathrm{A} + \mu \mathrm{B})^{-1}}{\mu} \right), \\
&\mathrm{P}_2 = \mathrm{I}_{\mathrm{X}} - \mathrm{P}_1, \qquad \mathrm{Q}_2 = \mathrm{I}_{\mathrm{Y}} - \mathrm{Q}_1.
\end{aligned} \tag{7}$$

[Filipkovska, M.S.: Two combined methods for the global solution of implicit semilinear differential equations with the use of spectral projectors and Taylor expansions. *Int. J. of Computing Science and Mathematics* **15**(1), 1–29 (2022)]

[Filipkovska M.S. Combined numerical methods for solving time-varying semilinear differential-algebraic equations with the use of spectral projectors and recalculation, 2022 (In review)]
`https://doi.org/10.48550/arXiv.2212.00012`

Let $X = \mathbb{R}^n$ and $Y = \mathbb{R}^m$.
Thus, we consider the DAE (1): $\frac{d}{dt}[Ax] + Bx = f(t,x)$, where $A, B \in L(\mathbb{R}^n, \mathbb{R}^m)$, $f \in C(\mathscr{T} \times D, \mathbb{R}^m)$, $\mathscr{T} \subseteq [0,\infty)$ is an interval, $D \subseteq \mathbb{R}^n$ is an open set.
The characteristic pencil $\lambda A + B$ is **singular** (i.e., $n \neq m$ or $n = m$ and $\mathrm{rk}(\lambda A + B) < n$).

The block form of a singular pencil of operators and the associated direct decompositions of spaces and projectors

Statement.
For operators $A, B \colon \mathbb{R}^n \to \mathbb{R}^m$, which form a singular pencil $\lambda A + B$, there exist the decompositions of the spaces

$$\mathbb{R}^n = X_s \dot{+} X_r = X_{s_1} \dot{+} X_{s_2} \dot{+} X_r, \quad \mathbb{R}^m = Y_s \dot{+} Y_r = Y_{s_1} \dot{+} Y_{s_2} \dot{+} Y_r \tag{8}$$

such that with respect to the decompositions $\mathbb{R}^n = X_s \dot{+} X_r$, $\mathbb{R}^m = Y_s \dot{+} Y_r$ the operators A, B have the block structure

$$A = \begin{pmatrix} A_s & 0 \\ 0 & A_r \end{pmatrix}, B = \begin{pmatrix} B_s & 0 \\ 0 & B_r \end{pmatrix} : X_s \dot{+} X_r \to Y_s \dot{+} Y_r \quad (X_s \times X_r \to Y_s \times Y_r), \tag{9}$$

where $A_s = A\big|_{X_s}, B_s = B\big|_{X_s} : X_s \to Y_s$ and $A_r = A\big|_{X_r}, B_r = B\big|_{X_r} : X_r \to Y_r$, that is, the pair of "singular" subspaces $\{X_s, Y_s\}$ and the pair of "regular" subspaces $\{X_r, Y_r\}$ are invariant under the operators A, B,
and (**if** $\mathrm{rank}(\lambda A + B) < n, m$) with respect to the decompositions

$$X_s = X_{s_1} \dot{+} X_{s_2}, \quad Y_s = Y_{s_1} \dot{+} Y_{s_2} \tag{10}$$

the "singular" blocks A_s, B_s have the block structure

$$A_s = \begin{pmatrix} A_{gen} & 0 \\ 0 & 0 \end{pmatrix}, B_s = \begin{pmatrix} B_{gen} & B_{und} \\ B_{ov} & 0 \end{pmatrix} : X_{s_1} \dot{+} X_{s_2} \to Y_{s_1} \dot{+} Y_{s_2}, \tag{11}$$

where the operator $A_{gen} : X_{s_1} \to Y_{s_1}$ has the inverse $A_{gen}^{-1} \in L(Y_{s_1}, X_{s_1})$ (if $X_{s_1} \neq \{0\}$), $B_{gen} : X_{s_1} \to Y_{s_1}$, $B_{und} : X_{s_2} \to Y_{s_1}$ and $B_{ov} : X_{s_1} \to Y_{s_2}$.
If $\mathrm{rank}(\lambda A + B) = m < n$, **then** the structure of the singular blocks takes the form

$$A_s = \begin{pmatrix} A_{gen} & 0 \end{pmatrix}, B_s = \begin{pmatrix} B_{gen} & B_{und} \end{pmatrix} : X_{s_1} \dot{+} X_{s_2} \to Y_s \tag{12}$$

and $Y_{s_1} = Y_s$, $Y_{s_2} = \{0\}$ in (8) and, accordingly, in (10).
If $\mathrm{rank}(\lambda A + B) = n < m$, **then** the structure of the singular blocks takes the form

$$A_s = \begin{pmatrix} A_{gen} \\ 0 \end{pmatrix}, B_s = \begin{pmatrix} B_{gen} \\ B_{ov} \end{pmatrix} : X_s \to Y_{s_1} \dot{+} Y_{s_2} \tag{13}$$

and $X_{s_1} = X_s$, $X_{s_2} = \{0\}$ in (8) and, accordingly, in (10).

The direct decompositions (8) generate the pair S, P, the pair F, Q, the pair S_1, S_2 and the pair F_1, F_2 of the mutually complementary projectors

$$S\colon \mathbb{R}^n \to X_s,\, P\colon \mathbb{R}^n \to X_r, \qquad F\colon \mathbb{R}^m \to Y_s,\, Q\colon \mathbb{R}^m \to Y_r, \tag{14}$$
$$S_i\colon \mathbb{R}^n \to X_{s_i}, \qquad F_i\colon \mathbb{R}^m \to Y_{s_i}, \quad i=1,2, \tag{15}$$

where $F_1 = F$, $F_2 = 0$ **if** $\operatorname{rank}(\lambda A + B) = m < n$, and $S_1 = S$, $S_2 = 0$ **if** $\operatorname{rank}(\lambda A + B) = n < m$. These projectors have the properties $FA = AS$, $FB = BS$, $QA = AP$, $QB = BP$, $AS_2 = 0$, $F_2A = 0$, $F_2BS_2 = 0$.

The converse assertion that there exist the pairs of mutually complementary projectors (14), (15) satisfying the properties indicated above, which generate the direct decompositions (8), is also true.

[1] Filipkovska M.S. Lagrange stability and instability of irregular semilinear differential-algebraic equations and applications. *Ukrainian Math. J.* **70**(6), 947–979 (2018). https://doi.org/10.1007/s11253-018-1544-6

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[3] Filipkovska M. Criterion of the global solvability of regular and singular differential-algebraic equations. *J. of Mathematical Sciences* (2024) [in Production]. https://doi.org/10.1007/s10958-024-07152-7

Introduce the extensions of the operators A_s, A_r, B_s, B_r from (9) to $\mathbb{R}^n$:

$$\mathcal{A}_s = FA, \quad \mathcal{A}_r = QA, \quad \mathcal{B}_s = FB, \quad \mathcal{B}_r = QB. \tag{16}$$

Then the operators $\mathcal{A}_s, \mathcal{B}_s, \mathcal{A}_r, \mathcal{B}_r \in L(\mathbb{R}^n, \mathbb{R}^m)$ act so that $\mathcal{A}_s, \mathcal{B}_s : \mathbb{R}^n \to Y_s$, $\mathcal{A}_r, \mathcal{B}_r : \mathbb{R}^n \to Y_r$, $X_r \subset \mathrm{Ker}(\mathcal{A}_s)$, $X_r \subset \mathrm{Ker}(\mathcal{B}_s)$, $X_s \subset \mathrm{Ker}(\mathcal{A}_r)$, $X_s \subset \mathrm{Ker}(\mathcal{B}_r)$, and

$$\mathcal{A}_s\big|_{X_s} = A_s, \quad \mathcal{A}_r\big|_{X_r} = A_r, \quad \mathcal{B}_s\big|_{X_s} = B_s, \quad \mathcal{B}_r\big|_{X_r} = B_r. \tag{17}$$

