A special fuzzy number vector space and applications

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X Partial differential equations, optimal design and numerics

Outline of the talk

1. Landscape of fuzzy world
2. Existing basic concepts of fuzziness
3. Space of Membership Functions
4. Sample Space: $(h, p) = (\tan, \mathcal{N}(\mu, 1))$
5. Upgrade: Ring
6. Vector space
7. References

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Landscape of fuzzy world

Fuzzy mathematics includes fuzzy set theory and fuzzy logic. First well-known paper is given by Zadeh[1], with 136368/152507 citations.

Fuzzy sets, fuzzy logic, artificial intelligence (AI), fuzzy semantics, languages, fuzzy algorithms, fuzzy control, fuzzy systems, approximate reasoning, fuzzy pattern recognition, machine learning.

[1] L. A. Zadeh, Fuzzy sets, Information and control, 15p, 1965

Landscape of fuzzy world

- Fuzzy probability and fuzzy statistics (based on T-norms and T-conorms) [Zadeh68, 84, Mesiar92,Riečan97,Navara05]
- Experts systems, aggregation of informations. [Xu2007]
- Fuzzy logic, fuzzy controller (Takagi-Sugeno model), etc.
- Application: success on industry such as automatic control (Full Self-Driving) Apollo by Baidu.

Landscape of fuzzy world

100 millions Mileage, 0 Accident with Primary Fault

put "safety first" as the top priority

83% +

Acceptance of Self-Driving Tech

Half a year

Complete the preparation for autonomous driving operation of a new city

Safety is the foundation for the commercialisation of autonomous driving. Over the past decade, Apollo has always put "safety first" as the top priority in the development of autonomous driving, which is also the ultimate goal of autonomous driving.

Currently, the total mileage of Apollo autonomous driving tests and operations has exceeded 100 million kilometres, and is growing by more than 100,000 kilometres per day, and there has never been a major safety accident reported by autonomous driving that resulted in injury or death.

Membership function: fuzzy numbers

CONERSTONE: Membership Function. Two perspectives

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1. Objective fuzzy number with uncertainty and fuzziness.

Different to the probability theory with random variable,

due to the lack of

The law of excluded middle, leads to accountable additivity

Membership function: fuzzy numbers

CONERSTONE: Membership Function. Two perspectives

1. Objective fuzzy number with uncertainty and fuzziness.

Different to the probability theory with random variable,

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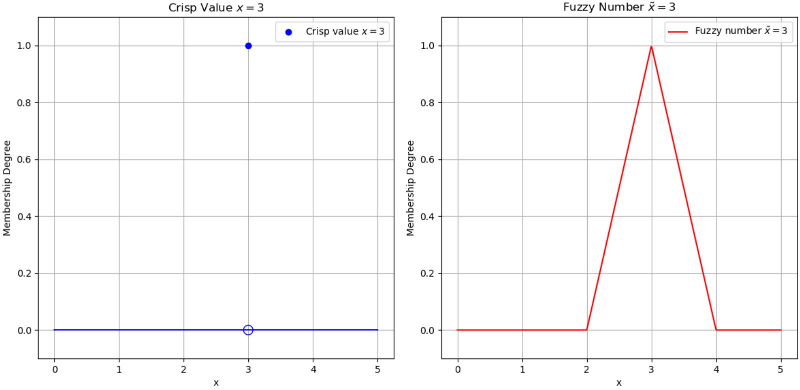
The law of excluded middle, leads to accountable additivity

- Step ①: Fuzzy number $\Longleftrightarrow$ Membership function Inf. Dim!!!.
- Step ②: $>, <, +, -, *, /$ on numbers becomes to on functions, ranking fuzzy numbers, fuzzy calculus, etc.

Membership function: fuzzy numbers

Comparison: 3 (Indicator Fcn) and approxim. 3 (Membership Fcn)



Membership function: fuzzy numbers

Question: Algorithms of Memb. func.?

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Popular fuzzy methodology:

Defuzzification+ Interval analysis+Fuzzification.

- I: Decomposition theorem. The α-cut of a fuzzy set A_α is a classical interval;
- II: Representation theorem. A bundle of classical α-cut set constructs a fuzzy set.

