# Hypoelliptic Kolmogorov operators

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# Prototype

Let $(W_t)_{t\geq 0}$ denotes a real Brownian motion, and consider the Stochastic process $(V_t, Y_t)_{t\geq 0}$

$$\begin{cases} V_t = v_0 + \sqrt{2} W_t, \\ Y_t = y_0 + \int_0^t V_s ds. \end{cases}$$

The density $p = p(v, y, v_0, y_0, t)$ is a solution to

$$\mathscr{L} p = \partial^2_{vv} p + v \partial_y p - \partial_t p = 0$$

that we notice to be a **degenerate** equation.

# Kolmogorov equation

$$\mathscr{L} p = \partial^2_{vv} p + v \partial_y p - \partial_t p$$

Kolmogorov (1934) provided us with the explicit expression of the density $p$ (that is the *fundamental solution* of the operator)

$$p = \frac{\sqrt{3}}{2\pi t^2} \exp\left( -\frac{(v - v_0)^2}{t} - 3\frac{(v - v_0)(y - y_0 - tv_0)}{t^2} - 3\frac{(y - y_0 - tv_0)^2}{t^3} \right).$$

We point out that despite the degeneracy of the equations the density is smooth, this indicating that the operator is **hypoelliptic**.

# Hypoelliptic operator

**Hypoellipticity**

An operator $\mathscr{L}$ is **hypoelliptic** if, for every distributional solution $u \in L^1_{\text{loc}}(\Omega)$ to the equation $\mathscr{L}u = f$, we have that

$$f \in C^\infty(\Omega) \implies u \in C^\infty(\Omega).$$

An hypoelliptic operator posses the same regularity property of elliptic operator with $C^\infty$ coefficients.

# Hypoelliptic operator

Starting from Kolmogorov's observations, Hörmander (1967) considered a more general class of operators on $\mathbb{R}^{N+1}$

$$\boxed{\mathscr{L} = \sum_{k=1}^{m} X_k^2 + Y,}$$

where $X_k$ and $Y$ are smooth vector fields of the form

$$\boxed{X_k = X_k(z) = \sum_{j=1}^{N+1} a_{j,k}(z)\partial_{z_j}, \qquad Y = Y(z) = \sum_{j=1}^{N+1} a_{j,m+1}(z)\partial_{z_j},}$$

with $a_{j,k}, a_{j,m+1} \in C^\infty(\Omega)$.

# Smooth vector fields

Given two vector fields $Z_1, Z_2$, their commutator is given by

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We identify every vector field $Z(z) = \sum_{j=1}^{N+1} a_j(z)\partial_{z_j}$ with the vector $(a_1(z), \dots, a_{N+1}(z))$.

# Hörmander's rank condition

**Theorem (Hörmander)**

Suppose that

$$\mathrm{rank}\,\mathrm{Lie}(X_1, \dots, X_m, Y)(z) = N + 1, \qquad \forall z \in \Omega.$$

Then the operator $\mathscr{L} = \sum_{k=1}^{m} X_k^2 + Y$, is hypoelliptic.

The Kolmogorov operator

$$\mathscr{L} = \partial_{vv}^2 + v\partial_y - \partial_t$$

is of Hörmander type with $X = \partial_v$, $Y = v\partial_y - \partial_t$ and $[X, Y] = \partial_y$.

# Lie group

Since the regularity properties of Hörmander operators are related to a Lie algebra, the natural framework for the regularity theory is the non-euclidean setting of the homogeneous Lie groups, as pointed out by Folland and Stein (1974).