Further, introduce extensions of the operators (blocks) from (11) to $\mathbb{R}^n$ as follows:

$$\mathcal{A}_{gen} = F_1 A, \quad \mathcal{B}_{gen} = F_1 B S_1, \quad \mathcal{B}_{und} = F_1 B S_2, \quad \mathcal{B}_{ov} = F_2 B S_1. \tag{18}$$

Then $\mathcal{A}_{gen}, \mathcal{B}_{gen}, \mathcal{B}_{und}, \mathcal{B}_{ov} \in L(\mathbb{R}^n, \mathbb{R}^m)$ act so that $\mathcal{A}_{gen}\mathbb{R}^n = \mathcal{A}_{gen} X_{s_1} = Y_{s_1}$ $(X_{s_2} \dot{+} X_r = \mathrm{Ker}(\mathcal{A}_{gen}))$, $\mathcal{B}_{gen} : \mathbb{R}^n \to Y_{s_1}$, $X_{s_2} \dot{+} X_r \subset \mathrm{Ker}(\mathcal{B}_{gen})$, $\mathcal{B}_{und} : \mathbb{R}^n \to Y_{s_1}$, $X_{s_1} \dot{+} X_r \subset \mathrm{Ker}(\mathcal{B}_{und})$, and $\mathcal{B}_{ov} : \mathbb{R}^n \to Y_{s_2}$, $X_{s_2} \dot{+} X_r \subset \mathrm{Ker}(\mathcal{B}_{ov})$, and

$$\mathcal{A}_{gen}\big|_{X_{s_1}} = A_{gen}, \; \mathcal{B}_{gen}\big|_{X_{s_1}} = B_{gen}, \; \mathcal{B}_{und}\big|_{X_{s_2}} = B_{und}, \; \mathcal{B}_{ov}\big|_{X_{s_1}} = B_{ov}. \tag{19}$$

Extensions of the operators (blocks) from (12) and (13) to $\mathbb{R}^n$ are introduced in a similar way.

The operator $\mathcal{A}_{gen}^{(-1)} \in L(\mathbb{R}^m, \mathbb{R}^n)$ defined by the relations

$$\mathcal{A}_{gen}^{(-1)} \mathcal{A}_{gen} = S_1, \quad \mathcal{A}_{gen} \mathcal{A}_{gen}^{(-1)} = F_1, \quad \mathcal{A}_{gen}^{(-1)} = S_1 \mathcal{A}_{gen}^{(-1)},$$

where $F_1 = F$ if $\mathrm{rank}(\lambda A + B) = m < n$ and $S_1 = S$ if $\mathrm{rank}(\lambda A + B) = n < m$, is the *semi-inverse* operator of $\mathcal{A}_{gen}$.

Assume that the regular block $\lambda A_r + B_r$ is a *regular pencil of index not higher than 1*. Then there exists the pairs $\tilde{P}_i : X_r \to X_i$, $i = 1,2$, $\tilde{Q}_j : Y_r \to Y_j$, $j = 1,2$, of mutually complementary projectors which generate the direct decompositions

$$X_r = X_1 \dot{+} X_2, \quad Y_r = Y_1 \dot{+} Y_2 \tag{20}$$

such that the pairs of subspaces X_1, Y_1 and X_2, Y_2 are invariant under A_r, B_r, and the restricted operators $A_i = A_r\big|_{X_i} : X_i \to Y_i$, $B_i = B_r\big|_{X_i} : X_i \to Y_i$, $i = 1,2$, are such that $A_2 = 0$ and there exist $A_1^{-1} \in L(Y_1, X_1)$ (if $X_1 \neq \{0\}$) and $B_2^{-1} \in L(Y_2, X_2)$ (if $X_2 \neq \{0\}$). We introduce the extensions P_i, Q_i of the projectors $\tilde{P}_i$, $\tilde{Q}_i$ so that $X_i = P_i \mathbb{R}^n$, $Y_i = Q_i \mathbb{R}^m$, $i = 1,2$, and the extensions of the operators A_i, B_i to $\mathbb{R}^n$

$$\mathcal{A}_i = Q_i A, \quad \mathcal{B}_i = Q_i B, \quad i = 1,2. \tag{21}$$

The extended operators $\mathcal{A}_1, \mathcal{B}_2 \in L(\mathbb{R}^n, \mathbb{R}^m)$ have the semi-inverse operators $\mathcal{A}_1^{(-1)}, \mathcal{B}_2^{(-1)} \in L(\mathbb{R}^m, \mathbb{R}^n)$.

Reduction of the singular (nonregular) DAE to a system of ordinary differential and algebraic equations

In what follows, it is assumed that the regular block $\lambda A_r + B_r$, where A_r, B_r from (9), is a *regular pencil of index not higher than 1*.
The pair P_1, P_2 and the pair S_1, S_2 of mutually complementary projectors generate the decomposition of the set D into the direct sum of subsets

$$D = D_{s_1} \dot{+} D_{s_2} \dot{+} D_1 \dot{+} D_2, \qquad D_{s_i} = S_i D, \quad D_i = P_i D, \quad i = 1,2, \tag{22}$$

($D_{s_i} \subseteq X_{s_i}$, $D_i \subseteq X_i$ $(i = 1,2)$, where X_{s_i}, X_i are defined in (8), (20)).
By using the above projectors, the singular semilinear DAE (1) is reduced to the equivalent system

$$\frac{d}{dt}(AS_1 x) = F_1[f(t,x) - Bx], \tag{23}$$
$$\frac{d}{dt}(AP_1 x) = Q_1[f(t,x) - Bx], \tag{24}$$
$$0 = Q_2[f(t,x) - Bx], \tag{25}$$
$$0 = F_2[f(t,x) - Bx], \tag{26}$$

where $F_1 = F$, $F_2 = 0$ if $\operatorname{rank}(\lambda A + B) = m < n$, and $S_1 = S$ $(S_2 = 0)$ if $\operatorname{rank}(\lambda A + B) = n < m$.

With respect to the decomposition $\mathbb{R}^n = X_s \dot{+} X_r = X_{s_1} \dot{+} X_{s_2} \dot{+} X_1 \dot{+} X_2$ any $x \in \mathbb{R}^n$ can be *uniquely represented* as

$$x = x_s + x_r = x_{s_1} + x_{s_2} + x_{p_1} + x_{p_2} \qquad (x_s = x_{s_1} + x_{s_2}, \quad x_r = x_{p_1} + x_{p_2}), \tag{27}$$

where $x_s = Sx \in X_s$, $x_r = Px \in X_r$, $x_{s_i} = S_i x \in X_{s_i}$, $x_{p_i} = P_i x \in X_i$, $i = 1,2$.
The system (23)–(26) is equivalent to

$$\dot{x}_{s_1} = \mathcal{A}_{gen}^{(-1)}\big(F_1 f(t,x) - \mathcal{B}_{gen} x_{s_1} - \mathcal{B}_{und} x_{s_2}\big), \tag{28}$$

$$\dot{x}_{p_1} = \mathcal{A}_1^{(-1)}\big(Q_1 f(t,x) - \mathcal{B}_1 x_{p_1}\big), \tag{29}$$

$$\mathcal{B}_2^{(-1)} Q_2 f(t,x) - x_{p_2} = 0, \tag{30}$$

$$F_2 f(t,x) - \mathcal{B}_{ov} x_{s_1} = 0, \tag{31}$$

where $\mathcal{A}_{gen}^{(-1)}$, $\mathcal{A}_1^{(-1)}$, $\mathcal{B}_2^{(-1)}$ are the semi-inverse operators and $x_{s_i} \in D_{s_i}$, $x_{p_i} \in D_i$.