Membership function: fuzzy numbers

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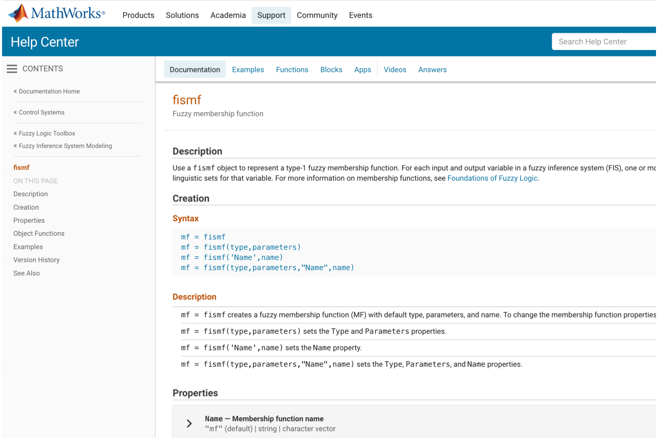
- II: Representation theorem. A bundle of classical α-cut set constructs a fuzzy set.

13 types of membership functions in MATLAB (control system, fuzzy logic toolbox, fuzzy inference system modelling)

Landscape of fuzzy world



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fismf

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fismf

Fuzzy membership function

Description

Use a `fismf` object to represent a type-1 fuzzy membership function. For each input and output variable in a fuzzy inference system (FIS), one or mo
linguistic sets for that variable. For more information on membership functions, see Foundations of Fuzzy Logic.

Creation

Syntax

```
mf = fismf
mf = fismf(type,parameters)
mf = fismf('Name',name)
mf = fismf(type,parameters,"Name",name)
```

Description

`mf = fismf` creates a fuzzy membership function (MF) with default type, parameters, and name. To change the membership function properties

`mf = fismf(type,parameters)` sets the Type and `Parameters` properties.

`mf = fismf('Name',name)` sets the Name property.

`mf = fismf(type,parameters,"Name",name)` sets the Type, `Parameters`, and Name properties.

Properties

Name — Membership function name
`"mf"` (default) | string | character vector

Landscape of fuzzy world

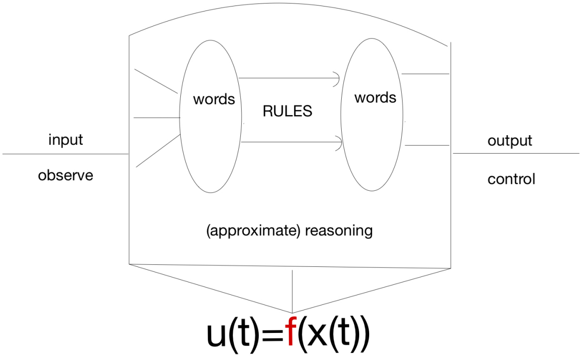

Figure: Control via human's experience. Nonlinear, hybrid, time-delay.

Membership fuction: words

Second perspective:

2. WORDS in language. AKA Linguistic Variable. Caution: They do not convey the same meaning at the operational level when being cast in various contexts.

There is NO profound uniformity as to the interpretation of what a membership stands for. objective + subjective.

Cannot be directly solved by probability theory.

Membership fuction: words

Example: Linguistic Variable

"Design" of Membership Function, Artificial Intelligence

Example: "Tall" for an NBA basketball player and a ten years old child. Based on the knowledge and perceptions of the observer, same word has significant different membership function.

Example: "Yellow, blue" words with metaphor.

Membership fuction: words

Example: Linguistic Variable

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Example: "Tall" for an NBA basketball player and a ten years old child. Based on the knowledge and perceptions of the observer, same word has significant different membership function.

Example: "Yellow, blue" words with metaphor.

Open problem on Large Language Models (ChatGPT).

Exploring the universe of WORDS instead of numbers.

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Function Space and Arithmetic operators

Cornerstone: Membership Function Space

Continuous Function Space, Sobolev Space,Probability,

Polynomials, Fourier Series, Wavelet, Barron space, etc.

Arithmetic operators on the functions

Crucial for the computation of the fuzzy number's algorithm.

Methodology of design

- Function from $\mathbb{R} \longrightarrow [0,1]$, with compact support?
- Function from $[a,b] \longrightarrow [0,1]$, how to overcome the Contradiction of **Subjective and Objective**?
- Composite functions $[a,b] \longrightarrow \mathbb{R} \longrightarrow [a,b]$, **PAIRS** h, p.

$$h : [a,b] \to \mathbb{R}.\text{Objective entity}$$

$$p : \mathbb{R} \to [0,1].\ \text{Subjective entity}$$

Space $X_{h,p}(\mathbb{R})$

We define a nonlinear mapping h from (a, b) to $\mathbb{R}$ as

Definition

Let h be a function defined on the interval (a, b) with range $\mathbb{R}$. We call h as a **Auxiliary function (AF)**, if h satisfies

a) $h(a) = -\infty, h(b) = +\infty,$

b) h is continuous on (a, b),

c) h is increasing on (a, b).

