Dispersive perturbation, compressible fluids and Navier-Stokes-Korteweg system

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Introduction

Compressible Navier-Stokes system

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0 & \text{in } (0,T) \times \Omega, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \mathcal{A}u + \nabla \mathrm{P} = \operatorname{div}(\mathcal{K}) & \text{in } (0,T) \times \Omega \end{cases}$$

Where

- $\mathcal{A}u := \operatorname{div}\left(2\mu(\rho)\nabla^S u\right) + \nabla\left(\nu(\rho)\operatorname{div}(u)\right)$ (Viscosity)
- P (Pressure)
- $\operatorname{div}(\mathcal{K}) := \rho\nabla\left(\kappa(\rho)\,\triangle\rho + \frac{1}{2}\kappa'(\rho)|\nabla\rho|^2\right)$ (Capillarity)

* *Compressible Navier-Stokes system*

The **compressible Navier-Stokes system** correspond to the case $\kappa = 0$ (thus $\mathsf{div}(\mathcal{K}) = 0$)

Take $\mu(\rho) := \mu_\star\rho$, $\nu(\rho) := \nu_\star\rho$ and $\kappa(\rho) := \kappa_\star/\rho$ and divide by $\rho > 0$

$\star$ Compressible Navier-Stokes system: hyperbolic-parabolic behavior

$$\begin{cases} \partial_t\rho + u\cdot\nabla\rho = \dots \\ \partial_t u - (\mu_\star \triangle + (\mu_\star+\nu_\star)\nabla\,\mathrm{div})\,u = \dots \end{cases}$$

$\star$ Navier-Stokes-Korteweg system: dispersive-parabolic behavior

$$\begin{cases} \partial_t\rho + u\cdot\nabla\rho = \dots \\ \partial_t u - (\mu_\star \triangle + (\mu_\star+\nu_\star)\nabla\,\mathrm{div})\,u \underline{+\kappa_\star\nabla\triangle\rho} = \dots \end{cases}$$

$\rightarrow$ In fact with have hidden parabolic behavior!

⋆ Cauchy problem

- **Derivations of the System :** Dunn and Serrin (1983), Brull and Méhats (2010)...
- **Weak solutions:** Bresch, Desjardins and Lin (2007), Antonelli and Spirito (2022)...
- **Strong solution:** Hattori and Li (1996), Danchin and Desjardins (2001), Haspot (2013), Charve, Danchni and Xu (2018), Tendani-Soler (2021), Paicu and Wen (2022), Bresch, Gisclon, Lacroix-Violet and Alexis Vasseur (2022)...

⋆ Main classical hypothesis

For a reference density $\rho_\star \in \mathbb{R}_+$, we suppose that

(1) $2\mu + \nu > 0$ and $\mu > 0$ near $\rho_\star$ (to get the dissipation for u)

(2) $P' > 0$ near $\rho_\star$ (use to get the dissipation for ρ and u, through energy estimates)

⋆ Dissipative properties

$\rightarrow$ Now under 1) and 2) by using : **enregy estimates** and/or **Fourier analysis methods**

The main results

For $\omega \subset \mathbb{T}^d$, we consider the following control system.

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = v_\rho \mathbf{1}_\omega & \text{in } (0,T) \times \mathbb{T}^d, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \mathcal{A}(\rho) u + \nabla(\mathrm{P}(\rho)) = \operatorname{div}(\mathcal{K}(\rho)) + v_u \mathbf{1}_\omega & \text{in } (0,T) \times \mathbb{T}^d, \end{cases} \tag{1}$$

Hypothesis

Let $\rho_\star > 0$, $u_\star \in \mathbb{R}^d$, μ, ν, κ and P such that

1. $\kappa(\rho_\star) > 0$, $\mu(\rho_\star) > 0$ *and* $2\mu(\rho_\star) + \nu(\rho_\star) > 0$
2. μ *and* ν *are* $\mathcal{C}^2$ ***in an neighborhood of*** $\rho_\star$
3. P *and* κ *are* $\mathcal{C}^3$ ***in an neighborhood*** $\rho_\star$

Theorem (T-S, 2023)