The derivative $\dot{V}_{(28),(29)}$ of a scalar function $V \in C^1(\mathscr{T} \times K_{s1}, \mathbb{R})$, where $K_{s1} \subseteq D_{s_1} \times D_1$ is an open set, along the trajectories of equations (28), (29) has the form

$$\begin{aligned}
&\dot{V}_{(28),(29)}(t,x_{s_1},x_{p_1}) = \partial_t V(t,x_{s_1},x_{p_1}) + \\
&+ \partial_{(x_{s_1},x_{p_1})} V(t,x_{s_1},x_{p_1}) \cdot \Upsilon(t,x_{s_1},x_{s_2},x_{p_1},x_{p_2}) = \\
&= \partial_t V(t,x_{s_1},x_{p_1}) + \partial_{x_{s_1}} V(t,x_{s_1},x_{p_1}) \cdot \Big[\mathcal{A}_{gen}^{(-1)}\big(F_1 f(t,x) - \mathcal{B}_{gen} x_{s_1} - \mathcal{B}_{und} x_{s_2}\big)\Big] + \\
&+ \partial_{x_{p_1}} V(t,x_{s_1},x_{p_1}) \cdot \Big[\mathcal{A}_1^{(-1)}\big(Q_1 f(t,x) - \mathcal{B}_1 x_{p_1}\big)\Big], \qquad (32)
\end{aligned}$$

$$\Upsilon(t,x_{s_1},x_{s_2},x_{p_1},x_{p_2}) = \begin{pmatrix} \mathcal{A}_{gen}^{(-1)}\big(F_1 f(t,x) - \mathcal{B}_{gen} x_{s_1} - \mathcal{B}_{und} x_{s_2}\big) \\ \mathcal{A}_1^{(-1)}\big(Q_1 f(t,x) - \mathcal{B}_1 x_{p_1}\big) \end{pmatrix},$$

where $x = x_{s_1} + x_{s_2} + x_{p_1} + x_{p_2}$ $(x_{s_i} = S_i x,\ x_{p_i} = P_i x,\ i = 1,2)$, $(x_{s_1},x_{p_1}) \in K_{s1}$, $x_{s_2} \in D_{s_2}$, $x_{p_2} \in D_2$.

Notice that the **regular** semilinear DAE (1) (with the characteristic pencil of index not higher than 1) can be reduced to the equivalent system

$$\dot{x}_{p_1} = \mathcal{A}_1^{(-1)}\big(Q_1 f(t,x) - \mathcal{B}_1 x_{p_1}\big), \tag{33}$$

$$\mathcal{B}_2^{(-1)} Q_2 f(t,x) - x_{p_2} = 0, \tag{34}$$

where $x_{p_i} = P_i x \in D_i,\ D_i = P_i D,\ i = 1,2,\ D = D_1 \dot{+} D_2,\ x = x_{p_1} + x_{p_2}$.

Definitions.

A solution $x(t)$ (of an equation or inequality) is called **global** if it exists on the interval $[t_0,\infty)$ (where t_0 is a given initial value).
A solution $x(t)$ **has a finite escape time** or *is blow-up in finite time* and is called **Lagrange unstable** if it exists on some finite interval $[t_0,T)$ and is unbounded, that is, there exists $T < \infty$ such that $\lim_{t\to T-0} \|x(t)\| = +\infty$.
A solution $x(t)$ is called **Lagrange stable** if it is global and bounded, that is, $x(t)$ exists on the interval $[t_0,\infty)$ and $\sup_{t\in[t_0,\infty)} \|x(t)\| < \infty$.
The DAE (1) is called **Lagrange unstable** (respectively, *Lagrange stable*) *for the initial point* (t_0,x_0) if the solution of IVP (1), (2) is Lagrange unstable (respectively, Lagrange stable) for this initial point. The DAE (1) is called **Lagrange unstable** (respectively, **Lagrange stable**) if each solution of IVP (1), (2) is Lagrange unstable (respectively, Lagrange stable).

Solutions of the equation (1) are called **ultimately bounded**, if there exists a constant $K > 0$ (K is independent of the choice of t_0, x_0) and for each solution $x(t)$ with an initial point (t_0, x_0) there exists a number $\tau = \tau(t_0, x_0) \geq t_0$ such that $\|x(t)\| < K$ for all $t \in [t_0 + \tau, \infty)$. The similar definition holds for solutions of equation (1) with the initial values $t_0 \in \mathscr{T}$, $x_0 \in M \subseteq D$.

The equation (1) is called **ultimately bounded** or **dissipative**, if for any consistent initial point (t_0, x_0) there exists a global solution of the initial value problem (1), (2) and all the solutions are ultimately bounded. If the number τ does not depend on the choice of t_0, then the solutions of (1) are called *uniformly ultimately bounded* and the equation (1) is called *uniformly ultimately bounded* or *uniformly dissipative*.

The equation (1) is called *ultimately bounded* or *dissipative for the initial points* $(t_0, x_0) \in \mathscr{T} \times M$, if these initial points are consistent and for the initial values $t_0 \in \mathscr{T}$, $x_0 \in M$ there exist global solutions of the IVP (1), (2) and the solutions are ultimately bounded.

The Lagrange stability and ultimate boundedness of explicit ordinary differential equations were studied in [La Salle J., Lefschetz S., Stability by Liapunov's Direct Method with Applications, 1961] and [Yoshizawa T. Stability theory by Liapunov's second method, 1966], respectively.

Consider the manifold associated with the *singular* semilinear DAE (1):

$$L_{t_*} = \{(t,x) \in [t_*,\infty) \times \mathbb{R}^n \mid (F_2 + Q_2)[f(t,x) - Bx] = 0\}, \tag{35}$$

where $t_* \in \mathscr{T}$. It can be represented as $L_{t_*} = \{(t,x) \in [t_*,\infty) \times \mathbb{R}^n \mid F_2[f(t,x) - Bx] = 0,\ Q_2[f(t,x) - Bx] = 0\}$ or $L_{t_*} = \{(t,x) \in [t_*,\infty) \times \mathbb{R}^n \mid (t,x)$ satisfies equations (30), (31)$\}$. Thus, a point $(t,x) \in \mathscr{T} \times D$ belongs to L_{t_*} if and only if it satisfies equations (30), (31) or the equivalent ones.

Also, consider the manifold associated with the *regular* semilinear DAE (1):

$$L_{t_*} = \{(t,x) \in [t_*,\infty) \times \mathbb{R}^n \mid Q_2[f(t,x) - Bx] = 0\}, \tag{36}$$

where $t_* \in \mathscr{T}$. If the DAE (1) is regular, then we can set $S_i = F_i = 0$, $i = 1,2$, and reduce the manifold (35) to (36).

For the singular semilinear DAEs we will consider the following results:

- **The criterion of the global solvability. Previously, theorems on the existence and uniqueness of global solutions and on the blow-up of solutions will be presented.**

  One of the advantages: the restrictions of the type of the global Lipschitz condition (including contractive mapping) are not used.

- **The conditions of the Lagrange stability and uniform ultimate boundedness (dissipativity).**

Mathematical models of nonlinear electrical circuits and gas networks, which are described by semilinear DAEs, are considered.

[Filipkovska M. Criterion of the global solvability of regular and singular differential-algebraic equations. *J. of Mathematical Sciences* (2024) [in Production] `https://doi.org/10.1007/s10958-024-07152-7`]

[Filipkovska M. Qualitative analysis of nonregular differential-algebraic equations and the dynamics of gas networks. *Journal of Mathematical Physics, Analysis, Geometry*, Vol. 19, No. 4, 719–765 (2023). `https://doi.org/10.15407/mag19.04.719`]

[Filipkovska2024] = [Filipkovska M. Criterion of the global solvability of regular and singular differential-algebraic equations. *J. of Mathematical Sciences* (2024) [in Production] `https://doi.org/10.1007/s10958-024-07152-7`]
Below, the theorems and corollaries from [Filipkovska2024] are presented.

