Schrödinger eigenfunctions sharing the same modulus and applications to the control of quantum systems

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X Partial differential equations, optimal design and numerics, Benasque

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The bilinear Schrödinger equation

The bilinear Schrödinger equation is

$$\begin{cases} i\partial_t \psi = (-\Delta_g + V)\psi + \sum_{i=1}^m u_i(t) Q_i(x)\psi & \text{in } (0,+\infty)\times M, \\ \text{Boundary conditions} & \text{on } (0,+\infty)\times \partial M, \\ \psi(0,\cdot) = \psi_0 & \text{in } M. \end{cases} \tag{S}$$

where $\psi(t) \in \mathcal{H} := L^2(M;\mathbb{C})$ state, $u(t) = (u_1(t), \dots, u_m(t)) \in \mathbb{R}^m$ the control.

- (M, g) smooth compact manifold, possibly with boundary.
- $\Delta_g = \mathrm{div}_{\omega_g} \circ \nabla_g$ the Laplace-Beltrami operator on (M, g).
- $V \in L^\infty(M;\mathbb{R})$ electric potential.
- $Q = (Q_1, \dots, Q_m) \in L^\infty(M;\mathbb{R})^m$ potentials of interactions.

The bilinear Schrödinger equation can be written as

$$\begin{cases} i\partial_t \psi = H_0\psi + \langle u(t), Q\rangle_{\mathbb{R}^m}\psi & \text{in } (0,+\infty)\times M, \\ \text{Boundary conditions} & \text{on } (0,+\infty)\times \partial M, \\ \psi(0,\cdot) = \psi_0 & \text{in } M. \end{cases} \tag{S}$$

Well-posedness and obstruction to exact controllability

$$\left\{\begin{array}{ll} i\partial_t \psi = H_0 \psi + \langle u(t), Q\rangle_{\mathbb{R}^m} \psi & \text{in } (0,+\infty)\times M, \\ \text{Boundary conditions} & \text{on } (0,+\infty)\times \partial M, \\ \psi(0,\cdot) = \psi_0 & \text{in } M. \end{array}\right. \tag{S}$$

- For every $T > 0$, $\psi_0 \in L^2(M)$ and $u \in L^2(0,T;\mathbb{R}^d)$, there exists a unique mild solution $\psi = \psi(\cdot;\psi_0,u) \in C([0,T];L^2(M))$ of (S), i.e.,

$$\psi(t) = e^{-itH_0}\psi_0 + \int_0^t e^{-i(t-s)H_0}\langle u(s), Q(x)\rangle \psi(s) ds, \qquad \forall t \in [0,T].$$

- If $\psi_0 \in \mathcal{S} = \{\psi \in L^2(M) \mid \|\psi\|_{L^2(M)} = 1\}$, then $\psi(t) \in \mathcal{S}$.

For $\psi_0 \in L^2(M)$, the **reachable space** is

$$\mathcal{R}(\psi_0) := \{\psi(t;\psi_0,u) \mid t \geq 0,\ u \in L^2(0,t;\mathbb{R}^d)\}.$$

Theorem (Ball, Marsden, Slemrod (1982), Turinici (2000))

For every $\psi_0 \in \mathrm{Dom}(H_0)\cap\mathcal{S}$, $\overline{(\mathcal{R}(\psi_0))^c} = \mathrm{Dom}(H_0)\cap\mathcal{S}$. This means that the interior of $\mathcal{R}(\psi_0)$ in $\mathrm{Dom}(H_0)\cap\mathcal{S}$ for the topology of $\mathrm{Dom}(H_0)$ is empty.

Small-time isomodulus approximate controllability

$$\begin{cases} i\partial_t \psi = H_0 \psi + \langle u(t), Q \rangle_{\mathbb{R}^m} \psi & \text{in } (0,+\infty) \times M, \\ \text{Boundary conditions} & \text{on } (0,+\infty) \times \partial M, \\ \psi(0,\cdot) = \psi_0 & \text{in } M. \end{cases} \qquad \text{(S)}$$

Due to BMS obstruction, people rather study:

- Exact controllability in regular spaces (Beauchard, Laurent (2010) ...).
- Large time approximate controllability (Boscain, Caponigro, Chambrion, Sigalotti (2012) ...)
- Small-time approximate controllability (Beauchard, Pozolli (2024) ...).

