The current state in the field of nonlocal conservation laws

A Talk at the Workshop-Summer School: X Partial differential equations, optimal design and numerics, Benasque

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26.08.2024

1. **Nonlocal conservation laws in applications and theory**
2. The singular limit problem
3. More on the singular limit problem
4. Open problems – future research

Why Nonlocal?

Figure: Star Trek: Picard, Episode 2 (“Maps and Legends”), aired at January 30th, 2020

Supply chains: semiconductor manufacturing systems

Governing equations

Consider the IBVP on $(0,T)\times(0,1)$

$$\partial_t q(t,x) + V\left(\int_0^1 q(t,y)\,\mathrm{d}y\right)\partial_x q(t,x) = 0$$
$$q(0,x) = q_0(x)$$
$$V\left(\int_0^1 q(t,y)\,\mathrm{d}y\right) q(t,0) = y(t).$$

- q_0 initial density • y boundary datum
- $V:\mathbb{R}\to\mathbb{R}$ processing speed
- $\int_0^1 q(t,y)\,\mathrm{d}y$ “work in progress”.

Bibliography

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[2] Coron, J. M., et al. Analysis of a conservation law modeling a highly re-entrant manufacturing system. **Disc. and Cont. Dyn. Sys.**, (2010).

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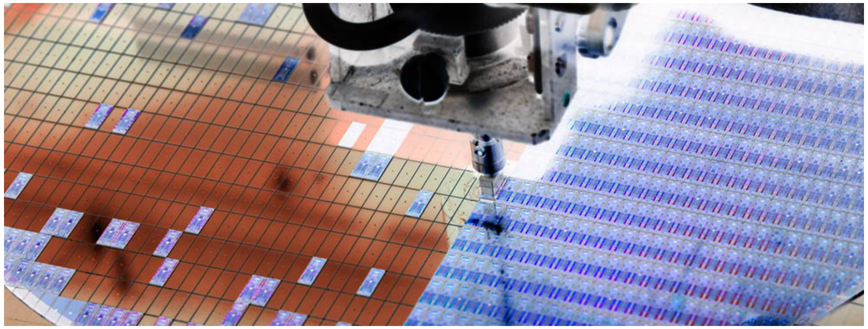

Figure: Semiconductor manufacturing system, © KISTLER

Traffic flow

Governing equations

For $(t,x)\in(0,T)\times(0,1)$ and $\eta\in\mathbb{R}_{>0}$ the look ahead parameter, consider

$$\partial_t q(t,x) = -\partial_x\Big(V\left(W[q](t,x)\right)q(t,x)\Big)$$
$$W[q](t,x) = \frac{1}{\eta}\int_x^{x+\eta}\gamma\Big(\frac{x-y}{\eta}\Big)q(t,y)\,\mathrm{d}y.$$

• q traffic density • V velocity and • γ a weight

Figure: Traffic at Interstate 80, Berkeley, CA, © Wikipedia

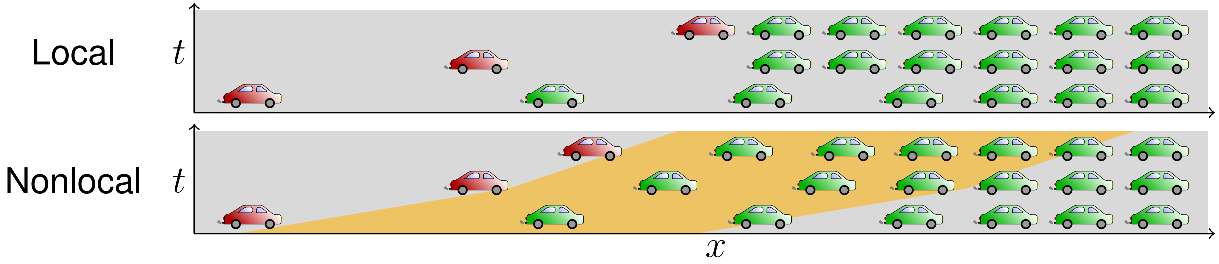

Figure: Local vs. nonlocal behavior in the case of a congestion ahead

Traffic flow

Governing equations

For $(t,x) \in (0,T) \times (0,1)$ and $\eta \in \mathbb{R}_{>0}$ the look ahead parameter, consider

$$\partial_t q(t,x) = -\partial_x\Big(V\left(W[q](t,x)\right) q(t,x)\Big)$$
$$W[q](t,x) = \tfrac{1}{\eta}\int_x^{x+\eta} \gamma\Big(\tfrac{x-y}{\eta}\Big) q(t,y)\,\mathrm{d}y.$$

• q traffic density • V velocity and • γ a weight

Figure: Traffic at Interstate 80, Berkeley, CA, © Wikipedia

Bibliography

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[8] Chiarello, F.A., Goatin P. Global entropy weak solutions for general non-local traffic flow models with anisotropic kernel. **ESAIM: Mathematical Modelling and Numerical Analysis**, (2018).