Space $X_{h,p}(\mathbb{R})$

The probability density function (PDF) is given by

Definition

We say p is an **probability density function (PDF)**, if

$p : \mathbb{R} \to [0, 1]$ satisfies

a) $$\int_{-\infty}^{+\infty} p(t)dt = 1,$$

b) $p(t) \geq 0, \forall t \in (-\infty, +\infty)$ and continuous.

Space $X_{h,p}(\mathbb{R})$

Definition

We call the function space

$$X_{h,p}(\mathbb{R}) := \{f_{h,p}(x) : \mathbb{R} \to [0,1] \text{ is as the form of } (1) : a \leq b \leq c\}$$

a **Probability Density Membership Function** space with

$$f_{h,p}(x) = \begin{cases} 0, & x \in (-\infty, a] \cup [c, +\infty) \\ \int_{-\infty}^{h^-(x)} p^-(y)dy, & x \in (a, b) \\ \int_{-\infty}^{h^+(x)} p^+(y)dy, & x \in (b, c) \end{cases} \quad (1)$$

Universal Approximation Theorem[2]

Theorem

Let

$$P = (x^-, y^-) \in (a, b) \times (0, 1), \qquad Q = (x^+, y^+) \in (b, c) \times (0, 1).$$

Then there exists at least one pair (h, p) *such that the graph of* $f_{h,p}$ *passes through* P *and* Q*, i.e.,* $f_{h,p}(x^-) = y^-$ *and* $f_{h,p}(x^+) = y^+$*.*

[2] H. Wang & C. Zheng, Fuzzy sets and systems, 2023

Universal Approximation Theorem[3]

Theorem

Let $P_i = (x_i^-, y_i^-) \in (a,b) \times (0,1), i = 1, \cdots, m$ *satisfy*

$a < x_1^- \cdots < x_m^- < b$ *and* $0 < y_1^- \cdots < y_m^- < 1$,

$Q_j = (x_j^+, y_j^+) \in (b,c) \times (0,1), j = 1, \cdots, n$ *satisfy*

$b < x_1^+ \cdots < x_n^- < c$ *and* $0 < y_n^+ < \cdots < y_1^+ < 1$.

Then there exists at least one pair (h, p) *such that the graph of* $f_{h,p}$ *passes through all points* P_i*'s and* Q_j*'s, i.e.,*

$f_{h,p}(x_i^-) = y_i^-, i = 1, \cdots, m$ *and* $f_{h,p}(x_j^+) = y_j^+, j = 1, \cdots, n$.

[3] H. Wang & C. Zheng, Fuzzy sets and systems, 2023

Space $X_{h,p}(\mathbb{R})$

Some properties and remarks:

1. The classical triangular fuzzy number space as the form (a, b, c) is a special PDMF space.

Space $X_{h,p}(\mathbb{R})$

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2. All 13 types of membership functions in MATLAB can be deduced by special PDMF spaces.

Space $X_{h,p}(\mathbb{R})$

Some properties and remarks:

1. The classical triangular fuzzy number space as the form (a, b, c) is a special PDMF space.
2. All 13 types of membership functions in MATLAB can be deduced by special PDMF spaces.
3. There are two tags/attribution (h, p) such that we can solve the contradiction between objectivity and subjectivity (from Baseball bat to Nunchaku).

Space $X_{h,p}(\mathbb{R})$

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Sample Space: $(h, p) = (\tan, \text{Gaussian})$

The mother structures in Bourbaki's Architecture manifesto:

- Algebraic structures,
- Order structures and topological structures (Ongoing work).

Sample Space: $(h, p) = (\tan, \text{Gaussian})$

The mother structures in Bourbaki's Architecture manifesto:

- Algebraic structures,
- Order structures and topological structures (Ongoing work).