Assume Hypothesis 0.1. Let $d \in \{1, 2, 3\}$ *and* ω *be a non-empty open subset of* $\mathbb{T}^d$*. Then, for any* $T > 0$*, there is* $\varepsilon > 0$ *such that for all* $(\rho_0, u_0) \in H^2 \times H^1$ *satisfying*

$$\|(\rho_0 - \rho_\star, u_0 - u_\star)\|_{H^2 \times H^1} \leq \varepsilon,$$

there exist a control $(v_\rho, v_u) \in L^2(0, T; H^2) \times L^2(0, T; H^1)$ *and a corresponding controlled trajectory* (ρ, u) *solving (1) and satisfying*

$$(\rho, u)_{|_{t=0}} = (\rho_0, u_0) \ \text{and} \ (\rho, u)_{|_{t=T}} = (\rho_\star, u_\star) \ \text{in} \ \mathbb{T}^d.$$

Besides, the controlled trajectory (ρ, u) *enjoys the following regularity*

$$\rho \in \mathcal{C}([0, T]; H^2) \cap L^2(0, T; H^3) \cap H^1(0, T; H^1),$$
$$u \in \mathcal{C}([0, T]; H^1(\mathbb{T}^d)) \cap L^2(0, T; H^2) \cap H^1(0, T; L^2),$$

and the following positivity condition

$$\inf_{(t,x) \in [0,T] \times \mathbb{T}^d} \rho(t, x) > 0.$$

** Controllability*

$\rightarrow$ To the best of my knowledge this is the first results on the controllability of Navier-Stokes-Korteweg system

** Main hypothesis*

(1) $2\mu + \nu > 0$ and $\mu > 0$ near $\rho_\star$

** Dissipative properties*

$\rightarrow$ This work give a new way to capture the **dissipation** in the Navier-Stokes-Korteweg system and the **different physical regimes** of coefficients
$\rightarrow$ We point out that the **controllability properties** of the Navier-Stokes-Korteweg system are of **parabolic** type (O is reachable at any positive times)

Strategy

The strategy is inspired from the work of **Ervedoza, Glass and Guerrero in 2015** on the controllability for compressible Navier-Stokes system

Step 1 *A priori* analysis of the controllability of the linearized system

Step 2 Controllability of the complex coefficients heat equation and estimates on suitable weighted Sobolev spaces

Step 3 Recover the controllability for the linearized systems from the controlability to the heat equation

Step 4 Estimates of the nonlinear terms and fixed point results

The proof

$$a := \frac{\rho}{\rho_\star} - 1.$$

We are then led to study the following system

$$\begin{cases} \partial_t a + \operatorname{div}(u) = f_a(a,u) + v_a \mathbf{1}_\omega & \text{in } (0,T)\times\mathbb{T}^d, \\ \partial_t u - \mu_\star \triangle u - (\mu_\star + \nu_\star)\nabla \operatorname{div}(u) + p_\star \nabla a - \kappa_\star \nabla \triangle a = f_u(a,u) + v_u \mathbf{1}_\omega & \text{in } (0,T)\times\mathbb{T}^d, \end{cases}$$

where

$$\kappa_\star := \rho_\star \kappa(\rho_\star), \quad \mu_\star := \rho_\star^{-1}\mu(\rho_\star), \quad \nu_\star := \rho_\star^{-1}\nu(\rho_\star), \quad p_\star := P'(\rho_\star)$$

$$\begin{cases} f_a(a,u) := -u\cdot\nabla a, \\ f_u(a,u) := f_u^1(a,u) + f_u^2(a,u) + f_u^3(a,u) + f_u^4(a) + f_u^5(a), \end{cases}$$

$$\begin{cases} f_u^1(a,u) := -(a+1)u\cdot\nabla u, \\ f_u^2(a,u) := \operatorname{div}(2\underline{\mu}(a)\nabla^S u)) + \nabla(\underline{\nu}(a) \operatorname{div} u), \\ f_u^3(a,u) := (\partial_t u)a, \\ f_u^4(a) := \underline{P}'(a)\nabla a, \\ f_u^5(a) := (a+1)\nabla\left(\underline{\kappa}(a)\triangle a + \nabla\underline{\kappa}(a)\cdot\nabla a\right) \end{cases}$$