The **small-time approximately reachable space** is

$$\overline{\mathcal{R}_0(\psi_0)} := \{\psi_1 \in \mathcal{S} \ ; \\ \forall \varepsilon, \tau > 0,\ \exists T \in (0,\tau],\ u \in L^2(0,T;\mathbb{R}^m),\ \|\psi(T;\psi_0,u) - \psi_1\|_{L^2(M)} < \varepsilon\}.$$

Here, we focus on

Definition

(S) is small-time isomodulus approximately controllable from $\psi_0 \in \mathcal{S}$ if

$$\{e^{i\theta}\psi_0 \mid \theta \in L^2(M;\mathbb{T})\} \subset \overline{\mathcal{R}_0(\psi_0)}.$$

Duca, Nersesyan's results

$$\begin{cases} i\partial_t \psi = (-\Delta_g + V)\psi + \sum_{i=1}^m u_i(t) Q_i(x)\psi & \text{in } (0,+\infty)\times M, \\ \text{Boundary conditions} & \text{on } (0,+\infty)\times\partial M, \\ \psi(0,\cdot) = \psi_0 & \text{in } M. \end{cases} \qquad \text{(S)}$$

Let $M = \mathbb{T}^d$ and assume

$$x \mapsto 1,\ x \mapsto \sin\langle x,k\rangle,\ x\mapsto \cos\langle x,k\rangle \ \in \operatorname{span}\{Q_1,\dots,Q_m\}, \qquad \forall k \in \mathcal{K}.$$

where $\mathcal{K} = \{(1,0,\dots,0),(0,1,0\dots,0),\dots,(0,\dots,0,1,0),(1,1,\dots,1)\} \subset \mathbb{R}^d$.

Theorem (Duca, Nersesyan (2023))

(S) *is small-time isomodulus approximately controllable.*

Theorem (Duca, Nersesyan (2023))

If $V = 0$ *then*

$$e^{\pm i\langle k,x\rangle} \in \overline{\mathcal{R}_0(e^{\pm i\langle l,x\rangle})}.$$

- Technique inspired by Agrachev, Sarychev (2005) for NS equations.
- Extension to NLS.

An abstract limit using Lie brackets

$$\begin{cases} i\partial_t \psi(t) = H_0 \psi(t) + \sum_{j=1}^m u_j(t) H_j \psi(t), \quad t \in (0, +\infty), \\ \psi(0) = \psi_0. \end{cases} \tag{S}$$

First directions: $\lim_{\delta \to 0} \exp\left(-i\delta\left(H_0 + \sum_{j=1}^m \frac{u_j}{\delta} H_j\right)\right)\psi_0 = \exp\left(-i\sum_{j=1}^m u_j H_j\right)\psi_0.$

Theorem (Chambrion, Pozolli (2023))

Let S be a bounded self-adjoint operator satisfying

$$[S, H_j] = 0, \quad j = 1, \ldots, m, \tag{Commutation}$$

$$S\mathrm{Dom}(H_0) \subset \mathrm{Dom}(H_0), \quad [S,[S,[S,H_0]]]\mathrm{Dom}(H_0) = 0. \tag{Stability}$$

Then, for each $\psi_0 \in \mathcal{H}$ and $u = (u_1, \ldots, u_m) \in \mathbb{R}^m$, the following limit holds in $\mathcal{H}$

$$\lim_{\delta \to 0} e^{-i\delta^{-1/2}S} \exp\left(-i\delta\left(H_0 + \sum_{j=1}^m \frac{u_j}{\delta} H_j\right)\right) e^{i\delta^{-1/2}S}\psi_0$$

$$= \exp\left(\frac{i}{2}[S,[S,H_0]] - i\sum_{j=1}^m u_j H_j\right)\psi_0. \tag{Second Lie bracket direction}$$