A sample space is given. Called as **Gaussian-PDMF**. Consider

$$\mathcal{F} = \{F(x) : \mathbb{R} \to [0,1] \,\big|\, F \text{ has the form in (2)}\},$$

$$F(x) = \begin{cases} F_{-}(x; \mu^{-}), & x \in (a, b) \\ F_{+}(x; \mu^{+}), & x \in (b, c) \\ 0 & x \in (-\infty, a] \cap [c, \infty) \end{cases} \tag{2}$$

Gaussian Probability Density Membership Function

with

$$\varphi_-(x) = \tan\left(\frac{\pi}{b-a}(x-a)-\frac{\pi}{2}\right), \quad x \in (a,b)$$

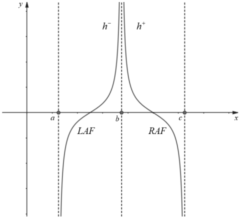
$$\varphi_+(x) = \tan\left(\frac{\pi}{b-c}(c-x)-\frac{\pi}{2}\right), \quad x \in (b,c)$$

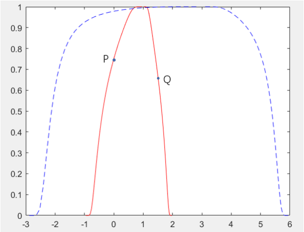
$$p(x;\mu) = \text{Probability Density}\left(p(x;\mu,1) = \frac{1}{\sqrt{2\pi}}e^{(x-\mu)^2}, \quad x \in \mathbb{R}\right),$$

$$F_-(x;\mu^-) = \frac{\pi}{b-a}\int_a^x p(\varphi_-(t);\mu^-)(1+\varphi_-^2(t))dt, \quad x \in (a,b)$$

$$F_+(x;\mu^+) = \frac{\pi}{c-b}\int_b^x p(\varphi_+(t);\mu^+)(1+\varphi_+^2(t))dt, \quad x \in (b,c)$$

Graph of Gaussian-PDFM





$$F(x) \Longleftrightarrow (a, b, c; P, Q) \Longleftrightarrow (b; d^-, d^+, \mu^-, \mu^+) \Longleftrightarrow \tilde{b}$$

Properties of Gaussian-PDMF

Lemma

$\tilde{b}$ with membership function in Gaussian-PDMF Space is a triangular fuzzy number. More precisely, it satisfies:

- $\tilde{b}$ is *normal*.

  $F(x) = 1$ *for* $x = b$ *and* $F(x) = 0$ *for* $x \leq a$ *and* $x \geq c$.

- $\tilde{b}$ is *upper semicontinuous with compact suppport.*

  $F(x)$ *is increasing in* $[a, b]$ *and is decreasing in* $[b, c]$.

- $\tilde{b}$ is *fuzzy convex, i.e.,*

  $F(\lambda x + (1 - \lambda)y)) \geq \min\{F(x), F(y)\}, \ \forall x, y \in \mathbb{R}, \lambda \in [0, 1].$

Operations of Gaussian-PDFM

Definition (FSS2023)

Let $\tilde{b}_1 = (a_1, b_1, c_1; \mu_1^-, \mu_1^+)$ and $\tilde{b}_2 = (a_2, b_2, c_2; \mu_2^-, \mu_2^+)$ be two Gaussian-PDFMs. Then

1. $\tilde{b}_1 \oplus \tilde{b}_2 = (a_1 + a_2, b_1 + b_2, c_1 + c_2; \mu_1^- + \mu_2^-, \mu_1^+ + \mu_2^+)$;
2. $\lambda \tilde{b}_1 = (\lambda a_1, \lambda b_1, \lambda c_1; \lambda \mu_1^-, \lambda \mu_1^+), \lambda \in \mathbb{R}^+$,
   $\lambda \tilde{b}_1 = (\lambda c_1, \lambda b_1, \lambda a_1; \lambda \mu_1^+, \lambda \mu_1^-), \lambda \in \mathbb{R}^-$;

Operations of Gaussian-PDMF

Theorem (FSS2023)

Let $\tilde{b}_1 = (a_1, b_1, c_1; \mu_1^-, \mu_1^+)$ *and* $\tilde{b}_2 = (a_2, b_2, c_2; \mu_2^-, \mu_2^+)$ *be two Gaussian-PDFMs. and let* $\tilde{b} = \tilde{a}_1 \oplus \tilde{a}_2$ *and* $\tilde{c} = \lambda \tilde{a}_1$*. Then both* $\tilde{b}$ *and* $\tilde{c}$ *are also Gaussian-PDFMs.*

Graph of the operations

$$\tilde{b}_1 \oplus \tilde{b}_2 = \langle(-1,0,1);0,0\rangle \oplus \langle(-1,1,4);0,0\rangle = \langle(-2,1,5);0,0\rangle.$$