Chemical ripening processes

Governing equations

Consider on $(0,T)\times\mathbb{R}^n_{>0}$

$$\begin{aligned}\partial_t q(t,\boldsymbol{x}) &= -\operatorname{div}\big(R(t,\boldsymbol{x},W[q](t))q(t,\boldsymbol{x})\big)\\ &\qquad + h(t,\boldsymbol{x}) - g(t,\boldsymbol{x})q(t,\boldsymbol{x})\\ q(0,\boldsymbol{x}) &= q_0(\boldsymbol{x})\\ W[q](t) &\coloneqq \iint_{\mathbb{R}^n_{>0}} \gamma(\boldsymbol{y})q(t,\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}.\end{aligned}$$

• q particle shape distribution • $\boldsymbol{x}$ shape parameters • h source term • g shape dependent outflow rate • q_0 initial particle shape distribution • R growth rate

Figure: Pigments

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[9] Ramkrishna, D., et al. Population balance modeling: Current status and future prospects. ***Annual Review of Chemical and Biomolecular Engineering***, (2014).

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Crowd dynamics, pedestrian flow and opinion formation

Governing equations

Consider for the $i \in \{1, \dots, n\}$ population on $(0,T) \times \mathbb{R}^n$

$$\partial_t q^i(t,\boldsymbol{x}) = -\operatorname{div}\left(q^i(t,\boldsymbol{x})\boldsymbol{V}^i[\boldsymbol{q}](t,\boldsymbol{x})\right)$$

$$\boldsymbol{V}^i[\boldsymbol{q}](t,\boldsymbol{x}) = v^i\left(\sum_{k=1}^{n} q^i * \gamma^i(t,\boldsymbol{x})\right)\boldsymbol{v}^i(\boldsymbol{x}).$$

• v^i nonlocal velocity • $\boldsymbol{v}^i$ general direction • γ^i nonlocal weight

Bibliography

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[13] Keimer, A., et al. Existence, uniqueness and regularity of multi-dimensional nonlocal balance laws with damping. ***Journal of Mathematical Analysis and Applications***, (2018).

Figure: A swarm of geese, © Wikipedia, and a crowd heading to an exit (right), © Hermes

Problem formulation for scalar nonlocal balance laws

Nonlocal balance laws

Consider for $(t,x) \in (0,T) \times \mathbb{R}$

$$\begin{aligned} q_t(t,x) + \partial_x\Big(V\big(t,x,W\big[q,\gamma,a,b\big](t,x)\big)q(t,x)\Big) &= h(t,x) \\ q(0,x) &= q_0(x) \end{aligned}$$

supplemented by the nonlocal term W, averaging the “density” in space

$$W\big[q,\gamma,a,b\big]\big(t,x\big) := \int_{a(x)}^{b(x)} \gamma(t,x,y)q(t,y)\,\mathrm{d}y.$$

- V velocity
- a, b boundaries of the nonlocal term
- γ nonlocal weight
- q_0 initial datum
- h space-time dependent source term

Main existence theorem

Remark

- *No fully local behavior anymore, i.e., solution has to be known between $a(x)$ and $b(x)$ for each $x \in \mathbb{R}$ to advance in time*
- *Still finite propagation of mass, but infinite speed of "information"*
- *None of the usual existence and uniqueness results (Kružkov, etc.) applicable*

Theorem (Existence and uniqueness for sufficiently small time)

Suppose

$\bullet\ q_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ $\bullet\ h \in L^\infty((0,T); L^\infty(\mathbb{R}))$ $\bullet\ a, b \in C^1(\mathbb{R})$ *with* $a', b' \in L^\infty(\mathbb{R})$ $\bullet\ \gamma$ *sufficiently smooth* $\bullet\ V$ *(locally) Lipschitz,*

there is $T^* \in (0,T]$ *so that a unique weak solution*

$$q \in C([0,T^*]; L^p(\mathbb{R})) \cap L^\infty((0,T^*); L^\infty(\mathbb{R}))$$

exists $(p \in [1,\infty))$*.*

Comments

Remark (Entropy condition)

No entropy condition required for uniqueness!

Lemma (Higher regularity)

Let q_0, V, γ, a, b be smooth, the solution q will be smooth as long as it exists.