Application of the second Lie bracket direction

Theorem (Chambrion, Pozolli (2023) - Boscain, L.B., Sigalotti (2024))

For every $\psi_0 \in L^2(M)$, $(u_1, \dots, u_m) \in \mathbb{R}^m$, $\varphi \in C^\infty(M;\mathbb{R})$ *such that* $\varphi Dom(H_0) \subset Dom(H_0)$,

$$\lim_{\delta \to 0} e^{-i\delta^{-1/2}\varphi} \exp\left(-i\delta\left(-\Delta_g + V + \sum_{j=1}^m \frac{u_j}{\delta} Q_j\right)\right) e^{i\delta^{-1/2}\varphi}\psi_0$$

$$= \exp\left(-ig(\nabla_g\varphi, \nabla_g\varphi) - i\sum_{j=1}^m u_j Q_j\right)\psi_0.$$

$$\mathcal{H}_0 = \{\varphi \in \text{span}\{Q_1, \dots, Q_m\} \mid \varphi \mathrm{Dom}(H_0) \subset \mathrm{Dom}(H_0)\} \subset L^2(M;\mathbb{C}),$$

$$\mathcal{H}_{N+1} = \{\varphi \in \mathcal{H}_N + \text{span}\{g(\nabla_g\psi, \nabla_g\psi) \mid \psi \in \mathcal{H}_N\} \mid \varphi \mathrm{Dom}(H_0) \subset \mathrm{Dom}(H_0)\}, \quad N \geq 0.$$

$$\mathcal{H}_\infty = \bigcup_{N \geq 0} \mathcal{H}_N.$$

Theorem (Chambrion, Pozolli (2023) - Boscain, L. B., Sigalotti (2024))

For every $\psi_0 \in L^2(M)$, *we have* $\{e^{i\phi}\psi_0 \mid \phi \in \mathcal{H}_\infty\} \subset \overline{\mathcal{R}_0(\psi_0)}$.
If $\mathcal{H}_\infty$ *is dense in* $L^2(M;\mathbb{R})$ *then,* (S) *is small-time isomodulus approximately controllable.*

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Eigenfunctions sharing the same modulus

- $H_0 = -\Delta_g + V$, $(H_0, \mathrm{Dom}(H_0))$ self-adjoint on $\mathcal{H}$ with compact resolvant.
- Basis of eigenfunctions $(\phi_k)_{k\geq 1}$ associated with the eigenvalues $(\lambda_k)_{k\geq 1}$.
- $E_{\lambda_k} = \mathrm{Ker}(H_0 - \lambda_k I)$ the eigenspace associated to λ_k.

Definition

$\phi_k \in E_{\lambda_k}$, $\phi_\ell \in E_{\lambda_\ell}$ share the same modulus if $|\phi_k(x)| = |\phi_\ell(x)|\ \forall x \in M$.

Several notions:

- For $k \geq 1$, H_0 may admit eigenfunctions sharing the same modulus inside the energy level λ_k, that is, there may exist two $\mathbb{C}$-linearly independent eigenfunctions in E_{λ_k} that share the same modulus;
- For $k, \ell \geq 1$, H_0 may admit two eigenfunctions $\phi_k \in E_{\lambda_k}$ and $\phi_\ell \in E_{\lambda_\ell}$ sharing the same modulus and corresponding to different energy levels λ_k and λ_ℓ;
- H_0 may admit eigenfunctions sharing the same modulus corresponding to all energy levels, that is, there may exist a subsequence $(\phi_{k_j})_{j\geq 1}$ of an orthonormal basis of eigenfunctions $(\phi_k)_{k\geq 1}$ such that the functions ϕ_{k_j} all share the same modulus and such that $\{\lambda_k \mid k \geq 1\} = \{\lambda_{k_j} \mid j \geq 1\}$.

Question: Conditions on (M, g, V) so that the Schrödinger eigenfunctions share the same modulus?

