Small-Time Local Controllability of the multi-input bilinear Schrödinger equation thanks to a quadratic term

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X Partial differential equations, optimal design and numerics
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One considers the affine system:

$$x' = \mathbf{f_0}(x) + uf_1(x) + vf_2(x), \tag{1}$$

with $f_0, f_1, f_2 \in \mathcal{C}^\omega(\mathbb{R}^d)$. The terms $\mathbf{f_0}$ is called the **drift**.

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We assume that $f_0(0) = 0$, *i.e.* $(0,(0,0))$ is an **equilibrium** trajectory of the system (1).

We focus on small time and small controls: the solution is well-defined, and we note it $x(\cdot;(u,v),0)$.

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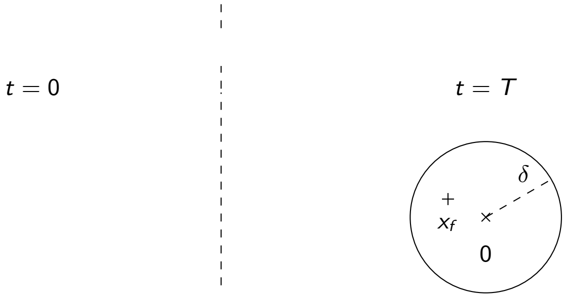

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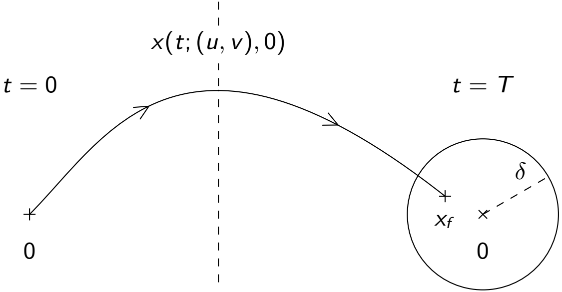

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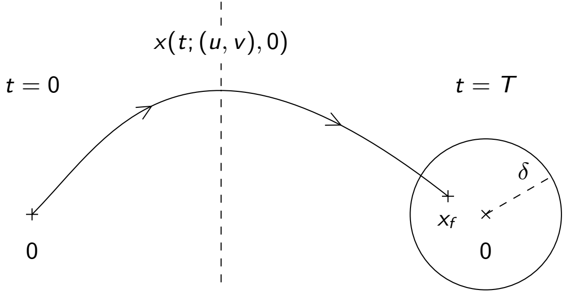
Historical definition : $E = L^\infty$.



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Let

$$\mathcal{F}_T : \left[\begin{array}{ccc} E^2 & \rightarrow & \mathbb{R}^d \\ (u,v) & \mapsto & x(T;(u,v),0) \end{array} \right].$$

Then,

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Definition (smooth-STLC)

(1) is **smooth-STLC** if (1) is $W^{m,\infty}-STLC$, for every $m \in \mathbb{N}$

Definition (Lie Brackets)

For f, g, regular vectors fields on $\mathbb{R}^d$, we define the vector field $[f, g]$ as :

$$[f, g] : x \in \mathbb{R}^d \mapsto Dg_x f(x) - Df_x g(x).$$

By induction, one defines : $\mathrm{ad}_f^0 g = g$ and $\forall k \in \mathbb{N}, \mathrm{ad}_f^{k+1}(g) = [f, \mathrm{ad}_f^k(g)]$.

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Remark

If $f \leftrightarrow op_f := f \cdot \nabla$, Lie brackets coincide with operator commutators.

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Example

One supposes $f_0(x) = \begin{pmatrix} x_2^2 \\ 0 \end{pmatrix}$ and $f_1(x) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Then,

$$[f_1, f_0](x) = \begin{pmatrix} 0 & 2x_2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2x_2 \\ 0 \end{pmatrix}.$$

$$\mathrm{ad}_{f_1}^2(f_0)(0) = [f_1, \mathrm{ad}_{f_1}^1(f_0)](0) = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 2e_1.$$

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$$\text{ad}^2_{f_1}(f_0)(0) = [f_1, \text{ad}^1_{f_1}(f_0)](0) = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 2e_1.$$

We want to prove **sufficient conditions** of controllability in terms of the evaluation at $x = 0$ of **Lie brackets** of f_0, f_1 and f_2.