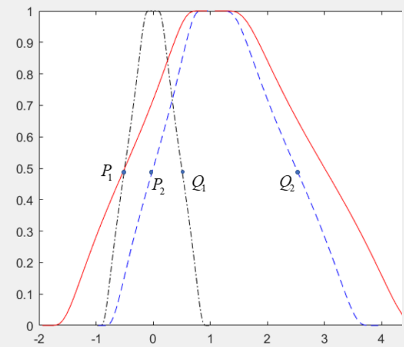


Figure: $\tilde{b}_1$ (pecked line), $\tilde{b}_2$ (dash line) and $\tilde{b}_1 \oplus \tilde{b}_2$ (solid line)

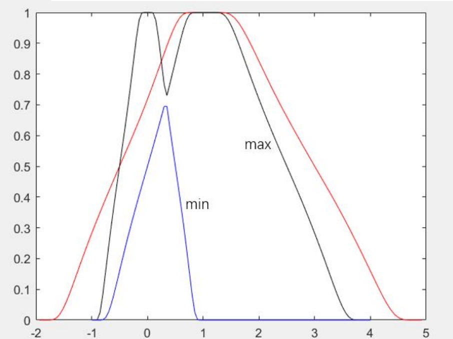


Figure: Three results based on MIN, MAX and ours

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Upgrade: Notation and Definition

For integer i, fuzzy number $\tilde{x}_i \in X_{h,p}(\mathbb{R})$ has the notation

$$\tilde{x}_i = (x_i - d_i^-, x_i, x_i + d_i^+, \mu_i^-, \mu_i^+) \Leftrightarrow \langle x_i; d_i^-, d_i^+, \mu_i^-, \mu_i^+ \rangle.$$

Definition (Preprint)

Let $\tilde{x}_1, \tilde{x}_2$ be two G-PDMFs in X, we define

1. $\tilde{x}_1 \oplus \tilde{x}_2 = \langle x_1 + x_2; d_1^- d_2^-, d_1^+ d_2^+, \mu_1^- + \mu_2^-, \mu_1^+ + \mu_2^+ \rangle.$
2. $\lambda \tilde{x} = \langle \lambda x; (d^-)^\lambda, (d^+)^\lambda, \lambda\mu^-, \lambda\mu^+ \rangle$
3. $\tilde{x}_1 \ominus \tilde{x}_2 = \langle x_1 - x_2; d_1^-(d_2^-)^{-1}, d_1^+(d_2^+)^{-1}, \mu_1^- - \mu_2^-, \mu_1^+ - \mu_2^+ \rangle.$
4. $\tilde{x}_1 \otimes \tilde{x}_2 = \langle x_1 x_2; (d_1^-)^{\ln d_2^-}, (d_1^+)^{\ln d_2^+}, \mu_1^- \mu_2^-, \mu_1^+ \mu_2^+ \rangle.$

Upgrade: Abelian Group

Theorem

X together with the binary operation $\oplus$ is an **abelian group***:*

1. *The binary operation $\oplus$ is* **associative***:*

   $(\tilde{x}_1 \oplus \tilde{x}_2) \oplus \tilde{x}_3 = \tilde{x}_1 \oplus (\tilde{x}_2 \oplus \tilde{x}_3)$

2. *There exists a unique two-sided identity element*

   $\tilde{0} = \langle 0; 1, 1, 0, 0 \rangle$ *such that* $\tilde{0} \oplus \tilde{x} = \tilde{x} \oplus \tilde{0}, \quad$ *for all* $\tilde{x} \in X$.

3. *For every $\tilde{x} \in X$ there exists a (two-sided) inverse element*

   $-\tilde{x} \in X$ *such that* $-\tilde{x} \oplus \tilde{x} = \tilde{x} \oplus -\tilde{x} = \tilde{0}$.

4. *The binary operation $\oplus$ is commutative:* $\tilde{x}_1 \oplus \tilde{x}_2 = \tilde{x}_2 \oplus \tilde{x}_1$.