Sketch of the proof

Solution formula gives for $(t, x) \in \Omega_T$

$$q(t,x) = q_0(\xi_W(t,x;0))\partial_2\xi_W(t,x;0)$$

with ξ_W the solution of

$$\xi(t,x,\tau) = x + \int_t^\tau V\left(W\Big[q,\gamma,a,b\Big](t,\xi(t,x;s)),t,\xi(t,x;s)\right)\,\mathrm{d}s$$

$$W[q,\gamma,a,b](t,x) = \int_{\xi_W(t,a(x);0)}^{\xi_W(t,b(x);0)} \gamma(t,x,\xi_W(0,y;t))q_0(y)\,\mathrm{d}y.$$

Literature

[14] A. Keimer and L. Pflug Existence, uniqueness and regularity results on nonlocal balance laws. ***Journal of Differential Equations (JDE)***, (2017).

Finite time horizon of existence

Blow-up in finite time: Nonlocal Burgers' with the wrong sided kernel

$$\partial_t q(t,x) + \partial_x \left(\int_x^{x+1} q(t,y)\,\mathrm{d}y\, q(t,x) \right) = 0$$

$$q_0(x) = \chi_{[0,1]}(x)$$

Solution blows up for $t \to 1$.

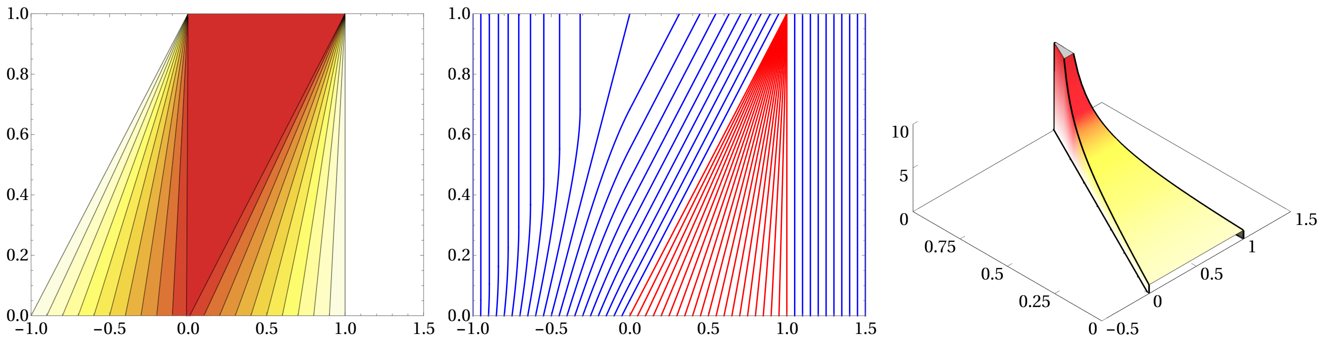

Figure: **Left:** Nonlocal impact $\int_x^{x+1} q(t,y)\,\mathrm{d}y$ **Middle:** Characteristics **Right:** Solution

Semi-global existence of solutions

Theorem (Existence of the solution for larger time)

Suppose that in addition one of the following items holds:

1. $a(x) = a$ *with* $a \in \mathbb{R} \cup \{\pm\infty\}$ *and* $b(x) = b$ *with* $b \in \mathbb{R} \cup \{\pm\infty\}$
2. $\operatorname{supp}(\gamma(t,x,\cdot)) \subsetneq (a(x), b(x))\ \forall x \in \mathbb{R}$
3.
   - $\eta \in \mathbb{R}_{>0}$, $a(x) = x, b(x) = x + \eta\ \forall x \in \mathbb{R}$
   - $\tilde{V} \in W^{1,\infty}_{\text{loc}}(\mathbb{R}):\ V' \leqq 0$ *not explicitly space-time dependent*
   - $\gamma(t,x,y) = \frac{1}{\eta}\tilde{\gamma}(\frac{y-x}{\eta})\ \forall (x,y) \in \mathbb{R}^2$ *with* $\tilde{\gamma} \in W^{1,\infty}(\mathbb{R})$ *monotone decreasing*
   - $q_0 \in L^\infty(\mathbb{R}; \mathbb{R}_{\geq 0})$

Then, the solution exists on every finite time horizon.

Corollary (A maximum principle)

In the case of the previous item 3, the solution satisfied a maximum-principle, i.e.

$$\inf_{x\in\mathbb{R}} q_0(x) \leq q(t,x) \leq \|q_0\|_{L^\infty(\mathbb{R})} \quad \forall (t,x) \in (0,T) \times \mathbb{R} \text{ a.e.}$$

Theorem (Stability)

L^1 *stability in the initial datum holds if the initial datum is total variation bounded.*

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The final problem or the singular limit

Nonlocal conservation law on $\mathbb{R}$

Recall for $\eta \in \mathbb{R}_{>0}$ the weak solution q_η of the nonlocal conservation law on $\mathbb{R}$ with **exponential kernel**

$$
\begin{aligned}
q_t(t,x) &= -\partial_x \Big(V\Big(W[q,\gamma_\eta](t,x)\Big)q(t,x)\Big) && (t,x) \in (0,T) \times \mathbb{R} \\
q(0,x) &= q_0(x) && x \in \mathbb{R} \\
W[q,\gamma_\eta](t,x) &:= \tfrac{1}{\eta} \int_x^\infty \mathrm{e}^{\frac{x-y}{\eta}} q(t,y)\,\mathrm{d}y && (t,x) \in (0,T) \times \mathbb{R}.
\end{aligned}
$$