Theorem (W.-L. Chow, 1939, P.K. Rashevski, 1938)

If $\mathbf{f_0} \equiv 0$ (no drift), then, the system (1) is $L^\infty - STLC$ **iff** LARC holds, i.e.

$$Lie(f_0, f_1, f_2)(0) = \mathbb{R}^d.$$

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This result is **false** in general. For example, $\begin{cases} x_1' & = & x_2^2 \geqslant 0 \\ x_2' & = & u \end{cases}$. Then, $f_0(x) = \begin{pmatrix} x_2^2 \\ 0 \end{pmatrix}$ and $f_1(x) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Thus, $\text{Span}(f_1(0), \text{ad}^2_{f_1}(f_0)(0)) = \mathbb{R}^2$. Nevertheless, the system is not controllable.

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Theorem (R. Hermann 1963, T. Nagano 1966)

If the system (1) is $L^\infty - STLC$, **then** LARC holds, i.e.

$$Lie(f_0, f_1, f_2)(0) = \mathbb{R}^d.$$

Theorem[1]

The solution of (1) is given by

$$x(T;(u,v),0) = \sum_{\substack{b\in\mathcal{B}_{[\![1,2]\!]},\\ |b|\leqslant L}} \underbrace{\xi_b(T,(u,v))}_{\text{explicit functional in (u,v)}} \times \underbrace{f_b}_{\in Lie(f_0,f_1,f_2)} (0) + \textit{remainders},$$

where $\mathcal{B}_{[\![1,2]\!]}$ is a set of brackets.

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The set $\mathcal{B}_{[\![1,2]\!]}$ is defined as:

$$\mathcal{B}_{[\![1,2]\!]} := \underbrace{\mathcal{B}_1}_{\substack{\text{linear terms: brackets}\\ \text{with } f_1 \text{ or } f_2 \text{ one time}}} \cup \underbrace{\mathcal{B}_{2,good} \cup \mathcal{B}_{2,bad}}_{\substack{\text{quadratic terms: brackets}\\ \text{with } f_1 \text{ or } f_2 \text{ two times}}}.$$

For $\tilde{b} \in \mathcal{B}_{2,bad}$,

$$\xi_{\tilde{b}}(t,(u,v)) \geqslant 0, \qquad \text{for example } \mathrm{ad}^2_{f_1}(f_0) \to \int_0^t \left(\int_0^s u(\sigma)\mathrm{d}\sigma\right)^2 \mathrm{d}s.$$

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For $\tilde{b} \in \mathcal{B}_{2,good}$,

$$\xi_{\tilde{b}}(t,(-u,v)) = -\xi_{\tilde{b}}(t,(u,v)).$$

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Theorem (Linear Test, R. Kalman 1960)

If $\{f_b(0),\ b \in \mathcal{B}_1\} = \mathbb{R}^d$, then system (1) is $W^{m,\infty} - STLC$, for every $m \in \mathbb{N}$.

Idea of the proof: For all $T > 0$,

$\mathrm{d}\mathcal{F}_T(0,0)(u,v) = X(T)$ is the solution of the **linearized system, starting from** 0.

However,

✓ $$\text{linearized system controllable} \underset{\substack{\text{Kalmam}\\ \text{condition}}}{\Longleftrightarrow} \{f_b(0),\ b \in \mathcal{B}_1\} = \mathbb{R}^d.$$

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✓ linearized system controllable $\underset{\text{inverse mapping theorem}}{\Longrightarrow}$ STLC.

$\mathcal{B}_1$ is good.