Laplace eigenfunctions on the torus $\mathbb{T}^d$

Let $M = \mathbb{T}^d$, $V = 0$, and $H_0 = -\Delta$. The eigenvalues, eigenfunctions are given by

$$\lambda_k = n_1^2 + \cdots + n_d^2, \qquad (n_1, \ldots, n_d) \in \mathbb{N}^d,$$

$$\Phi_k^{\pm}(x) = e^{\pm i \sum_{j=1}^{d} n_j x_j}, \qquad x = (x_1, \ldots, x_d) \in \mathbb{T}^d.$$

Proposition

The operator H_0 admits eigenfunctions sharing the same modulus inside each energy level $\lambda_k > 0$ and corresponding to all energy levels.

The spherical harmonics

Let $M = \mathbb{S}^2$, $V = 0$, and $H_0 = -\Delta_g$. The eigenvalues, eigenfunctions are given by

$$\lambda_k = l(l+1), \qquad l \in \mathbb{N},$$

$$Y_l^m(\alpha, \beta) = \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} P_l^m(\cos(\alpha)) e^{im\beta}, \qquad m \in \{-l, \dots, l\},$$

where P_l^m is the Legendre polynomial, (α, β) are the spherical coordinates on $\mathbb{S}^2$.

Proposition

For every $l \in \mathbb{N}$, $m \in \{-l, \dots, l\}$, Y_l^m and Y_l^{-m} share the same modulus. Then, for each $l \geq 1$, H_0 admits eigenfunctions sharing the same modulus inside the energy level $l(l+1)$.

Proof: $P_l^{-m} = (-1)^m \frac{(l-m)!}{(l+m)!} P_l^m$ then $Y_l^{-m}(\alpha, \beta) = (-1)^m e^{-2im\beta} Y_l^m(\alpha, \beta)$. □

Question: for $k, l \geq 0$, $k \neq l$, do there exist $\phi_k \in E_{k(k+1)}$ and $\phi_l \in E_{l(l+1)}$ sharing the same modulus?

The disk with Dirichlet boundary conditions

Let $\mathbb{D} = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$, $V = 0$, $H_0 = -\Delta$.
The eigenvalues, eigenfunctions are given by

$$\lambda_{0,k} = j_{0,k}^2, \qquad \lambda_{n,k} = j_{n,k}^2 \qquad \forall n, k \geq 1,$$

$$u_{0,k}(r, \theta) = \sqrt{\frac{1}{\pi}} \frac{1}{|J_0'(j_{0,k})|} J_0(j_{0,k} r)$$

$$u_{n,k}(r, \theta) = \sqrt{\frac{2}{\pi}} \frac{1}{|J_n'(j_{n,k})|} J_n(j_{n,k} r) \cos(n\theta) \text{ and } \sqrt{\frac{2}{\pi}} \frac{1}{|J_n'(j_{n,k})|} J_n(j_{n,k} r) \sin(n\theta),$$

$j_{n,k}$ k-th zero of the Bessel function J_n.

Proposition

Let $n, m \geq 0$ be such that $n \neq m$ and $k, l \geq 1$. Assume that $\phi_{n,k}$ and $\phi_{m,l}$ are eigenfunctions corresponding to $j_{m,l}^2$ and $j_{n,k}^2$, respectively. Then $\phi_{n,k}$ and $\phi_{m,l}$ do not share the same modulus.

On the other hand, for $n \geq 1$ and $k \geq 1$, there exist two $\mathbb{C}$-linearly independent eigenfunctions corresponding to the eigenvalue $j_{n,k}^2$ that share the same modulus.

Proof: Siegel's result tells us that J_n and J_m for $n \neq m$ have no common zeros.

Harmonic oscillator in $\mathbb{R}$

Let $M = \mathbb{R}$, $V(x) = |x|^2$, $H_0 = -\partial_x^2 + |x|^2$.
The eigenvalues, eigenfunctions are given by

$$\lambda_k = 2k + 1 \qquad \forall k \geq 0,$$

$$\Phi_k(x) = \frac{1}{\sqrt{2^k k! \sqrt{\pi}}} \left(x - \frac{d}{dx} \right)^k e^{-\frac{x^2}{2}} = H_k(x) e^{-x^2/2}, \qquad x \in \mathbb{R}, \quad k \in \mathbb{N},$$

H_k is the Hermite polynomial of degree k.

