Control of Parabolic Equations with Inverse Square Infinite Potential Wells

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X Partial Differential Equations, Optimal Design and Numerics
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Joint work with Alberto Enciso (ICMAT), Bruno Vergara (Brown).

Section 1

Introduction

The Main Setting

Main setting. Heat equation with critically singular potential:

$$-\partial_t v + \left(\Delta + \frac{\sigma}{d^2}\right) v = Y \cdot \nabla v + W v \qquad \text{on } (0, T) \times \Omega,$$
$$v|_{t=0} = v_0 \qquad \text{on } \Omega,$$
$$\text{“} v|_{(0,T)\times\Gamma} \text{''} = f \qquad \text{on } (0, T) \times \Gamma.$$

- $\Omega \subseteq \mathbb{R}^n$: open, bounded.
- $\Gamma := \partial\Omega \in C^2$.
- $d := d(\cdot, \Gamma)$: distance to boundary.
- $\sigma \in \mathbb{R}$: strength of singular potential.
- $Y \in C^1(\Omega; \mathbb{R}^n)$, $W \in d^{-1} L^\infty(\Omega; \mathbb{R})$: lower-order coefficients.

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Approximate controllability:

- *Given any initial data v_0, final data v_T, and $\varepsilon > 0$, is there a control f with*
$$\|v|_{t=T} - v_T\| < \varepsilon?$$

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- Carleman estimates: robust results, for general Y, W.

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Some motivations for $\sigma \neq 0$:

- Wave equations: AdS/CFT, holography.
- Heat equations: "playground" for understanding σ/d^2.

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Difficulty. Potential is critically singular:

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1. Modified asymptotics of solutions at Γ:

$$v \sim_\Gamma d^\kappa v_D + d^{1-\kappa} v_N, \qquad \kappa := \frac{1-\sqrt{1-4\sigma}}{2}, \quad \sigma \leq \frac{1}{4}.$$

- Dirichlet trace: $\mathcal{D}_\sigma v := d^{-\kappa} v|_\Gamma$.
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Remark. Threshold values of σ:

- $\sigma = \frac{1}{4}$ ($\kappa = \frac{1}{2}$): threshold for well-posedness and controllability.
- $\sigma \leq -\frac{3}{4}$ ($\kappa \leq -\frac{1}{2}$): Dirichlet branch $\notin L^2$.

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2. Shift of regularity for solutions at Γ.

- L^2-norm of $\mathcal{N}_\sigma v \Leftrightarrow H^{1+\delta(\sigma)}$-norm of solution.

The Case $n = 1$

Existing results only for $n = 1$:

$$-\partial_t v + \partial_x^2 v + \frac{\sigma}{x^2} v = 0, \quad \text{on } (0, T) \times (0, 1).$$

- Boundary null control at $x = 1$: Martinez-Vancostenoble
- Boundary null control at $x = 0$: Biccari, Cannarsa-Martinez-Vancostenoble, Gueye

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(Biccari, 2019) Boundary null controllability for $(-\frac{3}{4} <)\ \sigma < \frac{1}{4}$

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(Biccari, 2019) Several key open questions remain:

- Null controllability via global Carleman estimates?
- Potential critically singular at $x = 0$ and $x = 1$?
- Higher dimensions, $\Omega \subseteq \mathbb{R}^n$, $n > 1$?

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(Biccari-Zuazua, 2016) Interior null controllability for

$$-\partial_t v + \left(\Delta + \frac{\sigma}{d^2} \right) v = \dots .$$

- Via global Carleman estimate.
- Does not work for boundary control.

Theorem 1: Null Control

Theorem (Enciso-S-Vergara, 2023)

Assume:

- $Y \in C^1(\Omega)$, $d \cdot W \in L^\infty(\Omega)$.
- Γ *is* C^2 *and convex.*
- $-\frac{3}{4} < \sigma < 0$.

Then, $\forall\ T > 0$ *and* $\forall\ v_0 \in H^{-1}(\Omega)$, $\exists\ f \in L^2((0,T) \times \Gamma)$ *s.t. solution* v *of*

$$\begin{aligned} -\partial_t v + (\Delta + \tfrac{\sigma}{d^2})\, v &= Y \cdot \nabla v + W\, v && \text{on } (0,T) \times \Omega, \\ v|_{t=0} &= v_0 && \text{on } \Omega, \\ \mathcal{D}_\sigma v &= f && \text{on } (0,T) \times \Gamma, \end{aligned}$$

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First boundary control result for $n > 1$.

- First boundary control result for $Y, W \neq 0$ for any n.

Theorem 2: Approximate Control

Theorem (S-Vergara, 2024)

Assume:

- $Y \in C^1(\Omega)$, $d \cdot W \in L^\infty(\Omega)$.
- Γ *is* C^2, $\omega \subseteq \Gamma$ *open.*
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satisfies $\| v|_{t=T} - v_T \|_{H^{-1}(\Omega)} < \varepsilon$.