Remark: For **mono-control system**, $\mathcal{B}_2 = \mathcal{B}_{2,bad}$ ($\mathcal{B}_{2,good} = \emptyset$), [Beauchard, Marbach].

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Theorem

Let $L > 0$. One supposes that:

$$\text{Span}\left(f_b(0),\ b \in \mathcal{B}_1 \cup \mathcal{B}_{2,good},\quad |b| \leqslant L\right) = \mathbb{R}^d.$$

$$\text{For all } b \in \mathcal{B}_{2,bad},\ |b| \leqslant L \Rightarrow f_b(0) \in \mathcal{B}_1(f)(0).$$

Then, the system (1) is **smooth**$-STLC$, *i.e.* $W^{m,\infty} - STLC$, for every $m \in \mathbb{N}$.

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Example

A typical example is the following one:

$$\begin{cases} x_1' &= u \\ x_2' &= x_1 \\ y_1' &= v \\ z_1' &= x_1 y_1 - 7 x_2 y_1 \\ z_2' &= x_1^2 + x_1 \end{cases}$$

If we want to change the hypothesis as:

$$\text{For all } b \in \mathcal{B}_{2,bad}, \ |b| \leqslant L \Rightarrow f_b(0) \in \mathcal{B}_1(f)(0) + \mathcal{B}_{2,\mathbf{good}}(f)(0).$$

we can have problems !

Example

$$\begin{cases} x_1' &= u \\ y_1' &= v \\ z_1' &= x_1^2 + 2y_1^2 + \frac{3}{2}x_1 y_1 \end{cases},$$

Indeed,

$$z_1' = \left(x_1 + \frac{3}{4}y_1\right)^2 + \frac{23}{16}y_1^2 \geqslant 0.$$

Work in progress..!

✗ Included in the H. Sussmann's $\mathcal{S}(\theta)$ condition (1987), with $\theta \to 0$.

One considers a basis of $\mathbb{R}^d$ given by the LARC:

$$\mathbb{R}^d = \mathcal{B}_1(f)(0) \oplus \text{Spn}\left(f_{b_{r+1}}(0), \cdots, f_{b_d}(0)\right),$$

with $r = \dim\left(\mathcal{B}_1(f)(0)\right)$ and $b_{r+1}, \cdots, b_d \in \mathcal{B}_{2,good}$. Let $m \in \mathbb{N}$.

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Let $j \in [\![r+1, d]\!]$. It is sufficient to prove that we can create a motion along $f_{b_j}(0)$, *i.e.* there exists a continuous map $\Xi : [0, +\infty[\to \mathbb{R}^d$ with $\Xi(0) = f_{b_j}(0)$ such that for all $T > 0$, there exists $C, \rho, s_j > 0$ and a continuous map $z \in (-\rho, \rho) \mapsto (u_z, v_z) \in W^{m,\infty}(0,T)^2$ such that,

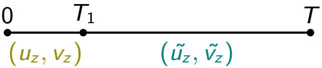
$$\forall z \in (-\rho, \rho), \quad \|x(T; (u_z, v_z), 0) - z\Xi(T)\| \leqslant C|z|^{1+s_j},$$

with

$$\|(u_z, v_z)\|_{W^{m,\infty}} \leqslant C|z|^{s_j}.$$

Then, the **Brouwer fixed-point theorem** gives the STLC result.

Idea of the proof: Let $j \in [\![r+1, d]\!]$. One considers $\mathbb{P}$, the linear projection on $\text{Span}\left(f_{b_i}(0)\right)_{r+1 \leqslant i \leqslant d}$ parallel to $\mathcal{B}_1(f)(0)$.