Application: fuzzy linear equation

Theorem

Let $\tilde{b}, \tilde{c} \in X$ *with* $\tilde{b} = \langle b; d_b^-, d_b^+, \mu_b^-, \mu_b^+ \rangle$ *and* $\tilde{c} = \langle c; d_c^-, d_c^+, \mu_c^-, \mu_c^+ \rangle$. *Let* $a \in \mathbb{R}$ *and* $a \neq 0$. *Set* $\tilde{x} = \langle x; d^-, d^+, \mu^-, \mu^+ \rangle$ *be an undetermined fuzzy number such that*

$$a\tilde{x} \oplus \tilde{b} = \tilde{c}.$$

Then there exists a unique fuzzy number $\tilde{x}$ *belonging to* X *satisfying the above equation. More precisely,*

$$\tilde{x} = \frac{1}{a}(\tilde{c} \ominus \tilde{b}) = \left\langle \frac{c-b}{a}; \left(\frac{d_c^-}{d_b^-}\right)^{\frac{1}{a}}, \left(\frac{d_c^+}{d_b^+}\right)^{\frac{1}{a}}, \frac{\mu_c^- - \mu_b^-}{a}, \frac{\mu_c^- - \mu_b^-}{a} \right\rangle.$$

Theorem

X together with two binary operations $\oplus$ and $\otimes$ is a **commutative ring with identity**. *More precisely,*

1. $(X, \oplus)$ *is an abelian group;*
2. $(\tilde{x}_1 \otimes \tilde{x}_2) \otimes \tilde{x}_3 = \tilde{x}_1 \otimes (\tilde{x}_2 \otimes \tilde{x}_3)$ *(***associative multiplication***);*
3. $\tilde{x}_1 \otimes (\tilde{x}_2 \oplus \tilde{x}_3) = \tilde{x}_1 \otimes \tilde{x}_2 \oplus \tilde{x}_1 \otimes \tilde{x}_3$ *and* $(\tilde{x}_1 \oplus \tilde{x}_2) \otimes \tilde{x}_3 = \tilde{x}_1 \otimes \tilde{x}_3 \oplus \tilde{x}_2 \otimes \tilde{x}_3$ *(***left and right distributive laws***);*
4. $\tilde{x}_1 \otimes \tilde{x}_2 = \tilde{x}_2 \otimes \tilde{x}_1$ *(***commutative***);*
5. *X contains an identity element* $1_X = \langle 1; e, e, 1, 1 \rangle$ *such that* $1_X \otimes \tilde{x} = \tilde{x} \otimes 1_X$.

Note that for $\tilde{x}_1, \tilde{x}_2 \in X$, the following assertion is false:

$$\tilde{x}_1 \otimes \tilde{x}_2 = \tilde{0} \implies \tilde{x}_1 = \tilde{0} \quad \text{or} \quad \tilde{x}_2 = \tilde{0}.$$

In fact, any $\tilde{x}_2 = \langle 0; d_2^-, d_2^+, \mu_2^-, \mu_2^+ \rangle$ with $x_2 = 0$ is a left (and also right) zero divisor of the nonzero element $\tilde{x}_1 = \langle x_1; 1, 1, 0, 0 \rangle$. One can easily prove that, the following two expressions are equivalent:

$$\tilde{x}_1 \otimes \tilde{x}_2 = \tilde{0} \iff$$

$$x_1 x_2 = 0, \ \ln d_1^- \ln d_2^- = 1; \ \ln d_1^+ \ln d_2^+ = 1; \ \mu_1^- \mu_2^- = 0; \ \mu_1^+ \mu_2^+ = 0.$$

Definition

An element $\tilde{x}$ in X with is said to be **left** (resp. right) **invertible** if there exists $\tilde{y} \in X$ (resp. $\tilde{z} \in X$) such that $\tilde{y} \otimes \tilde{x} = 1_X$ (reps. $\tilde{x} \otimes \tilde{z} = 1_X$). The element $\tilde{y}$ (resp. $\tilde{z}$) is called a **left** (resp. right) **inverse** of $\tilde{x}$. An element $\tilde{x} \in X$ that is both left and right invertible is denoted by $\tilde{x}^{-1}$ and said to be **invertible** or to be a **unit**.

Theorem

The set of all units in X is given by

$$U(X) = \{\tilde{x} = \langle x; d^-, d^+, \mu^-, \mu^+ \rangle \in X \mid x \ln d^- \ln d^+ \mu^- \mu^+ \neq 0\}.$$

Moreover, for $\tilde{x} \in U(X)$,

$$\tilde{x} \otimes \tilde{y} = \tilde{0} \text{ or } \tilde{y} \otimes \tilde{x} = \tilde{0} \implies \tilde{y} = \tilde{0}.$$

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Note: $U(X)$ is a integral domain. Suffice to play Gaussian Elimination method for solving fuzzy polynomial equations.