## Homogeneous Lie groups

A Lie group $\mathbb{G} = (\mathbb{R}^{N+1}, \circ, \delta_r)$ is a group on $\mathbb{R}^{N+1}$ with a smooth composition law $\circ$ and a dilation law $\{\delta_r\}_{r\geq 0}$ that is an automorphism of the group

$$\delta_r(x \circ y) = \delta_r(x) \circ \delta_r(y), \qquad \forall x, y \in \mathbb{R}^{N+1}, r \geq 0.$$

# Lie group

$$\mathscr{L} = \partial_{vv}^2 + v\partial_y - \partial_t$$

Let consider on $\mathbb{R}^3$ an homogeneous Lie group with composition law and dilation law given by

$$(v, y, t) \circ (v_0, y_0, t_0) = (v + v_0, y + y_0 - tv_0, t + t_0),$$

$$\delta_r(v, y, t) = (rv, r^3 y, r^2 t).$$

$\mathscr{L}$ is left invariant w.r.t. $\circ$ and homogeneous of degree 2 with respect to $\delta_r$. This also appears from the fundamental solution

$$p = \frac{\sqrt{3}}{2\pi t^2} \exp\left(-\frac{(v - v_0)^2}{t} - 3\frac{(v - v_0)(y - y_0 - tv_0)}{t^2} - 3\frac{(y - y_0 - tv_0)^2}{t^3}\right).$$

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# Kolmogorov operators

Starting from these observation, some mathematicians (Lanconelli, Polidoro, Pascucci, Pagliarani, Manfredini...) investigated a wide class of Kolmogorov operators

$$\boxed{\mathscr{L} = \mathrm{Tr}(AD^2) + \langle Bx, D\rangle - \partial_t, \quad (x,t) \in \mathbb{R}^{N+1},}$$

with $A = A^T \geq 0$. There are many condition for the hypoellipticity of $\mathscr{L}$ to hold. One is the *Kalman's rank condition*

$$\boxed{\mathrm{rank}\ \left(A^{\frac{1}{2}}, BA^{\frac{1}{2}}, B^2A^{\frac{1}{2}}, ..., B^{N-1}A^{\frac{1}{2}}\right) = N.}$$

This is the starting point to study more general operators (variable coefficients, non-local terms...). The matrices $A$ and $B$ defines the Lie group structure useful to study these operators.

# Intrinsic regularity space

When working with Hörmander vector fields, is useful to work in the regularity framework that they induce. Let $Z$ be a vector field, we denote by $s \mapsto \mathrm{e}^{sZ}(z)$ the integral curve of $Z$, that is the unique solution to

$$\begin{cases} \frac{d}{ds}\mathrm{e}^{sZ}(z) = Z(\mathrm{e}^{sZ}(z)), \\ \mathrm{e}^{sZ}(z)\big|_{s=0} = z. \end{cases}$$

A function $u$ is $Z$-Lie differentiable if the function $s \mapsto u(\mathrm{e}^{sZ}(z))$ is differentiable.

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# Intrinsic regularity space

We closely inspect the prototype Kolmogorov operator

$$\mathscr{L} = \partial^2_{vv} + v\partial_y - \partial_t = X^2 + Y.$$

The integral curve of $X$ is $\mathrm{e}^{sX}(v, y, t) = (v + s, y, t)$, while the integral curve of $Y$ is $\mathrm{e}^{sY}(v, y, t) = (v, y + sv, t - s)$.
The intrinsic regularity space of classical solutions is the space of functions $u$ with two continuous derivatives w.r.t. the non-degenerate variable $v$, and with continuous Lie-derivative $Yu$

$$Yu := \lim_{s \to 0} \frac{u(v, y + sv, t - s) - u(v, y, t)}{s}$$

# Kuramoto model with inertia

These operators appear for instance in the following Kuramoto-type model, that describes the synchronization of coupled oscillators.
Let us consider a continuum of coupled oscillators, whose natural frequencies are distributed according to a function $g(\Omega)$. The density function $\rho$ that describes the fraction of oscillators at phase $\theta$, frequency $\omega$, natural frequency $\Omega$ at time $t$ solves

$$\frac{\partial^2 \rho}{\partial \omega^2} + \frac{\partial}{\partial \omega}\left[(\omega - \Omega - K_\rho(\theta, t))\rho\right] - \omega \frac{\partial \rho}{\partial \theta} - \frac{\partial \rho}{\partial t} = 0,$$

$$K_\rho(\theta, t) = K \int_{\mathbb{R}} \int_{\mathbb{R}} \int_0^{2\pi} g(\Omega') \sin(\theta' - \theta) \rho(\omega', \theta', \Omega', t) \, d\theta' d\omega' d\Omega'.$$

where $K$ represents the strength coupling between oscillators.