The proof is divised in two steps:

1. We construct (u_z, v_z) such that:

$$\mathbb{P}(x(T_1; (u_z, v_z), 0)) = z f_{b_j}(0) + \mathcal{O}\left(|z|^{1+s_j}\right), \text{ with } s_j > 0.$$

2. STLC in $\mathcal{B}_1(f)(0)$.

Step 1 : Let $\bar{u}, \bar{v} \in \mathcal{C}_c^\infty((0,1),\mathbb{R})$

Let $T_1(z) > 0$, $\varepsilon(z), \varepsilon'(z) > 0$ and $u_z, v_z : t \in (0, T_1) \mapsto \varepsilon \bar{u}\left(\frac{t}{T_1}\right), \varepsilon' \bar{v}\left(\frac{t}{T_1}\right)$.
Then, with the Magnus formula,

$$\mathbb{P}(x(T_1; (u_z, v_z), 0)) = \mathbb{P}\left(\sum_{b \in \mathcal{B}_1, |b| \leqslant L}\right) + \mathbb{P}\left(\sum_{b \in \mathcal{B}_{2,bad}, |b| \leqslant L}\right) + \varepsilon\varepsilon' \sum_{\substack{b \in \mathcal{B}_{2,good}, \\ |b| \leqslant L}} T_1^{|b|} \xi_b(1, (u,v)) \, \mathbb{P}(f_b(0)) + \textit{remainders}.$$

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Then,

$$\mathbb{P}(x(T_1;(u_z,v_z),0)) = \varepsilon\varepsilon' T_1^{|b_j|} f_{b_j}(0) + \mathcal{O}\left(\varepsilon\varepsilon' T_1^{|b_j|+1} + (\varepsilon+\varepsilon')^3 T_1^3\right).$$

Taking $\varepsilon = \text{sgn}(z)|z|^{\sigma_1}$, $\varepsilon' = |z|^{\sigma_2}$, and $T_1 = \varepsilon = |z|^{\sigma_3}$, with $\sigma_1, \sigma_2, \sigma_3 = f^{\theta}(|b_j|, m)$, well chosen, one has: $\mathbb{P}(x(T_1; (u_z, v_z), 0)) = z f_{b_j}(0) + \mathcal{O}\left(|z|^{1+s_j}\right)$.

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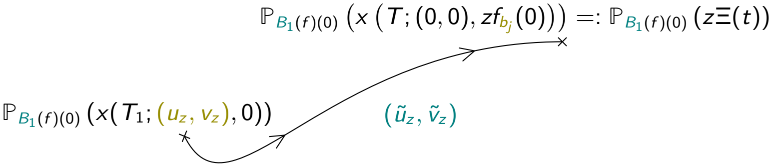
Step 2: Thanks linear test, one considers $(\tilde{u}_z, \tilde{v}_z)$ s.t.



Note that $\mathbb{P}_{B_1(f)(0)} = I - \mathbb{P}$. Then,

$$\|x(T; (U_z, V_z), 0) - z\Xi(t)\| = \|\mathbb{P}\left(x(T; (U_z, V_z), 0)\right) - z\mathbb{P}\left(z\Xi(t)\right)\|.$$

Using the explicit form of $\mathcal{B}_1$, one proves that the new step doesn't destroy the first step.

We consider the following PDE:

$$\begin{cases} i\partial_t\psi = -\partial^2_{xx}\psi - \left(u(t)\mu_1(x) + v(t)\mu_2(x)\right)\psi, & (t,x)\in(0,T)\times(0,1) \\ \psi(t,0)=\psi(t,1)=0, & t\in(0,T) \\ \psi(0,x)=\psi_0(x), & x\in(0,1) \end{cases} \tag{2}$$

$$i\partial_t\psi = f_0(\psi) + uf_1(\psi) + vf_2(\psi),$$

with

$$f_0(\psi) = -\partial^2_{xx}\psi, \qquad f_i(\psi) = \mu_i \times \psi, \quad i\in\{1,2\}.$$