⋆ We aim to the **null controllability** of the following system

$$\begin{cases} \partial_t a + \operatorname{div}(u) = f_a + v_a \mathbf{1}_\omega & \text{in } (0,T)\times\mathbb{T}^d, \\ \partial_t u - \mu_\star \triangle u - (\mu_\star + \nu_\star)\nabla \operatorname{div}(u) + p_\star \nabla a - \kappa_\star \nabla \triangle a = f_u + v_u \mathbf{1}_\omega & \text{in } (0,T)\times\mathbb{T}^d, \end{cases}$$

⋆ The null controllability is equivalent to the **observability** of the adjoint system

$$\begin{cases} -\partial_t \sigma - p_\star \operatorname{div}(z) + \kappa_\star \triangle \operatorname{div}(z) = g_\sigma & \text{in } (0,T)\times\mathbb{T}^d, \\ -\partial_t z - \nabla\sigma - \mu_\star \triangle z - (\mu_\star + \nu_\star)\nabla \operatorname{div}(z) = g_z & \text{in } (0,T)\times\mathbb{T}^d, \end{cases}$$

⋆ The main idea to catch the parabolic behavior is to consider the **observability** of

$$\begin{cases} -\partial_t \sigma - p_\star q + \kappa_\star \triangle q = g_\sigma & \text{in } (0,T)\times\mathbb{T}^d, \\ -\partial_t q - \triangle\sigma - (2\mu_\star + \nu_\star)\triangle q = g_q & \text{in } (0,T)\times\mathbb{T}^d, \end{cases}$$

where

$$q := \operatorname{div}(z) \text{ and } g_q := \operatorname{div}(g_z)$$

- We are looking for the **observability** of a system of the form

$$-\partial_t U + A \triangle U + BU = F$$

where

$$A = \begin{pmatrix} 0 & \kappa_\star \\ -1 & -(2\mu_\star + \nu_\star) \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & -p_\star \\ 0 & 0 \end{pmatrix}$$

- Assume in this subsection that A is diagonalizable. These eigenvalues are

$$\zeta_+ := \frac{(2\mu_\star + \nu_\star) - D}{2} \text{ and } \quad \zeta_- := \frac{(2\mu_\star + \nu_\star) + D}{2} \text{ where } D := \sqrt{(2\mu_\star + \nu_\star)^2 - 4\kappa_\star}$$

- Note that

$$\Re(\zeta_\pm) > 0.$$

- The matrix $A \sim \mathrm{diag}(\zeta_+, \zeta_-)$. This can be done through a invertible matrix Q.

$\star$ We are looking for the **observability** of

$$\begin{cases} -\partial_t y^+ - \zeta_+ \triangle y^+ = g_{y^+} + \alpha_1 y^+ + \alpha_2 y^- & \text{in } (0,T) \times \mathbb{T}^d, \\ -\partial_t y^- - \zeta_- \triangle y^- = g_{y^-} + \alpha_3 y^+ + \alpha_4 y^- & \text{in } (0,T) \times \mathbb{T}^d, \end{cases}$$

with

$$\begin{pmatrix} g_{y^+} \\ g_{y^-} \end{pmatrix} := Q \begin{pmatrix} g_\sigma \\ g_q \end{pmatrix} \text{ and } \begin{pmatrix} y^+ \\ y^- \end{pmatrix} := Q \begin{pmatrix} \sigma \\ q \end{pmatrix}.$$

⋆ By duality we are looking for the following **control** problem:
Given (r_0^+, r_0^-) *in* $H^2 \times H^2$*, find two control* (v_{r^+}, v_{r^-}) *in* $L^2(H^1) \times L^2(H^1)$ *such that the solution* (r^+, r^-) *of*

$$\begin{cases} \partial_t r^+ - \overline{\zeta}_+ \triangle r = f_{r^+} + \overline{\alpha}_1 r^+ + \overline{\alpha}_3 r^- + \chi_0 v_{r^+} & \text{in } (0,T) \times \mathbb{T}^d, \\ \partial_t r^- - \overline{\zeta}_- \triangle r^- = f_{r^-} + \overline{\alpha}_2 r^+ + \overline{\alpha}_4 r^- + \chi_0 v_{r^-} & \text{in } (0,T) \times \mathbb{T}^d, \end{cases}$$

and belongs to $L^2(H^3) \times L^2(H^3)$ *and satisfies*

$$(r^+, r^-)_{|_{t=0}} = (r_0^+, r_0^-) \quad \text{and} \quad (r^+, r^-)_{|_{t=T}} = (0,0) \ \text{ in } \ \mathbb{T}^d.$$