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# Assumptions

① the initial condition $\rho_0$ is continuous, strictly positive, $2\pi$ periodic in $\theta$ and for every $\Omega \in \mathbb{R}$ verifies

$$\int_{]0,2\pi[\times\mathbb{R}} \rho_0(\omega,\theta,\Omega)\, d\theta\, d\omega = 1.$$

② $\rho_0$ has an exponential decay in $\omega$

$$\rho_0(\omega,\theta,\Omega,t) \le C\mathrm{e}^{-M\omega^2}.$$

③ $g$ is a non-negative, normalized function such that

$$\int_{\mathbb{R}} g(\Omega)\mathrm{e}^{|\Omega|^\beta}\, d\Omega < +\infty, \qquad \text{for some } \beta > 2.$$

# Existence result

## Theorem (P., Polidoro, Vernia)

Under assumptions ①, ②, ③ there exists a strictly positive classical solution $\rho$ in $\mathbb{R}^3 \times [0, +\infty[$ such that

$$\int_{]0,2\pi[\times\mathbb{R}} \rho(\omega, \theta, \Omega, t) d\theta d\omega = 1, \qquad \text{for every} \quad t \geq 0, \ \Omega \in \mathbb{R}.$$

Moreover $\rho$ is $2\pi$ periodic in $\theta$, continuously depends on $\Omega$ and verifies the following bounds

$$\rho(\omega, \theta, \Omega, t) \leq C_\Omega \mathrm{e}^{-\overline{M}\omega^2}, \qquad |\partial\rho(\omega, \theta, \Omega, t)| \leq t^{-k/2} C_\Omega \mathrm{e}^{-\overline{M}\omega^2}$$

for $k = 1$ ($\partial_\omega$) and $k = 2$ ($\partial_\omega^2$ and $Y$). Furthermore, if $g$ has compact support $\rho$ is the unique solution satisfying the properties above.

# Numerical method

We apply the Kolmogorov operator structure to define a *stable* numerical scheme. We use a finite difference scheme based on the approximation of the Lie derivative

$$Y\rho = \left(-\omega\frac{\partial}{\partial\theta} - \frac{\partial}{\partial t}\right)\rho$$

$$\Downarrow$$

$$\frac{\rho(\omega, \theta - \omega\Delta t, \Omega, t - \Delta t) - \rho(\omega, \theta, \Omega, t)}{\Delta t}$$

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# Test parameters

We test this numerical method evaluating the following quantity

$$|r(t)| = \left| \int_0^{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\theta} \rho(\theta, \omega, \Omega, t) g(\Omega) d\Omega d\omega d\theta \right|$$

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This parameter (that lives in the interval $[0, 1]$) give us key informations about phase synchronization: a value close to 0 implies low synchronization, while a value closer to 1 has the opposite meaning.

# Results

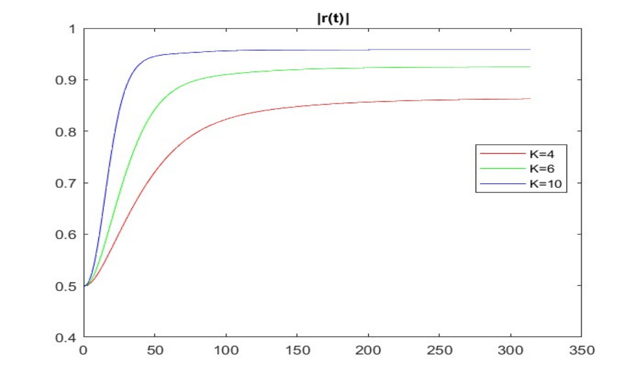


Figure 1: Time evolution of the phase coherence, $T = 10, \Delta t = 0.0317$

# That's all!

## Thanks for the attention!

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