We consider the following PDE:

$$\begin{cases} i\partial_t\psi = -\partial^2_{xx}\psi - (u(t)\mu_1(x) + v(t)\mu_2(x))\,\psi, & (t,x)\in(0,T)\times(0,1) \\ \psi(t,0)=\psi(t,1)=0, & t\in(0,T) \\ \psi(0,x)=\psi_0(x), & x\in(0,1) \end{cases} \tag{2}$$

$$i\partial_t\psi = f_0(\psi) + uf_1(\psi) + vf_2(\psi),$$

with

$$f_0(\psi) = -\partial^2_{xx}\psi, \qquad f_i(\psi) = \mu_i \times \psi, \quad i\in\{1,2\}.$$

Well-posedness

Let $T>0$, $\mu_1,\mu_2 \in H^3((0,T),\mathbb{R})$, $u,v\in L^2((0,T),\mathbb{R})$, and $\psi_0\in H^3_{(0)}(0,1)$. There exists a unique weak solution of (2), *i.e.* a function $\psi\in\mathcal{C}^0\left([0,T],H^3_{(0)}(0,1)\right)$ s.t., in $H^3_{(0)}$ for every $t\in[0,T]$:

$$\psi(t) = e^{-iAt}\psi_0 + i\int_0^t e^{-iA(t-s)}\left((u(s)\mu_1 + v(s)\mu_2)\psi(s)\right)\mathrm{d}s.$$

Functional analysis: $A := -\frac{\mathrm{d}^2}{\mathrm{d}x^2}$, $D(A) = H^2(0,1) \cap H_0^1(0,1)$.

1. eigenvalues: $\lambda_j = (j\pi)^2$, $j \geqslant 1$.
2. eigenvectors: $\varphi_j := \sqrt{2}\sin(j\pi\cdot)$, $j \geqslant 1$.
3. $(\varphi_j)_{j\geqslant 1}$ orthonormal basis of $L^2(0,1)$.

Ground state: $\psi_1(t,x) := \varphi_1(x)e^{-i\lambda_1 t} = \psi(t;(0,0),\varphi_1)$.

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Definition ($L^2 - STLC$)

(2) is $\mathbf{L^2}$ **– STLC in** $H^3_{(0)}(0,1)$ **around the ground state** if: for all $T > 0$, $\varepsilon > 0$, there exists $\delta > 0$ such that, for all target $\psi_f \in \mathcal{S} \cap H^3_{(0)}(0,1)$ such that $\|\psi_f - \psi_1(T)\|_{H^3} \leqslant \delta$, there exists $u, v \in L^2(0,T)$ with $\|(u,v)\|_{L^2} \leqslant \varepsilon$ such that $\psi(T;(u,v),\varphi_1) = \psi_f$.

Theorem (Linear Test)[2]

Let $\mu_1, \mu_2 \in H^3((0,1),\mathbb{R})$ such that

$$\exists c > 0, \quad \forall j \in \mathbb{N}^*, \quad \left\| \left(\langle \mu_i \varphi_1, \varphi_j \rangle \right)_{1 \leqslant i \leqslant 2} \right\| \geqslant \frac{c}{j^3}.$$

Then, the bilinear Schrödinger equation (2) is L^2-STLC in $H^3_{(0)}(0,1)$.

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[2] Karine Beauchard and Camille Laurent. "Local controllability of 1D linear and nonlinear Schrödinger equations with bilinear control". In: *Journal de Mathématiques Pures et Appliquées* 94.5 (2010), pp. 520–554.

[3] Mégane Bournissou. "Quadratic behaviors of the 1D linear Schrödinger equation with bilinear control". In: *Journal of Differential Equations* 351 (2023), pp. 324–360.

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Mégane Bournissou: Quadratic obstructions for the bilinear Schrödinger equation with **single-input system**[3].