Theorem 1 (the global solvability).
Let $f \in C(\mathscr{T} \times D, \mathbb{R}^m)$, where $D \subseteq \mathbb{R}^n$ is some open set and $\mathscr{T} = [t_+, \infty) \subseteq [0, \infty)$, and let the operator pencil $\lambda A + B$ be a singular pencil such that its regular block $\lambda A_r + B_r$, where A_r, B_r are defined in (9), is a regular pencil of index not higher than 1. Assume that there exists an open set $M_{s1} \subseteq D_{s_1} \dot{+} D_1$ and sets $M_{s_2} \subseteq D_{s_2}$, $M_2 \subseteq D_2$ such that the following holds:

1. For any fixed $t \in \mathscr{T}$, $x_{s_1} + x_{p_1} \in M_{s1}$, $x_{s_2} \in M_{s_2}$ there exists a unique $x_{p_2} \in M_2$ such that $(t, x_{s_1} + x_{s_2} + x_{p_1} + x_{p_2}) \in L_{t_+}$ (the manifold L_{t_+} has the form (35) where $t_* = t_+$).
2. A function $f(t,x)$ satisfies locally a Lipschitz condition with respect to x on $\mathscr{T} \times D$. For any fixed $t_* \in \mathscr{T}$, $x_* = x^*_{s_1} + x^*_{s_2} + x^*_{p_1} + x^*_{p_2}$ $(x^*_{s_i} = S_i x_*,\ x^*_{p_i} = P_i x_*,\ i = 1,2)$ such that $x^*_{s_1} + x^*_{p_1} \in M_{s1}$, $x^*_{s_2} \in M_{s_2}$, $x^*_{p_2} \in M_2$ and $(t_*, x_*) \in L_{t_+}$, there exists a neighborhood $N_\delta(t_*, x^*_{s_1}, x^*_{s_2}, x^*_{p_1}) = U_{\delta_1}(t_*) \times U_{\delta_2}(x^*_{s_1}) \times N_{\delta_3}(x^*_{s_2}) \times U_{\delta_4}(x^*_{p_1}) \subset \mathscr{T} \times D_{s_1} \times D_{s_2} \times D_1$, an open neighborhood $U_\varepsilon(x^*_{p_2}) \subset D_2$ (the numbers $\delta, \varepsilon > 0$ depend on the choice of t_*, x_*) and an invertible operator $\Phi_{t_*,x_*} \in L(X_2, Y_2)$ such

that for each $(t,x_{s_1},x_{s_2},x_{p_1}) \in N_\delta(t_*,x^*_{s_1},x^*_{s_2},x^*_{p_1})$ and each $x^i_{p_2} \in U_\varepsilon(x^*_{p_2})$, $i=1,2$, the mapping

$$\widetilde{\Psi}(t,x_{s_1},x_{s_2},x_{p_1},x_{p_2}) := Q_2 f(t,x_{s_1}+x_{s_2}+x_{p_1}+x_{p_2}) - \\ - B\big|_{X_2} x_{p_2} : \mathscr{T} \times D_{s_1} \times D_{s_2} \times D_1 \times D_2 \to Y_2 \quad (37)$$

satisfies the inequality

$$\|\widetilde{\Psi}(t,x_{s_1},x_{s_2},x_{p_1},x^1_{p_2}) - \widetilde{\Psi}(t,x_{s_1},x_{s_2},x_{p_1},x^2_{p_2}) - \Phi_{t_*,x_*}[x^1_{p_2}-x^2_{p_2}]\| \le q(\delta,\varepsilon)\|x^1_{p_2}-x^2_{p_2}\|, \quad (38)$$

where $q(\delta,\varepsilon)$ is such that $\lim\limits_{\delta,\varepsilon\to 0} q(\delta,\varepsilon) < \|\Phi^{-1}_{t_*,x_*}\|^{-1}$.

3. If $M_{s1} \neq X_{s_1} \dot{+} X_1$, then the following holds.

   The component $x_{s_1}(t)+x_{p_1}(t) = (S_1+P_1)x(t)$ of each solution $x(t)$ with the initial point $(t_0,x_0) \in L_{t_+}$, for which $(S_1+P_1)x_0 \in M_{s1}$, $S_2x_0 \in M_{s_2}$ and $P_2x_0 \in M_2$, can never leave M_{s1} (i.e., it remains in M_{s1} for all t from the maximal interval of existence of the solution).

4. If M_{s1} is unbounded, then the following holds.

   There exists a number $R > 0$ (R can be sufficiently large), a function $V \in C^1(\mathscr{T} \times M_R, \mathbb{R})$ positive on $\mathscr{T} \times M_R$, where $M_R = \{(x_{s_1}, x_{p_1}) \in X_{s_1} \times X_1 \mid x_{s_1} + x_{p_1} \in M_{s1}, \|x_{s_1} + x_{p_1}\| > R\}$, and a function $\chi \in C(\mathscr{T} \times (0,\infty), \mathbb{R})$ such that:

   (4.a) $\lim\limits_{\|(x_{s_1}, x_{p_1})\| \to +\infty} V(t, x_{s_1}, x_{p_1}) = +\infty$ uniformly in t on each finite interval $[a,b) \subset \mathscr{T}$;

   (4.b) for each $t \in \mathscr{T}$, $(x_{s_1}, x_{p_1}) \in M_R$, $x_{s_2} \in M_{s_2}$, $x_{p_2} \in M_2$ such that $(t, x_{s_1} + x_{s_2} + x_{p_1} + x_{p_2}) \in L_{t_+}$, the derivative (32) of the function V along the trajectories of equations (28), (29) satisfies the inequality

$$\dot{V}_{(28),(29)}(t, x_{s_1}, x_{p_1}) \leq \chi\big(t, V(t, x_{s_1}, x_{p_1})\big); \qquad (39)$$

   (4.c) the differential inequality $\dot{v} \leq \chi(t,v)$ ($t \in \mathscr{T}$) does not have positive solutions with finite escape time.

Then for each initial point $(t_0, x_0) \in L_{t_+}$ such that $(S_1 + P_1)x_0 \in M_{s1}$, $S_2 x_0 \in M_{s_2}$ and $P_2 x_0 \in M_2$, IVP (1), (2) has a unique global solution $x(t)$ for which the choice of the function $\phi_{s_2} \in C([t_0,\infty), M_{s_2})$ with the initial value $\phi_{s_2}(t_0) = S_2 x_0$ uniquely defines the component $S_2 x(t) = \phi_{s_2}(t)$ when $\mathrm{rank}(\lambda A + B) < n$ (when $\mathrm{rank}(\lambda A + B) = n$, the component $S_2 x$ is absent).

Theorem 2 (the global solvability).
Theorem 1 remains valid if condition 2 is replaced by

2. A function $f(t,x)$ has the continuous partial derivative with respect to x on $\mathscr{T}\times D$. For any fixed $t_*\in\mathscr{T}$, $x_* = x^*_{s_1}+x^*_{s_2}+x^*_{p_1}+x^*_{p_2}$ such that $x^*_{s_1}+x^*_{p_1}\in M_{s1}$, $x^*_{s_2}\in M_{s_2}$, $x^*_{p_2}\in M_2$ and $(t_*,x_*)\in L_{t_+}$, the operator
$$\Phi_{t_*,x_*} := [\partial_x(Q_2 f)(t_*,x_*) - B]\,P_2 : X_2 \to Y_2 \qquad (40)$$
has the inverse $\Phi^{-1}_{t_*,x_*}\in L(Y_2,X_2)$.

Corollary 1. Theorem 1 remains valid if condition 3 is replaced by condition 3 given in Corollary 3.4 from [Filipkovska2024].

Corollary 2. Theorem 1 remains valid if condition 4 is replaced by

4. If M_{s1} is unbounded, then the following holds.
There exists a number $R>0$, a function $V\in C^1\big(\mathscr{T}\times M_R,\mathbb{R}\big)$ positive on $\mathscr{T}\times M_R$, where $M_R=\{(x_{s_1},x_{p_1})\in X_{s_1}\times X_1 \mid x_{s_1}+x_{p_1}\in M_{s1}, \|x_{s_1}+x_{p_1}\|>R\}$, and functions $k\in C(\mathscr{T},\mathbb{R})$, $U\in C(0,\infty)$ such that: $\lim\limits_{\|(x_{s_1},x_{p_1})\|\to+\infty} V(t,x_{s_1},x_{p_1}) = +\infty$ uniformly in t on each finite interval $[a,b)\subset\mathscr{T}$; for each $t\in\mathscr{T}$, $(x_{s_1},x_{p_1})\in M_R$, $x_{s_2}\in M_{s_2}$, $x_{p_2}\in M_2$ such that $(t,x_{s_1}+x_{s_2}+x_{p_1}+x_{p_2})\in L_{t_+}$, the inequality $\dot V_{(28),(29)}(t,x_{s_1},x_{p_1})\le k(t)\,U\big(V(t,x_{p_1})\big)$ holds; $\int\limits_{v_0}^{\infty}\dfrac{dv}{U(v)}=\infty$ ($v_0>0$ is a constant).