Approximate control is weaker, but result is definitive:

- Can localise control f to arbitrarily small $\omega \subseteq \Gamma$.
- Handles full range of σ.

Section 2

Proof of Null Control

Duality

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- Holds for $-\frac{3}{4} < \sigma < \frac{1}{4}$.
- "New" for all Y, W.

Observability:

$$\partial_t u + \left(\Delta + \tfrac{\sigma}{d^2}\right) u = X \cdot \nabla u + V\, u,$$
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HUM $\Rightarrow$ controllability follows from observability-side estimates:

Observability

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Main objective. Prove the lemma!

- Focus on global Carleman estimate (key step and contribution).

The HUM Machinery

Rough sketch. Define functional:

$$I_\sigma : H_0^1(\Omega) \rightarrow \mathbb{R}, \qquad I_\sigma(u_T) := \frac{1}{2} \int_{(0,T)\times\Gamma} |\mathcal{N}_\sigma u|^2 - \int_\Omega u(0) v_0 .$$

- Lemma, upper bound $\Rightarrow$ I_σ is continuous.
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Thus, I_σ has minimiser $\tilde{u}_T$:

- Null control given by $\mathcal{N}_\sigma \tilde{u}$.

Carleman Overview

Goal. Weighted spacetime estimate (roughly):

$$C' \lambda \int_{(0,T) \times \Gamma} (\mathcal{N}_\sigma u)^2 + \int_{(0,T) \times \Omega} e^{-2\lambda F} \left(\partial_t u + \Delta u + \frac{\sigma}{d^2} u \right)^2$$
$$\geq C \lambda \int_{(0,T) \times \Omega} e^{-2\lambda F} \left(|\nabla u|^2 + \frac{1}{d^2} u^2 \right).$$

- $F = F(t, x)$: specially chosen weight.
- $\lambda \gg 1$: large free parameter.
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Very rough derivation. Integrate by parts:

$$\int_{(0,T)\times\Omega} e^{-\lambda F} (\partial_t + \Delta)(e^{\lambda F} w)\, Sw, \qquad w := e^{-\lambda F} u.$$

- $Sw := \partial_t w + \lambda \nabla F \cdot \nabla w + \dots$: multiplier.
- Good choice of F, large $\lambda \Rightarrow$ positive bulk term.

A Boundary-Adapted Weight

(Biccari-Zuazua, 2016) Carleman weight roughly of form (near Γ)

$$F_l(t,x) :\approx \frac{1}{t(T-t)}\left[C - d^2(x) - d^s(x) e^{s\, d(x)}\right], \qquad s \gg 1.$$

- Does not capture $\mathcal{N}_\sigma u$ at boundary.
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Idea. Need special power of d in F to capture $\mathcal{N}_\sigma u$:

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Lemma. Boundary only sees $\mathcal{N}_\sigma u$—assuming $u_T \in H^1_0(\Omega)$:

$$d^{-1+\kappa} u|_\Gamma = \frac{1}{1-2\kappa} \mathcal{N}_\sigma u, \qquad \int_{(0,T)\times\Gamma} e^{-2\lambda F} \, \partial_t (\mathcal{D}_\sigma u) \, \mathcal{N}_\sigma u = 0.$$

The Global Weight

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Proposition. There exists "boundary-defining function" $0 < y \in C^2(\Omega)$ such that:

- Near-boundary ($d < \delta_0$): $y = d$, and $-\nabla^2 y \geq 0$.
- Intermediate ($\delta_0 \leq d \leq 2\delta_0$): $|\nabla y| \geq c$, and $-\nabla^2 y \geq -\epsilon'$.
- Far region ($d > 2\delta_0$): $-\nabla^2 y \geq \epsilon$, and y has unique critical point x_*.

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Idea. Replace d by y in Carleman weight

$$F(t,x) := \frac{1}{t(T-t)} \left[\frac{1}{1+2\kappa} \, y(x)^{1+2\kappa} + \beta \right].$$

- Also work with smoother operator $\partial_t + \Delta + \sigma y^{-2}$.
- $y = d_\Gamma$ near $\Gamma \Rightarrow$ estimate still captures $\mathcal{N}_\sigma u$ on $(0, T) \times \Gamma$.
- Γ convex $\Rightarrow y$ "almost-convex" $\Rightarrow$ controls $\dot{H}^1$-norm on all of $(0, T) \times \Omega$.
- L^2-terms contain many singular weights, but most leading terms positive.

Double Carleman

Problem. Estimate does not control L^2-norm of u near critical point x_*!

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$$+ C \lambda^3 \int_{(0,T) \times [\Omega \setminus B_\delta (x_*)]} e^{-2 \lambda F} \dots u^2 .$$

- L^2-part positive only away from x_* (contains $|\nabla y|^2$-weight).

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Idea. Construct two boundary-defining functions y_1 and y_2, with $x_{*,1} \neq x_{*,2}$.

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Balance β_1 and β_2, take λ large enough:

- Near $x_{*,1}$: positive L^2-part from y_2-bound absorbs negative L^2-part from y_1-bound.
- Near $x_{*,2}$: positive L^2-part from y_1-bound absorbs negative L^2-part from y_2-bound.