Framework of the article: $\exists K \geqslant 2$ such that $\langle \mu_1 \varphi_1, \varphi_K \rangle = \langle \mu_2 \varphi_1, \varphi_K \rangle = 0$.
$\rightarrow$ use quadratic expansion of the solution to recover this direction

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Theorem (T.G., 2024)

One considers μ_1, μ_2 such that:

1. $\mu_1, \mu_2 \in H^3((0,1), \mathbb{R})$.
2. $\langle \mu_1 \varphi_1, \varphi_K \rangle = \langle \mu_2 \varphi_1, \varphi_K \rangle = 0$.
3. $\exists c > 0, \quad \forall j \in \mathbb{N}^* \setminus \{K\}, \quad \left\| ((\mu_i \varphi_1, \varphi_j))_{1 \leqslant i \leqslant 2} \right\| \geqslant \frac{c}{j^3}$.
4. $A_1^1 := \langle [\mu_1, [\mu_1, \Delta]] \varphi_1, \varphi_K \rangle = 0$.
5. $A_1^2 := \langle [\mu_2, [\mu_2, \Delta]] \varphi_1, \varphi_K \rangle = 0$.
6. $\gamma_1 := \langle [\mu_2, [\mu_1, \Delta]] \varphi_1, \varphi_K \rangle \neq 0$.

The equation (2) is L^2-STLC around the ground state in $H^3_{(0)}$.

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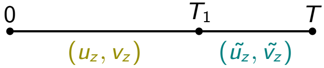
The equation (2) is L^2-STLC around the ground state in $H^3_{(0)}$.

- Point 1: well-posedness.
- Point 3: related to control in projection.
- Point 4 and 5: prevents the system from a drift.
- Point 6: allows us to use the bracket to recover the direction.

Idea of the proof:



The proof is divised in two steps:

1. $\langle \psi(T_1; (u_z, v_z), \varphi_1), \psi_K(T_1) \rangle = iz + \mathcal{O}\left(|z|^{\frac{13}{12}} \right)$.
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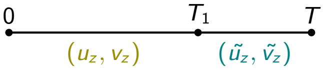

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$+$ Brouwer fixed-point theorem

Step 1: Let $\bar{u}, \bar{v} \in L^2((0,1), \mathbb{R})$ be such that, $\int_0^1 \bar{u}(t)\mathrm{d}t = \int_0^1 \bar{v}(t)\mathrm{d}t = 0$. Let $T_1(z) > 0$, $\varepsilon(z), \varepsilon'(z) > 0$ and $u_z, v_z : t \in (0, T_1) \mapsto \varepsilon \bar{u}'\left(\frac{t}{T_1}\right), \varepsilon' \bar{v}'\left(\frac{t}{T_1}\right)$. Then,

$$\langle \psi(T_1; (u_z, v_z), \varphi_1), \psi_K(T_1) \rangle = \mathcal{F}_{T_1}(u_z) + \mathcal{G}_{T_1}(u_z, v_z) + \mathcal{F}_{T_1}(v_z) + \mathcal{O}\left(\|(u_z, v_z)\|_{L^2}^3\right).$$

A direct computation gives:

$$\mathcal{F}_{T_1}(u_z) = -i\varepsilon^2 T_1^3 A_1^1 \int_0^1 \bar{u}(t)^2 \mathrm{d}t + \mathcal{O}\left(\varepsilon^2 T_1^4\right) = \mathcal{O}\left(\varepsilon^2 T_1^4\right).$$

Similarly, $\mathcal{F}_{T_1}(v_z) = \mathcal{O}\left(\varepsilon'^2 T_1^4\right)$.

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Similarly, $\mathcal{F}_{T_1}(v_z) = \mathcal{O}\left(\varepsilon'^2 T_1^4\right)$. Moreover,

$$\mathcal{G}_{T_1}(u_z, v_z) = i\varepsilon\varepsilon' T_1^3 \gamma_1 \int_0^1 \bar{u}(t)\bar{v}(t)\mathrm{d}t + \mathcal{O}\left(\varepsilon\varepsilon' T_1^4\right).$$