Corollary 3. If in the conditions of Theorem 1 the sets M_{s1}, M_{s_2} and M_2 are bounded, then equation (1) is Lagrange stable for the initial points $(t_0,x_0) \in L_{t_+}$ for which $(S_1+P_1)x_0 \in M_{s1}$, $S_2x_0 \in M_{s_2}$ and $P_2x_0 \in M_2$.

Remark 1. Note that if the conditions of Corollary 2 hold, then equation (1) is uniformly ultimately bounded (uniformly dissipative) for the initial points $(t_0,x_0) \in L_{t_+}$ for which $(S_1+P_1)x_0 \in M_{s1}$, $S_2x_0 \in M_{s_2}$ and $P_2x_0 \in M_2$.

Remark 2. The sets M_{s1}, M_{s_2}, M_2 can be considered as attracting sets in the sense that if a solution starts in the set $M_{s1} \dot{+} M_{s_2} \dot{+} M_2$ (i.e., $(S_1+P_1)x_0 \in M_{s1}$, $S_2x_0 \in M_{s_2}$ and $P_2x_0 \in M_2$), then it can never thereafter leave it.

Theorem 3 (the blow-up of solutions (Lagrange instability) of singular semilinear DAEs). Let $f \in C(\mathscr{T} \times D, \mathbb{R}^m)$, where $D \subseteq \mathbb{R}^n$ is some open set and $\mathscr{T} = [t_+, \infty) \subseteq [0, \infty)$, and let the operator pencil $\lambda A + B$ be a singular pencil such that its regular block $\lambda A_r + B_r$, where A_r, B_r are defined in (9), is a regular pencil of index not higher than 1. Assume that there exists an open (unbounded) set $M_{s1} \subseteq D_{s_1} \dot{+} D_1$ and sets $M_{s_2} \subseteq D_{s_2}$, $M_2 \subseteq D_2$ such that condition 1 of Theorem 1, condition 2 of Theorem 1 (or condition 2 of Theorem 2) and condition 3 of Theorem 1 (or condition 3 of Corollary 1) hold and:

4. There exists a function $V \in C^1(\mathscr{T} \times \widehat{M}_{s1}, \mathbb{R})$ positive on $\mathscr{T} \times \widehat{M}_{s1}$, where $\widehat{M}_{s1} = \{(x_{s_1}, x_{p_1}) \in X_{s_1} \times X_1 \mid x_{s_1} + x_{p_1} \in M_{s1}\}$, and a function $\chi \in C(\mathscr{T} \times (0, \infty), \mathbb{R})$ such that:

   (4.a) for each $t \in \mathscr{T}$, $(x_{s_1}, x_{p_1}) \in \widehat{M}_{s1}$, $x_{s_2} \in M_{s_2}$, $x_{p_2} \in M_2$ such that $(t, x_{s_1} + x_{s_2} + x_{p_1} + x_{p_2}) \in L_{t_+}$, the derivative (32) of the function V along the trajectories of equations (28), (29) satisfies the inequality

$$\dot{V}_{(28),(29)}(t, x_{s_1}, x_{p_1}) \geq \chi\big(t, V(t, x_{s_1}, x_{p_1})\big); \qquad (41)$$

   (4.b) the differential inequality $\dot{v} \geq \chi(t, v)$ $(t \in \mathscr{T})$ does not have global positive solutions.

Then for each initial point $(t_0, x_0) \in L_{t_+}$, for which $(S_1 + P_1)x_0 \in M_{s1}$, $S_2 x_0 \in M_{s_2}$ and $P_2 x_0 \in M_2$, IVP (1), (2) has a unique solution $x(t)$ for which the choice of the function $\phi_{s_2} \in C([t_0, \infty), M_{s_2})$ with the initial value $\phi_{s_2}(t_0) = S_2 x_0$ uniquely defines the component $S_2 x(t) = \phi_{s_2}(t)$ when $\mathrm{rank}(\lambda A + B) < n$ (when $\mathrm{rank}(\lambda A + B) = n$, the component $S_2 x$ is absent), and this solution has a finite escape time (i.e., is blow-up in finite time).

Corollary 4. Theorem 3 remains valid if condition 4 is replaced by

4. There exists a function $V \in C^1\big(\mathscr{T} \times \widehat{M}_{s1}, \mathbb{R}\big)$ positive on $\mathscr{T} \times \widehat{M}_{s1}$, where $\widehat{M}_{s1} = \{(x_{s_1}, x_{p_1}) \in X_{s_1} \times X_1 \mid x_{s_1} + x_{p_1} \in M_{s1}\}$, and functions $k \in C(\mathscr{T}, \mathbb{R})$, $U \in C(0,\infty)$ such that: for each $t \in \mathscr{T}$, $(x_{s_1}, x_{p_1}) \in \widehat{M}_{s1}$, $x_{s_2} \in M_{s_2}$, $x_{p_2} \in M_2$ such that $(t, x_{s_1} + x_{s_2} + x_{p_1} + x_{p_2}) \in L_{t_+}$ the inequality $\dot{V}_{(28),(29)}(t, x_{s_1}, x_{p_1}) \geq k(t)\, U\big(V(t, x_{s_1}, x_{p_1})\big)$ holds; $\int\limits_{k_0}^{\infty} k(t)dt = \infty$ and $\int\limits_{v_0}^{\infty} \frac{dv}{U(v)} < \infty$ ($k_0, v_0 > 0$ are constants).

Theorem 4 (The criterion of global solvability of singular semilinear DAEs).
Let $f \in C(\mathscr{T} \times D, \mathbb{R}^m)$, where $D \subseteq \mathbb{R}^n$ is some open set and $\mathscr{T} = [t_+, \infty) \subseteq [0, \infty)$, and let the operator pencil $\lambda A + B$ be a singular pencil such that its regular block $\lambda A_r + B_r$, where A_r, B_r are defined in (9), is a regular pencil of index not higher than 1. Let there exist an open set $M_{s1} \subseteq D_{s_1} \dot{+} D_1$ and sets $M_{s_2} \subseteq D_{s_2}$, $M_2 \subseteq D_2$ such that conditions 1, 2 and 3 of Theorem 1 hold.
Then for each initial point $(t_0, x_0) \in L_{t_+}$ such that $(S_1 + P_1)x_0 \in M_{s1}$, $S_2 x_0 \in M_{s_2}$ and $P_2 x_0 \in M_2$, IVP (1), (2) has a unique solution $x(t)$ for which the choice of the function $\phi_{s_2} \in C([t_0, \infty), M_{s_2})$ with the initial value $\phi_{s_2}(t_0) = S_2 x_0$ uniquely defines the component $S_2 x(t) = \phi_{s_2}(t)$ when $\mathrm{rank}(\lambda A + B) < n$ (when $\mathrm{rank}(\lambda A + B) = n$, the component $S_2 x$ is absent), and this solution is global if condition 4 of Theorem 1 holds and has a finite escape time if condition 4 of Theorem 3 holds.

Corollary 5. Theorem 4 remains valid if any of the following replacements (or all of them) take place:

- condition 2 of Theorem 1 is replaced by condition 2 of Theorem 2;
- condition 3 of Theorem 1 is replaced by condition 3 of Corollary 1;
- condition 4 of Theorem 1 is replaced by condition 4 of Corollary 2,
- condition 4 of Theorem 3 is replaced by condition 4 of Corollary 4.