Proposition

Let $d = 1$. For every $k_1, k_2 \in \mathbb{N}$, $k_1 \neq k_2$, H_0 does not admit two eigenfunctions corresponding to the energy levels $2k_1 + 1$ and $2k_2 + 1$ that share the same modulus.

Proof: Degree's argument. □

Genericity results

- $H_0 = -\Delta_g + V$, $(H_0, \mathrm{Dom}(H_0))$ self-adjoint on $\mathcal{H}$ with compact resolvant.

Lemma

If λ_k and λ_ℓ are simple and distinct eigenvalues of H_0 with corresponding eigenfunctions ϕ_k and ϕ_ℓ, then ϕ_k and ϕ_ℓ cannot share the same modulus.

Corollary

Let M be a compact connected C^∞ manifold M without boundary of dimension larger than or equal to 2. Then, generically with respect to the Riemanniann metric g, no pair of $\mathbb{C}$-linearly independent eigenfunctions of the Laplace–Beltrami operator $-\Delta_g$ share the same modulus.

Same result when considering the genericity with respect to the potential V (with no restriction on the dimension of M).

Ingredient: The spectrum of the Schrödinger operator is known to be generically simple with respect to g (Uhlenbeck (1976), Tanikawa (1979)) or V (Mason, Sigalotti (2010)).

Main result in 1-D

Theorem (Boscain, L.B., Sigalotti (2024))

If M is one-dimensional and the Schrödinger operator H_0 admits two $\mathbb{C}$-linearly independent eigenfunctions ϕ_k and ϕ_ℓ sharing the same modulus, then necessarily M is a closed curve and ϕ_k, ϕ_ℓ are nowhere vanishing on M. If, moreover, the two eigenfunctions correspond to distinct eigenvalues, then V is constant.

Proof: Four possibilities for M

- M is isometric to the line $\mathbb{R}$,
- M is isometric to the half-line $[0, +\infty)$,
- M is isometric to a compact interval $[0, L]$ for some $L > 0$,
- M is a closed curve isometric to the quotient $\mathbb{R}/L\mathbb{Z}$ for some $L > 0$.

Then ϕ_k and ϕ_ℓ share the same modulus $\rho := |\phi_k| = |\phi_\ell| \in C(M, [0, +\infty))$. Set $M_\rho = \{x \in M \mid \rho(x) \neq 0\}$ and let $\theta_k, \theta_\ell : M_\rho \to \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ be such that

$$\phi_k(x) = \rho(x)e^{i\theta_k(x)}, \quad \phi_\ell(x) = \rho(x)e^{i\theta_\ell(x)}, \qquad x \in M_\rho. \qquad \text{(Polar form)}$$

Write the equations for ϕ_k, ϕ_l, θ_k and θ_l. “Solve” them in 1-D. □

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Sum up, perspectives

$$\begin{cases} i\partial_t \psi = -\Delta_g \psi + V\psi + \langle u(t), Q\rangle_{\mathbb{R}^m}\psi & \text{in } (0,+\infty)\times M, \\ \text{Boundary conditions} & \text{on } (0,+\infty)\times \partial M, \\ \psi(0,\cdot) = \psi_0 & \text{in } M. \end{cases} \tag{S}$$

- Saturation property on Q, then small-time isomodulus approximate controllability of (S), i.e. $\{e^{i\theta}\psi_0 \mid \theta \in L^2(M;\mathbb{T})\} \subset \overline{\mathcal{R}_0(\psi_0)}$ (Duca, Nersesyan (2023)).
- $\Rightarrow$ **Main question**: Conditions on (M, g, V) so that the Schrödinger eigenfunctions share the same modulus?
- Explicit examples included the torus, the sphere, the disk...
- Generically, the spectrum is simple so the answer is negative.
- Full treatment of the one-dimensional case.
- Examples of quantum graphs that exhibit more complex structures.