Theorem

Let $\tilde{a} \in U(X), \tilde{b}, \tilde{c} \in X$ *with* $\tilde{a} = \langle a; d_a^-, d_a^+, \mu_a^-, \mu_a^+ \rangle$, $\tilde{b} = \langle b; d_b^-, d_b^+, \mu_b^-, \mu_b^+ \rangle$ *and* $\tilde{c} = \langle c; d_c^-, d_c^+, \mu_c^-, \mu_c^+ \rangle$. *Set* $\tilde{x} = \langle x; d^-, d^+, \mu^-, \mu^+ \rangle$ *be an undetermined fuzzy number such that*

$$\tilde{a}\tilde{x} \oplus \tilde{b} = \tilde{c}.$$

Then there exists a unique fuzzy number solution $\tilde{x}$ *in* X *with*

$$\tilde{x} = \left\langle \frac{c-b}{a}; \exp\left(\frac{\ln d_c^- - \ln d_b^-}{\ln d_a^-}\right), \exp\left(\frac{\ln d_c^+ - \ln d_b^+}{\ln d_a^+}\right), \frac{\mu_c^- - \mu_b^-}{\mu_a^-}, \frac{\mu_c^- - \mu_b^-}{\mu_a^+} \right\rangle.$$

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First step: vector space

The function is projected to the 5-dimensional vector space.

Parameters:

Theorem

X is a vector space on $\mathbb{R}$ satisfying the ten properties (axioms):

1. *The sum of $\tilde{x}_1, \tilde{x}_2$, denoted by $\tilde{x}_1 \oplus \tilde{x}_2$, is in X.*
2. $\tilde{x}_1 \oplus \tilde{x}_2 = \tilde{x}_2 \oplus \tilde{x}_1$.
3. $(\tilde{x}_1 \oplus \tilde{x}_2) \oplus \tilde{x}_3 = \tilde{x}_1 \oplus (\tilde{x}_2 \oplus \tilde{x}_3)$.
4. *There is a **zero** vector $\tilde{0}$ in X such that $\tilde{x} \oplus \tilde{0} = \tilde{x}$.*
5. *There is a vector $-\tilde{x}$ in X such that $\tilde{x} \oplus (-\tilde{x}) = \tilde{0}$.*

Theorem

1. *The scalar multiplication of* $\tilde{x}$ *by* λ, *denoted by* $\lambda\tilde{x}$, *is in* X.
2. $\lambda(\tilde{x}_1 \oplus \tilde{x}_2) = \lambda\tilde{x}_1 \oplus \lambda\tilde{x}_2$.
3. $(\lambda_1 + \lambda_2)\tilde{x} = \lambda_1\tilde{x} \oplus \lambda_2\tilde{x}$.
4. $\lambda_1(\lambda_2\tilde{x}) = (\lambda_1\lambda_2)\tilde{x}$.
5. $1\tilde{x} = \tilde{x}$.

We construct a basis for the linear space X:

Definition

An indexed set of vectors $\mathcal{X} = \{\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_p\}$ in X is said to be **linearly independent** if the equation

$$\lambda_1 \tilde{x}_1 \oplus \lambda_2 \tilde{x}_2 \oplus \cdots \oplus \lambda_p \tilde{x}_p = \tilde{0}$$

has only the trivial solution, $\lambda_1 = 0, \cdots, \lambda_p = 0$.

Under the above definition, we give a basis for X:

Theorem

A set of vectors $\mathcal{X} = \{\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4, \tilde{x}_5\}$ *in* X *as the form*

$$\tilde{x}_1 = \langle 1; 1, 1, 0, 0 \rangle,\ \tilde{x}_2 = \langle 0; e, 1, 0, 0 \rangle,\ \tilde{x}_3 = \langle 0; 1, e, 0, 0 \rangle,$$

$$\tilde{x}_4 = \langle 0; 1, 1, 1, 0 \rangle,\ \tilde{x}_5 = \langle 0; 1, 1, 0, 1 \rangle$$

is a basis for X. *More precisely,* $\mathcal{X}$ *is a linear independent set, and* X *can be spanned by* $\mathcal{X}$*, i.e. for any* $\tilde{x} \in X$*, there exists* $\lambda_i \in \mathbb{R}, i = 1, 2, 3, 4, 5$ *such that* $\tilde{x} = \lambda_1 \tilde{x}_1 \oplus \lambda_2 \tilde{x}_2 \oplus \lambda_3 \tilde{x}_3 \oplus \lambda_4 \tilde{x}_4 \oplus \lambda_5 \tilde{x}_5$. *Moreover, the set of scalars* $\lambda_1, \cdots, \lambda_5$ *is unique.*