Step 2: Carleman estimates

- (space wheight) Let ψ be in $\mathcal{C}^2(\mathbb{T}^d,\mathbb{R})$ such that

$$6 < \psi < 7 \;\text{ and }\; \inf_{\mathbb{T}^d \setminus \overline{\omega_0}} \{|\nabla \psi|\} > 0.$$

- (time wheight, **Badra, Ervedoza and Guerrero** in 2014) We choose $T_0 > 0$ and $\frac{1}{4} \geq T_1 > 0$ small enough, so that

$$T_0 + 2T_1 < T.$$

For any $m \geq 2$, we introduce a weight function $\theta_m \in \mathcal{C}^2([0,T))$ such that

$$\theta_m(t) = \begin{cases} 1 + \left(1 - \dfrac{t}{T_0}\right)^{m} & \text{for all } t \in [0,T_0], \\ 1 & \text{for all } t \in [T_0, T-2T_1], \\ \theta_m \text{ is increasing} & \text{in } [T-2T_1, T-T_1], \\ \dfrac{1}{T-t} & \text{for all } t \in [T-T_1, T). \end{cases}$$

- Then we consider the following weight function, given for $s \geq 1$ and $\lambda \geq 1$, and for any $(t,x) \in [0,T) \times \mathbb{T}^d$ by

$$\varphi_{s,\lambda}(t,x) := \theta_m(t)(\lambda e^{12\lambda} - e^{\lambda\psi(x)}), \quad \text{where } m = s\lambda^2 e^{2\lambda} \geq 0.$$

Step 2: Carleman estimates

Similarly to the work of **Ervedoza, Glass and Guerrero** we obtain the following Carleman estimates

Lemma (T.-S. 2015)

Let $T > 0$ and ζ a complex number satisfying $\Re(\zeta) > 0$. There exist three positive constants C, $s_0 \geq 1$ and $\lambda_0 \geq 1$, large enough, such that for any smooth function w on $[0, T] \times \mathbb{T}^d$ and for all $s \geq s_0$, we have

$$s^{\frac{3}{2}} \|\theta^{\frac{3}{2}} w e^{-s\varphi}\|_{L^2(L^2)} + s^{\frac{1}{2}} \|\theta^{\frac{1}{2}} \nabla w e^{-s\varphi}\|_{L^2(L^2)} + s\|w(0) e^{-s\varphi(0)}\|_{L^2} \leq C \left(\|(\partial_t + \overline{\zeta} \triangle) w e^{-s\varphi}\|_{L^2(L^2)} + s^{\frac{3}{2}} \|\theta^{\frac{3}{2}} \chi_0 w e^{-s\varphi}\|_{L^2(L^2)} \right).$$

• (Heat equation coefficient) Let $\zeta \in \mathbb{C}$ such that

$$\Re(\zeta) > 0.$$

• (control zone) In order to add a margin on the control zone ω, we introduce a non-negative smooth cut-off function χ_0 such that there exist two proper open subsets ω_0 and ω_1 of $\mathbb{T}^d$ such that

$$\omega_0 \subset supp(\chi_0) \subset \omega_1 \Subset \omega \ \text{ and } \ \chi_0 = 1 \text{ on } \omega_0.$$

• (control problem) We consider the following controllability problem:
Given r_0 and f, find a control function v_r such that the solution r of

$$\begin{cases} \partial_t r - \zeta \triangle r = f + v_r \chi_0 & in\ (0,T) \times \mathbb{T}^d, \\ r_{|_{t=0}} = r_0 & in\ \mathbb{T}^d, \end{cases}$$

satisfies

$$r_{|_{t=T}} = 0 \ \ in\ \mathbb{T}^d. \qquad (2)$$

Theorem

Let $T > 0$. There exist constants $C > 0$ and $s_0 \geq 1$ such that for all $s \geq s_0$, for all $f \in L^2(0, T; L^2(\mathbb{T}^d))$ satisfying