Several examples demonstrating the verification of the conditions of the obtained theorems and their effectiveness are presented in
[Filipkovska M. Criterion of the global solvability of regular and singular differential-algebraic equations. *J. of Mathematical Sciences* (2024) [in Production] `https://doi.org/10.1007/s10958-024-07152-7`]

In addition, in this paper, a relationship with the results of the paper [Filipkovska M. Qualitative analysis of nonregular differential-algebraic equations and the dynamics of gas networks. *Journal of Mathematical Physics, Analysis, Geometry*, Vol. 19, No. 4, 719–765 (2023). `https://doi.org/10.15407/mag19.04.719`] is described.

The model of a radio engineering device

A voltage source $e(t)$,
nonlinear resistances φ, φ_0, ψ,
a nonlinear conductance h,
a linear resistance r,
a linear conductance g,
an inductance L and
a capacitance C are given.

Let $e(t) \in C([0,\infty),\mathbb{R})$,
$\varphi(y), \varphi_0(y), \psi(y), h(y) \in C^1(\mathbb{R},\mathbb{R})$,
$r,\ g,\ L,\ C > 0$.

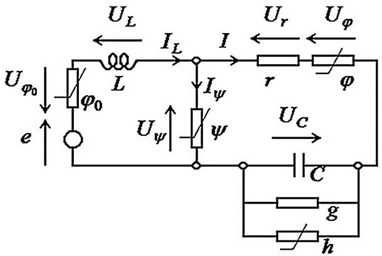

Fig. 1. The diagram of the electric circuit

The model of the circuit Fig. 1 is described by the system with the variables
$x_1 = I_L,\ x_2 = U_C,\ x_3 = I$:

$$L\frac{d}{dt}x_1 + x_2 + r\,x_3 = e(t) - \varphi_0(x_1) - \varphi(x_3), \quad (42)$$

$$C\frac{d}{dt}x_2 + gx_2 - x_3 = -h(x_2), \quad (43)$$

$$x_2 + rx_3 = \psi(x_1 - x_3) - \varphi(x_3). \quad (44)$$

$$A = \begin{pmatrix} L & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & 1 & r \\ 0 & g & -1 \\ 0 & 1 & r \end{pmatrix}$$

The vector form of the system is the DAE

$$\frac{d}{dt}[Ax] + Bx = f(t,x), \qquad (45)$$

where $x = (x_1, x_2, x_3)^T \in \mathbb{R}^3$

$$f(t,x) = \begin{pmatrix} e(t) - \varphi_0(x_1) - \varphi(x_3) \\ -h(x_2) \\ \psi(x_1 - x_3) - \varphi(x_3) \end{pmatrix}$$

This model has been studied in [Filipkovska M.S. Lagrange stability of semilinear differential-algebraic equations and application to nonlinear electrical circuits. *J. of Math. Phys., Anal., Geom.*, Vol. 14, No. 2, 169–196 (2018). `https://doi.org/10.15407/mag14.02.169`]. Below we present some results from this paper.

Lagrange stability of the model of a radio engineering device.

The particular cases.

$$\varphi_0(y) = \alpha_1 y^{2k-1},\ \varphi(y) = \alpha_2 y^{2l-1},\ \psi(y) = \alpha_3 y^{2j-1},\ h(y) = \alpha_4 y^{2s-1}, \tag{46}$$
$$\varphi_0(y) = \alpha_1 y^{2k-1},\ \varphi(y) = \alpha_2 \sin y,\ \psi(y) = \alpha_3 \sin y,\ h(y) = \alpha_4 \sin y, \tag{47}$$

$k,l,j,s \in \mathbb{N}$, $\alpha_i > 0$, $i = \overline{1,4}$, $y \in \mathbb{R}$.

For each initial point (t_0,x^0) satisfying $x_2^0 + r x_3^0 = \psi(x_1^0 - x_3^0) - \varphi(x_3^0)$, there exists a unique global solution of the IVP (45), $x(t_0) = x^0$ $(x(t_0) = (I_L(t_0), U_C(t_0), I(t_0))^T)$ for the functions of the form (46), if $j \le k$, $j \le s$ and α_3 is sufficiently small, and for the functions of the form (47), if $\alpha_2 + \alpha_3 < r$.

If, additionally, $\sup\limits_{t\in[0,\infty)} |e(t)| < +\infty$ or $\int\limits_{t_0}^{+\infty} |e(t)|\,dt < +\infty$, then for the initial points (t_0,x^0) the DAE (45) is Lagrange stable (in both cases), i.e., every solution of the

DAE is bounded. In particular, these requirements are fulfilled for voltages of the form

$$e(t) = \beta(t+\alpha)^{-n}, e(t) = \beta e^{-\alpha t}, e(t) = \beta e^{-\frac{(t-\alpha)^2}{\sigma^2}}, e(t) = \beta \sin(\omega t + \theta), \quad (48)$$

where $\alpha > 0$, $\beta, \sigma, \omega \in \mathbb{R}$, $n \in \mathbb{N}$, $\theta \in [0, 2\pi]$.

Lagrange stability. The numerical solution

$L = 500 \cdot 10^{-6}$, $C = 0.5 \cdot 10^{-6}$, $r = 2$, $g = 0.2$, $t_0 = 0$, $x_0 = (10, -10, 5)^T$
$\varphi_0(y) = y^3$, $\varphi(y) = \sin y$, $\psi(y) = \sin y$, $h(y) = \sin y$, $e(t) = (2t + 10)^{-2}$

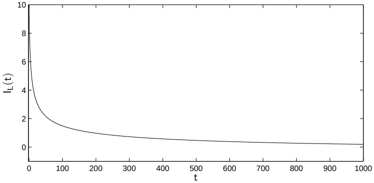

Fig. 2. The current $I_L(t)$

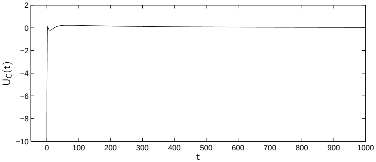

Fig. 3. The voltage $U_C(t)$

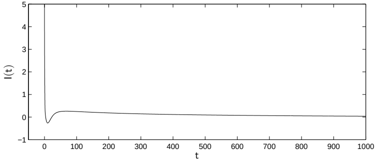

Fig. 4. The current $I(t)$

Lagrange stability. The numerical solution

$L = 500 \cdot 10^{-6}$, $C = 0.5 \cdot 10^{-6}$, $r = 2$, $g = 0.2$, $t_0 = 0$, $x_0 = (0,0,0)^T$,
$\varphi_0(y) = y^3$, $\varphi(y) = y^3$, $h(y) = y^3$, $\psi(y) = y^3$, $e(t) = 100\, e^{-t} \sin(5t)$

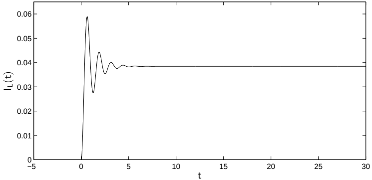

Fig. 5. The current $I_L(t)$

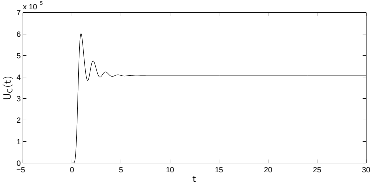

Fig. 6. The voltage $U_C(t)$

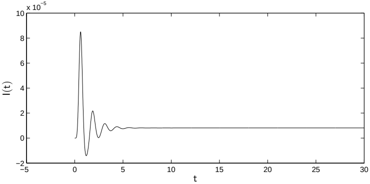

Fig. 7. The current $I(t)$

The global solution. The numerical solution

$L = 1000\cdot 10^{-6}$, $C = 0.5\cdot 10^{-6}$, $r = 2$, $g = 0.3$, $t_0 = 0$, $x^0 = (0,0,0)^T$
$\varphi_0(y) = y^3$, $\varphi(y) = y^3$, $\psi(y) = y^3$, $h(y) = y^3$, $e(t) = -t^2$