We call that the **coordinates of** $\tilde{x}$ **relative to the basis** $\mathcal{X}$ (or the $\mathcal{X}$**-coordinates of** $\tilde{x}$) are the weight $\lambda_1, \cdots, \lambda_5$ such that $\tilde{x} = \lambda_1\tilde{x}_1 \oplus \lambda_2\tilde{x}_2 \oplus \lambda_3\tilde{x}_3 \oplus \lambda_4\tilde{x}_4 \oplus \lambda_5\tilde{x}_5$.

We call the vector in $\mathbb{R}^n$ $[\tilde{x}]_{\mathcal{X}} = [\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5]$ the $\mathcal{X}$**-coordinate vector of** $\tilde{x}$.

The mapping $x \mapsto [\tilde{x}]_{\mathcal{X}}$ is the **coordinate mapping determined by** $\mathcal{X}$.

The coordinate mapping $x \mapsto [\tilde{x}]_{\mathcal{X}}$ is a one-to-one linear map (isomorphism) from X onto $\mathbb{R}^5$.

Example 1

Let $\tilde{b}, \tilde{c} \in X$ with $\tilde{1} = \langle 1; 1, 1, 0, 0 \rangle$ and $\tilde{3} = \langle 3; 4, 1, 1, 2 \rangle$. Let $a = 2$. Set $\tilde{x} = \langle x; d^-, d^+, \mu^-, \mu^+ \rangle$ be an undetermined fuzzy number in X,

$$2\tilde{x} \oplus \tilde{1} = \tilde{3}.$$

Solution *A*. $\tilde{x}$ is given by

$$\tilde{x} = \frac{1}{2}(\tilde{3} \ominus \tilde{1}) = \langle 1; 2, 1, \frac{1}{2}, 1 \rangle.$$

Example 1

Solution *B*. We compute the $\mathcal{X}$-coordinate vector of $\tilde{x}$ as

$$[\tilde{1}]_{\mathcal{X}} = [1, 0, 0, 0, 0]_{\mathcal{X}}, \quad [\tilde{3}]_{\mathcal{X}} = [3, \ln 4, 0, 1, 2]_{\mathcal{X}}$$

The solution holds if and only if

$$2[\tilde{x}]_{\mathcal{X}} + [\tilde{1}]_{\mathcal{X}} = [\tilde{3}]_{\mathcal{X}}.$$

It is a standard arithmetic equation and can be solved as

$$[\tilde{x}]_{\mathcal{X}} = [1, \ln, 0, \frac{1}{2}, 1]_{\mathcal{X}},$$

It means that

$$\tilde{x} = \langle 1; 2, 1, \frac{1}{2}, 1 \rangle.$$

Example 2

Let $\tilde{b}, \tilde{c} \in X$ with $\tilde{1} = \langle 1; 1, 1, 0, 0 \rangle$ and $\tilde{3} = \langle 3; 4, 1, 1, 2 \rangle$. Let $\tilde{a} = \langle 2; e, e, 1, 1 \rangle$. Set $\tilde{x} = \langle x; d^-, d^+, \mu^-, \mu^+ \rangle$ be an undetermined fuzzy number s.t.

$$\tilde{2}\tilde{x} \oplus \tilde{1} = \tilde{3}.$$

Solution. We first compute

$$\tilde{2}^{-1} = \langle \frac{1}{2}; 1, 1, 1, 1 \rangle, \quad \tilde{3} \ominus \tilde{1} = \langle 2; 4, 1, 1, 2 \rangle.$$

Hence

$$\tilde{x} = \tilde{2}^{-1} \otimes (\tilde{3} \ominus \tilde{1}) = \langle 1; 1, 1, 1, 2 \rangle.$$

Ongoing works

- Fuzzy linear equations $A\tilde{x} = \tilde{b}$, with fuzzy vector $\tilde{x}, \tilde{b} \in X^n$
- Fuzzy linear programming and fuzzy simplex method.

  min $f(\tilde{x}), s.t. A\tilde{x} \leq \tilde{b}$ (need the second structure: ordering)
- Fuzzy multi-criteria group decision making (MCGDM), fuzzy entropy.
- Fuzzy differentiation and integration and corresponding equations.

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THANK YOU!