Thus,

$$\langle \psi(T_1; (u_z, v_z), \varphi_1), \psi_K(T_1)\rangle = i\varepsilon\varepsilon' T_1^3 \gamma_1 \int_0^1 \bar{u}(t)\bar{v}(t)\mathrm{d}t + \mathcal{O}\left((\varepsilon+\varepsilon')^2 T_1^4 + \left(\varepsilon^3 + \varepsilon'^3\right) T_1^{\frac{3}{2}}\right).$$

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Let $\rho > 0$ and $z \in (-\rho, \rho)$. With $\varepsilon = sgn(z)|z|^{\frac{3}{8}}$, $\varepsilon' = |z|^{\frac{3}{8}}$ and $T_1 = |z|^{\frac{1}{12}}$, $(\bar{u}, \bar{v}) \in \mathcal{C}_c^\infty(0,1)^2$ such that $\int_0^1 \bar{u}(t)\bar{v}(t)\mathrm{d}t = \frac{1}{\gamma_1}$, one obtains:

$$\langle \psi(T_1; (u_z, v_z), \varphi_1), \psi_K(T_1)\rangle = iz\gamma_1 \int_0^1 \bar{u}\bar{v}' + \mathcal{O}\left(|z|^{\frac{13}{12}}\right) = iz + \mathcal{O}\left(|z|^{\frac{13}{12}}\right).$$

Theorem (T.G., 2024)

Let $n \geqslant 1$, $m, p \geqslant 0$, $K \geqslant 2$ such that $\lfloor \frac{n}{2} \rfloor \leqslant p$. Let μ_1, μ_2 such that:

1. $\mu_1, \mu_2 \in H^{2(p+m)+3}((0,1),\mathbb{R})$ with $\mu^{(2k+1)}\big|_{\{0,1\}} = 0$, for $0 \leqslant k \leqslant p-1$.
2. $\langle \mu_1 \varphi_1, \varphi_K \rangle = \langle \mu_2 \varphi_1, \varphi_K \rangle = 0$.
3. $\exists c > 0, \quad \forall j \in \mathbb{N}^* \setminus \{K\}, \quad \left\| \left((\mu_i \varphi_1, \varphi_j) \right)_{1 \leqslant i \leqslant 2} \right\| \geqslant \frac{c}{j^{2p+3}}$.
4. $\forall k \in [\![1, \lfloor \frac{n+1}{2} \rfloor]\!]$, $A_k^1 := \langle [ad_\Delta^{k-1}(\mu_1), ad_\Delta^k(\mu_1)]\varphi_1, \varphi_K \rangle = 0$.
5. $\forall k \in [\![1, \lfloor \frac{n+1}{2} \rfloor]\!]$, $A_k^2 := \langle [ad_\Delta^{k-1}(\mu_2), ad_\Delta^k(\mu_2)]\varphi_1, \varphi_K \rangle = 0$.
6. $\gamma_n := \left\langle [ad_\Delta^{\lfloor \frac{n+1}{2} \rfloor}(\mu_1), ad_\Delta^{\lfloor \frac{n}{2} \rfloor}(\mu_2)]\varphi_1, \varphi_K \right\rangle \neq 0$.

The equation (2) is H_0^m-STLC around the ground state in $H_{(0)}^{2(p+m)+3}(0,1)$: for all $T > 0$, $\varepsilon > 0$, there exists $\delta > 0$ such that, for all target $\psi_f \in \mathcal{S} \cap H_{(0)}^{2(p+m)+3}(0,1)$ such that $\|\psi_f - \psi_1(T)\|_{H^{2(p+m)+3}} \leqslant \delta$, there exists $u, v \in H_0^m(0,T)$ with $\|(u,v)\|_{H_0^m} \leqslant \varepsilon$ such that $\psi(T;(u,v),\varphi_1) = \psi_f$.

Perspectives:

1. Several lost directions (as in finite dimension) ? An infinite number ?
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Théo Gherdaoui. "Small-Time Local Controllability of the multi-input bilinear Schrödinger equation thanks to a quadratic term". In: *Preprint* (2024)

Thank you for your attention !