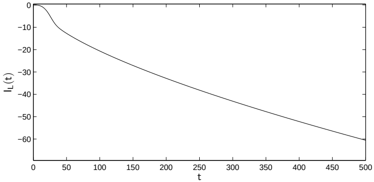

Fig. 8. The current $I_L(t)$

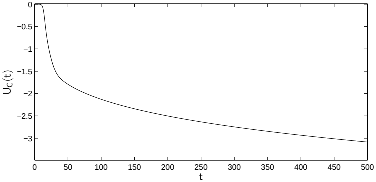

Fig. 9. The voltage $U_C(t)$

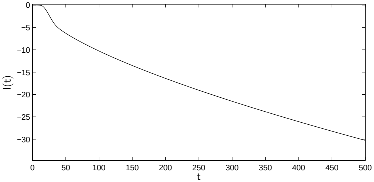

Fig. 10. The current $I(t)$

Lagrange instability of the radio engineering device model

Consider the system (42)–(44) with the nonlinear resistances and conductance

$$\varphi_0(y) = -y^2,\ \varphi(y) = y^3,\ \psi(y) = y^3,\ h(y) = y^2. \tag{49}$$

It is assumed that there exists $M_e = \sup\limits_{t\in[t_0,\infty)} |e(t)| < +\infty$. Choose

$$\Omega = \Big\{ (x_1, x_2)^T \in \mathbb{R}^2 \mid x_1 > m_1, m_1 = \max\Big\{ 1+\sqrt{M_e},\ \sqrt[3]{g+r^{-1}}, 3CL^{-1}, \sqrt{\max\{3^{-1}(L(rC)^{-1}-r), 0\}} \Big\}, x_2 < -rx_1 - x_1^3 - m_2, \quad m_2 = \max\{g - 2CL^{-1}r, 0\} \Big\}. \tag{50}$$

Then for any initial moment t_0 and any initial currents and voltage $I_L(t_0)$, $U_C(t_0)$, $I(t_0)$ satisfying $U_C(t_0) + rI(t_0) = \psi(I_L(t_0) - I(t_0)) - \varphi(I(t_0))$ and such that $(I_L(t_0), U_C(t_0))^T \in \Omega$ there exists a unique distribution of the currents and voltages in the circuit Fig. 1 only for $t_0 \le t < T$ ($[t_0, T)$ is some finite interval) and the currents and voltages are unbounded.

It means that there exists a unique solution of the Cauchy problem for the DAE (45) with the functions (49), $e(t)$ such that $\sup\limits_{t\in[t_0,\infty)} |e(t)| < +\infty$, and the initial condition $x(t_0) = (I_L(t_0), U_C(t_0), I(t_0))^T$, and this *solution has a finite escape time*.

Lagrange instability. The numerical solution

$L = 10 \cdot 10^{-6}$, $C = 0.5 \cdot 10^{-6}$, $r = 2$, $g = 0.2$,
$\varphi_0(x_1) = -x_1^2$, $\varphi(x_3) = x_3^3$, $h(x_2) = x_2^2$, $\psi(x_1 - x_3) = (x_1 - x_3)^3$, $e(t) = 2\sin t$,
$t_0 = 0$, $x_0 = (2.45, -20.625125, 2.5)^T$

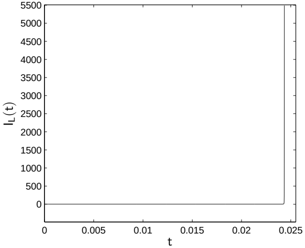

Fig. 11. The current $I_L(t)$

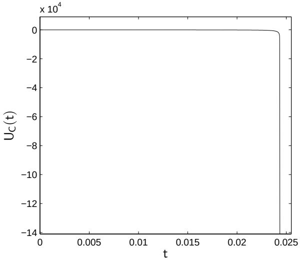

Fig. 12. The voltage $U_C(t)$

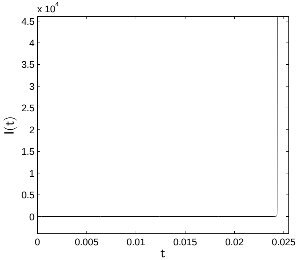

Fig. 13. The current $I(t)$

Model of a gas flow for a single pipe

We consider the mathematical model of a gas pipeline which consists of the **isothermal Euler equations** of the form

$$\partial_t \rho = -\partial_x \varphi, \tag{51}$$

$$\partial_t \varphi = -\partial_x p - g\rho s_{\text{lope}} - 0.5\lambda D^{-1}\varphi|\varphi|\rho^{-1} \tag{52}$$

and the **equation of state for a real gas** in the form

$$p = RT_0 \rho z(p), \tag{53}$$

- $x \in [0,L]$, $t \in [0,t_1) \subseteq [0,\infty)$, where $[t_0,t_1)$ is the time interval, $L < \infty$ is the pipe length and T_0 is the temperature
- $\rho = \rho(t,x)$, $\varphi = \varphi(t,x)$ ($\varphi := \rho v$, v is the velocity) and $p = p(t,x)$ are respectively the density, flow rate and pressure
- g is the gravitational constant, and R is the specific gas constant
- λ is the pipe friction coefficient, and D is the pipe diameter
- $s_{\text{lope}}(x) = dh(x)/dx$ denotes the slope of the pipe, where $h = h(x)$ is the height profile of the pipe over ground
- $z = z(p)$ is the compressibility factor

The modeling of gas networks is described, e.g., in [P. Benner, S. Grundel, C. Himpe, C. Huck, T. Streubel, C. Tischendorf. *Gas Network Benchmark Models*, 2018]

Denote $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & -\frac{d}{dx} & 0 \\ -g\, s_{lope} & 0 & -\frac{d}{dx} \\ 0 & 0 & -1 \end{pmatrix}$, $f(u) = \begin{pmatrix} 0 \\ -\frac{\lambda}{2D}\frac{\varphi|\varphi|}{\rho} \\ RT_0\rho z(p) \end{pmatrix}$ and $u = (\rho, \varphi, p)^T$. Then we can write the system (51)–(53) as:

$$A\frac{d}{dt}u(t) + Bu(t) = f(u(t)), \tag{54}$$

where $u = u(t)(x) = (\rho(t,x), \varphi(t,x), p(t,x))^T$, $x \in [0,L]$, $t \in [0,t_1)$. The initial condition has the form:

$$u(0) = u_0, \qquad u_0 = u_0(x) = (\rho(0,x), \varphi(0,x), p(0,x))^T, \quad x \in [0,L], \tag{55}$$

where $p(0,x)$ is chosen so as to satisfy the equation (53) for $t = 0$, $x \in [0,L]$. We will assume that $u(t,x)$ satisfies suitable boundary conditions, for example,

$$\varphi(t,0) = \varphi_l(t), \quad p(t,0) = p_l(t), \quad t \in [0,t_1), \tag{56}$$

i.e., $u(t)(0) = u_l(t) = (\rho(t,0), \varphi_l(t), p_l(t))^T$, where $\varphi_l(t)$ and $p_l(t)$ are given.

A model of a gas network (in the isothermal case)

Describe a gas network as oriented connected graph $G = (\mathscr{V}, \mathscr{E})$, where $\mathscr{V}$ denotes a set of nodes (vertices), $\mathscr{E}$ denotes a set of edges, and each edge joins two distinct nodes (i.e., there are no self-loops). We fix the orientation of edge $e \in \mathscr{E}$, denoting its endpoints by v_l and v_r and assuming that the edge is oriented from the left node v_l to the right node v_r.

We collect all nodes with a fixed pressure in $\mathscr{V}_{pset}$ and refer to them as *pressure nodes*. All other nodes we collect in $\mathscr{V}_{qset}$. Accordingly, $\mathscr{V} = \mathscr{V}_{pset} \cup \mathscr{V}_{qset}$.

We denote the sets of edges corresponding to the pipes, valves and regulating elements (regulators and compressors) by $\mathscr{E}_{pip}$, $\mathscr{E}_{val}$ and $\mathscr{E}_{reg}$, respectively. Thus, $\mathscr{E} = \mathscr{E}_{pip} \cup \mathscr{E}_{val} \cup \mathscr{E}_{reg}$.