$$\|\theta^{-\frac{3}{2}} f e^{s\varphi}\|_{L^2(L^2)} < +\infty \tag{3}$$

and $r_0 \in L^2(\mathbb{T}^d)$, there exists a solution (r, v_r) of the control problem which furthermore satisfies the following estimate:

$$s^{\frac{3}{2}} \|re^{s\varphi}\|_{L^2(L^2)} + \|\theta^{-\frac{3}{2}} \chi_0 v_r e^{s\varphi}\|_{L^2(L^2)} + s^{\frac{1}{2}} \|\theta^{-1} \nabla r e^{s\varphi}\|_{L^2(L^2)} \leq C \left(\|\theta^{-\frac{3}{2}} f e^{s\varphi}\|_{L^2(L^2)} + s^{\frac{1}{2}} \|r_0 e^{s\varphi(0)}\|_{L^2} \right).$$

Moreover, the solution (r, v_r) can be obtained through a linear operator in (r_0, f).

- We aim to obtain the following observability inequality

$$\|\sigma e^{-s_0\Phi}\|_{L^2(L^2)} + \|\sigma(0)e^{-s_0\Phi(0)}\|_{L^2} + \|qe^{-\frac{4s_0\Phi}{3}}\|_{L^2(L^2)} + \|q(0)e^{-\frac{4s_0\Phi(0)}{3}}\|_{L^2}$$
$$\lesssim \|(g_\sigma, g_q)e^{-\frac{3s_0\Phi}{4}}\|_{L^2(L^2)\times L^2(L^2)} + \|\chi(\sigma, q)e^{-\frac{3s_0\Phi}{4}}\|_{L^2(L^2)\times L^2(L^2)},$$

where (σ, z) is a solution of the following adjoint system

$$\begin{cases} -\partial_t\sigma - p_\star q + \kappa_\star \triangle q = g_\sigma & \text{in } (0,T)\times\mathbb{T}^d, \\ -\partial_t q - \triangle\sigma - (2\mu_\star + \nu_\star)\triangle q = g_q & \text{in } (0,T)\times\mathbb{T}^d, \end{cases} \tag{4}$$

with $(g_\sigma, g_q) \in L^2(L^2)\times L^2(L^2)$.

Step 3: Observability of (σ, z)

- Recall that

$$\|(y^+,y^-)e^{-s_0\Phi}\|_{L^2(H^{-1})} + \|(y^+(0),y^-(0))e^{-s_0\Phi(0)}\|_{H^{-2}}$$
$$= \sup_{\substack{\|(f_{r^+},f_{r^-})e^{s_0\Phi}\|_{L^2(L^2)}\le 1 \\ \|(r_0^+,r_0^-)e^{s_0\Phi(0)}\|_{L^2}\le 1}} \{\langle (f_{r^+},f_{r^-}),(y^+,y^-)\rangle_{L^2(L^2)} + \langle (r_0^+,r_0^-),(y^+(0),y^-(0))\rangle_{L^2}\}.$$

- By duality, we have

$$\langle (f_{r^+},f_{r^-}),(y^+,y^-)\rangle_{L^2(L^2)}) + \langle (r_0^+,r_0^-),(y^+(0),y^-(0))\rangle_{L^2}$$
$$= \langle (g_{y^+},g_{y^-}),(r^+,r^-)\rangle_{L^2(L^2)} + \Re(\langle (y^+,y^-),\chi_0(v_{r^+},v_{r^-})\rangle_{L^2(L^2)}.$$

$$\|(y^+,y^-)e^{-s_0\Phi}\|_{L^2(L^2)} + \|(y^+(0),y^-(0))e^{-s_0\Phi(0)}\|_{L^2}$$
$$\lesssim \|(g_{y^+},g_{y^-})e^{-\frac{3s_0\Phi}{4}}\|_{L^2(L^2)} + \|\chi_0(y^+,y^-)e^{-\frac{3s_0\Phi}{4}}\|_{L^2(L^2)}.$$

- We simply remind that solutions (y^+,y^-) correspond to solutions (σ,q) through the transform

$$\begin{pmatrix}\sigma\\ q\end{pmatrix} := Q^{-1}\begin{pmatrix}y^+\\ y^-\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}g_\sigma\\ g_q\end{pmatrix} := Q^{-1}\begin{pmatrix}g_{y^+}\\ g_{y^-}\end{pmatrix},$$

Merci!

Thanks !