The Double Carleman Estimate

Theorem. Let F_j be the Carleman weight from y_j. Then,

$$C'\lambda \int_{(0,T)\times\Gamma} (\mathcal{N}_\sigma u)^2 + \sum_{j=1}^{2} \int_{(0,T)\times\Omega} e^{-2\lambda F_j} \left(\partial_t u + \Delta u + \frac{\sigma}{y_j^2} u \right)^2$$

$$\geq C\lambda \sum_{j=1}^{2} \int_{(0,T)\times\Omega} e^{-2\lambda F_j} \left(|\nabla u|^2 + \frac{\lambda^2}{y_j^2} u^2 \right).$$

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Recently. Can address all three points for approximate control.

Section 3

Proof of Approximate Control

The HUM Revisited

Proof via same duality/HUM setup as before:

- **Main difference.** Need unique continuation property from ω, rather than observability.

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Crucial properties. For any solution u of

$$\begin{aligned} \partial_t u + \left(\Delta + \tfrac{\sigma}{d^2}\right) u &= X \cdot \nabla u + V u, \\ u|_{t=T} &= u_T \in H_0^1(\Omega), \\ \mathcal{D}_\sigma u &= 0, \end{aligned}$$

then $\mathcal{N}_\sigma u$ is well-defined in $L^2((0,T) \times \Gamma)$, and

- $\|\mathcal{N}_\sigma u\|_{L^2((0,T)\times\Gamma)} \lesssim \|u_T\|_{H^1(\Omega)}$.
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Main objective. Prove the lemma!

- Hidden regularity: same proof as before.
- Unique continuation property: new local Carleman estimate (near $(0,T) \times \omega$).

The HUM Machinery

Rough sketch. Can assume $v_0 \equiv 0$. Define functional:

$$I_{\sigma,\varepsilon} : H_0^1(\Omega) \to \mathbb{R}, \qquad I_\sigma(u_T) := \varepsilon \| u_T \|_{H^1(\Omega)} + \frac{1}{2} \int_{(0,T)\times\Gamma} |\mathcal{N}_\sigma u|^2 + \int_\Omega u_T v_T .$$

- Lemma, upper bound $\Rightarrow$ $I_{\sigma,\varepsilon}$ is continuous.
- Lemma, unique continuation $\Rightarrow$ $I_{\sigma,\varepsilon}$ is coercive.

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Thus, I_σ has minimiser $\tilde{u}_T$:

- Approximate control given by $\mathcal{N}_\sigma \tilde{u}|_{(0,T)\times\omega}$.
- Extra term in $I_{\sigma,\varepsilon} \Rightarrow$ need less for coercivity, minimizer only approximate control.

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Idea. Consider local Carleman weight near ω:

$$F(t,x) := \frac{1}{t(T-t)}\left[\frac{1}{1+2\kappa}\, d(x)^{1+2\kappa} + |w(x)|^2\right].$$

- $w := (w_1, \ldots, w_{n-1})$ local coordinates on Γ near $x_0 \in \omega$, with $w(x_0) = 0$.
- w constant along integral curves of ∇y.
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Observation. $F \geq 0$, and $F = 0$ only at $(0, T) \times \{x_0\}$.

- Leads to unique continuation from near $(0, T) \times \{x_0\}$ (rather than from $(0, T) \times \Gamma$).

Avoiding Convexity

Can remove convexity assumption on Γ:

- **Observation.** d^{-1} very large near $\Gamma \Rightarrow$ positive bulk terms.
- Stronger than negative terms from concavity of d.
- $w \ll d^{-1}$ cannot interfere with positivity.

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Remark. In some ways, localisation makes estimate easier:

- Do not need to replace d by y.
- Only need one Carleman estimate.

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Theorem. The following estimate holds:

$$C' \lambda \int_{(0,T) \times [\Gamma \cap B_\varepsilon(x_0)]} \left[\frac{1}{d^{q_1}} (\mathcal{N}_\sigma u)^2 + \frac{1}{d^{q_0}} (\mathcal{D}_\sigma u)^2 \right] + \int_{(0,T) \times B_\varepsilon(x_0)} e^{-2\lambda F} \left(\partial_t u + \Delta u + \frac{\sigma}{d^2} u \right)^2$$

$$\geq C \lambda \sum_{j=1}^{2} \int_{(0,T) \times B_\varepsilon(x_0)} e^{-2\lambda F} \left(|\nabla u|^2 + \frac{\lambda^2}{d^2} u^2 \right).$$

- Leads to unique continuation property.

Thank You

Thank you for your attention!

A. Enciso, A. Shao, B. Vergara, *Controllability of parabolic equations with inverse square infinite potential wells via global Carleman estimates*, arXiv: 2112.04457

A. Shao, B. Vergara, *Approximate boundary controllability for parabolic equations with inverse square infinite potential wells*, arXiv: 2311.01628