Introduce the vector p of the pressures of nodes $u \in \mathscr{V}_{pset}$, and the vectors $q_{pip,r}$, $q_{pip,l}$, q_{val} and q_{reg} of flows at the right ends of pipes, at the left ends of pipes, through valves and through regulating elements, respectively.

At the pressure nodes $u \in \mathscr{V}_{pset}$, the pressure function $p^{set}(t) = (\dots, p_u^{set}(t), \dots)^T_{u \in \mathscr{V}_{pset}}$ is given. At the nodes $u \in \mathscr{V}_{qset} = \mathscr{V} \setminus \mathscr{V}_{pset}$ (which include junction, demand and source nodes), the function $q^{set}(t) = (\dots, q_u^{set}(t), \dots)^T_{u \in \mathscr{V}_{qset}}$, which specifies the relationships between the flows $q_{pip,r}$, $q_{pip,l}$, q_{val} and q_{reg} in a Kirchhoff-type flow balance equation (see (61) below), is given.

The mathematical model of a gas network consisting of pipes, valves, regulators and compressors after applying spatial discretization (more precisely, a topologically adaptive discretization of the isothermal Euler equations for pipes and pipelines) has the form:

$$
\begin{aligned}
A_{pip,r}^T \frac{d}{dt}\phi(p) + D_q(q_{pip,r} - q_{pip,l}) &= 0, && (57)\\
\frac{d}{dt} q_{pip,l} + D_p(A_{pip,r}^T + A_{pip,l}^T)p + f_{pip}(p, q_{pip,l}, t) &= 0, && (58)\\
D_{val} \frac{d}{dt} q_{val} + f_{val}(p, q_{val}, t) &= 0, && (59)\\
D_{reg} \frac{d}{dt} q_{reg} - f_{reg}(p, q_{reg}, t) &= 0, && (60)\\
A_{pip,l} q_{pip,l} + A_{pip,r} q_{pip,r} + A_{val} q_{val} + A_{reg} q_{reg} &= q^{set}(t), && (61)\\
f_{pb}(p) &= 0, && (62)\\
f_{qb}(q_{pip,l}, q_{pip,r}, q_{val}, q_{reg}) &= 0, && (63)
\end{aligned}
$$

where $A_{pip,l} := (a_{ij}^{pip,l})_{\substack{i=1,\dots,|\mathscr{V}_{qset}|,\\ j=1,\dots,|\mathscr{E}_{pip}|}}$, $A_{pip,r} := (a_{ij}^{pip,r})_{\substack{i=1,\dots,|\mathscr{V}_{qset}|,\\ j=1,\dots,|\mathscr{E}_{pip}|}}$, $A_{val} := (a_{ij}^{val})_{\substack{i=1,\dots,|\mathscr{V}_{qset}|,\\ j=1,\dots,|\mathscr{E}_{val}|}}$ and $A_{reg} := (a_{ij}^{reg})_{\substack{i=1,\dots,|\mathscr{V}_{qset}|,\\ j=1,\dots,|\mathscr{E}_{reg}|}}$ are constant incidence matrices with the entries presented in [KSSTW22], $D_q := \mathrm{diag}\{\dots,\frac{\kappa_e}{L_e},\dots\}_{e\in\mathscr{E}_{pip}}$, $D_p := \mathrm{diag}\{\dots,\frac{S_e}{L_e},\dots\}_{e\in\mathscr{E}_{pip}}$, $D_{val} := \mathrm{diag}\{\dots,\mu_e,\dots\}_{e\in\mathscr{E}_{val}}$ and $D_{reg} := \mathrm{diag}\{\dots,\mu_e,\dots\}_{e\in\mathscr{E}_{reg}}$ are constant diagonal matrices, where $\mu_e \geq 0$, $\kappa_e = R_s T_0/S_e$ (as above, $T_0 = \mathrm{const}$ is the temperature and R_s is the specific gas constant), S_e and L_e are the cross-sectional area and the length of pipe e, respectively. Here p, $q_{pip,r}$, $q_{pip,l}$, q_{val} and q_{reg} are unknown and the remaining functions and parameters are given. $f_{pip}(p,q_{pip,l},t)$, $f_{val}(p,q_{val},t)$ and $f_{reg}(p,q_{reg},t)$ are functions specified in [KSSTW22, p.5–7]; $f_{pb}(p)$ and $f_{qb}(q_{pip,l},q_{pip,r},q_{val},q_{reg})$ are given continuous functions.

[KSSTW22] = [T. Kreimeier, H. Sauter, S.T. Streubel, C. Tischendorf, and A. Walther, *Solving Least-Squares Collocated Differential Algebraic Equations by Successive Abs-Linear Minimization – A Case Study on Gas Network Simulation*, Humboldt-Universität zu Berlin, 2022, preprint].

We introduce an additional variable $\rho = \begin{pmatrix} \vdots \\ \rho_{\mathrm{u}} \\ \vdots \end{pmatrix}_{\mathrm{u} \in \mathscr{V}_{\mathrm{qset}}}$, and instead of (57) we use the system

$$\mathrm{A}_{\mathrm{pip,r}}^{\mathrm{T}} \frac{\mathrm{d}}{\mathrm{dt}} \rho + \mathrm{D_q}(\mathrm{q_{pip,r}} - \mathrm{q_{pip,l}}) = 0,$$
$$\rho = \phi(\mathrm{p}),$$

which is equivalent to (57), taking into account the coefficient κ_{e}. Also, we rewrite the function $\mathrm{f_{pip}}(\mathrm{p},\mathrm{q_{pip,l}},\mathrm{t})$, without changing its notation, as $\mathrm{f_{pip}}(\rho,\mathrm{q_{pip,l}},\mathrm{t})$.

These system can be written in the form of the singular (nonregular) DAE

$$\frac{\mathrm{d}}{\mathrm{dt}}[\mathrm{Ax}] + \mathrm{Bx}(\mathrm{t}) = \mathrm{f}(\mathrm{t},\mathrm{x}), \tag{64}$$

where

$$x = \begin{pmatrix} \rho \\ q_{pip,l} \\ q_{val} \\ q_{reg} \\ q_{pip,r} \\ p \end{pmatrix}, f(t,x) = \begin{pmatrix} 0 \\ -f_{pip}(\rho, q_{pip,l}, t), \\ -f_{val}(p, q_{val}, t) \\ f_{reg}(p, q_{reg}, t), \\ q^{set}(t) \\ \phi(p) \\ f_{pb}(p) \\ f_{qb}(q_{pip,l}, q_{pip,r}, q_{val}, q_{reg}) \end{pmatrix}$$

$$A = \begin{pmatrix} A_{pip,r}^T & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & D_{val} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & D_{reg} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$
B = \begin{pmatrix}
0 & -D_q & 0 & 0 & D_q & 0 \\
0 & 0 & 0 & 0 & 0 & D_p(A_{pip,r}^T + A_{pip,l}^T) \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & A_{pip,l} & A_{val} & A_{reg} & A_{pip,r} & 0 \\
I & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}. \qquad (65)
$$

The initial condition for the DAE (64) has the form

$$
x(0) = x_0, \qquad (66)
$$

where $x_0 = (\rho^0, q_{pip,l}^0, q_{val}^0, q_{reg}^0, q_{pip,r}^0, p^0)^T$ is chosen so that the values t_0, x_0 satisfy the consistency condition.

[Filipkovska M. Qualitative analysis of nonregular differential-algebraic equations and the dynamics of gas networks. *Journal of Mathematical Physics, Analysis, Geometry*, Vol. 19, No. 4, 719–765 (2023). `https://doi.org/10.15407/mag19.04.719`]

Discussions

For the abstract semilinear DAE (1) with the regular characteristic pencil, the criterion of the global solvability is obtained in a preprint. Here we suppose that the pencil $\mathrm{P}(\lambda)$ is a regular pencil of index ν, where $\nu \in \mathbb{N}$ is some number. Thus, we consider higher-index regular abstract DAEs .

Thank you for your attention!