Corresponding local conservation law on $\mathbb{R}$

Consider the local counter-part q as the weak entropy solution of

$$
\begin{aligned}
\partial_t q(t,x) &= -\partial_x\Big(V(q(t,x))q(t,x)\Big) && (t,x) \in (0,T) \times \mathbb{R} \\
q(0,x) &= q_0(x) && x \in \mathbb{R}.
\end{aligned}
$$

The singular limit problem

Do we have in “some sense”

$$q_\eta \overset{\eta \to 0}{\to} q?$$

TV bounds and the nonlocal term

No TV bounds to be expected for the solution

[15] M. Colombo, G. Crippa, E. Marconi, and Laura V. Spinolo Local limit of nonlocal traffic models: Convergence results and total variation blow-up ***Annales de l'Institut Henri Poincaré C, Analyse non linéaire***, (2021).

But maybe for the nonlocal term?

The nonlocal term (thanks to the exponential weight) satisfies

$$\partial_x W[q_\eta](t,x) = \tfrac{1}{\eta} W[q_\eta](t,x) - \tfrac{1}{\eta} q_\eta(t,x) \quad \forall (t,x) \in \Omega_T.$$

Theorem (A nonlocal transport equation for the nonlocal term)

The nonlocal term (call it from now on W_η) satisfies the following Cauchy problem

$$\begin{aligned} \partial_t W_\eta + V(W_\eta)\partial_x W_\eta &= -\tfrac{1}{\eta}\int_x^\infty \exp(\tfrac{x-y}{\eta}) V'(W_\eta(t,y))\partial_y W_\eta(t,y) W_\eta(t,y)\,\mathrm{d}y \quad (t,x) \in \Omega_T \\ W_\eta(0,x) &= \tfrac{1}{\eta}\int_x^\infty \exp(\tfrac{x-y}{\eta}) q_0(y)\,\mathrm{d}y. \end{aligned}$$

TV bound on nonlocal term

Theorem (Uniform TV bound)

It holds that

$$|W_\eta(t,\cdot)|_{TV(\mathbb{R})} \leq |W_\eta(0,\cdot)|_{TV(\mathbb{R})} \leq |q_0|_{TV(\mathbb{R})}\ \forall \eta \in \mathbb{R}_{>0}\ \forall t \in [0,T],$$

and thus

$$\Big\{W_\eta \in C\big([0,T]; L^1_{\text{loc}}(\mathbb{R})\big) :\ \eta \in \mathbb{R}_{>0}\Big\} \overset{c}{\hookrightarrow} C\big([0,T]; L^1_{\text{loc}}(\mathbb{R})\big).$$

Theorem (Convergence to a limit point)

Modulo subsequences there exists $q^* \in C([0,T]; L^1_{\text{loc}}(\mathbb{R}))$ *so that*

$$\lim_{\eta\to 0} \|q_\eta - q^*\|_{C([0,T];L^1_{loc}(\mathbb{R}))} = 0\ \wedge\ \lim_{\eta\to 0} \|W_\eta - q^*\|_{C([0,T];L^1_{loc}(\mathbb{R}))} = 0,$$

when q^* *is a weak solution of the local conservation law, i.e. it satisfies*
$\forall \phi \in C^1_c((-42,T)\times\mathbb{R})$

$$\iint_{\Omega_T} \partial_t\phi(t,x)q^*(t,x) + \partial_x\phi(t,x)V(q^*(t,x))q^*(t,x)\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}} \phi(0,x)q_0(x)\,\mathrm{d}x = 0.$$

Sketch of proof

Recall the identity for $(t,x) \in \Omega_T$

$$\partial_x W_\eta(t,x) = \tfrac{1}{\eta} W_\eta(t,x) - \tfrac{1}{\eta} q_\eta(t,x)$$

implies

$$\eta |W_\eta(t,\cdot)|_{TV(\mathbb{R})} = \int_{\mathbb{R}} |W_\eta(t,x) - q_\eta(t,x)| \,\mathrm{d}x$$

and $\eta \to 0$ gives the claim.
Passing to the limit in the weak solution is possible due to the strong L^1 convergence.

Convergence result

Theorem (Convergence nonlocal – local)

Given that $x \mapsto xV(x)$ is strictly convex or concave and $V' \leqq 0$, we have

$$\lim_{\eta \to 0} \|q_\eta - q\|_{C([0,T];L^1_{\text{loc}}(\mathbb{R}))} = 0,$$

where q is the local Entropy solution. Additionally, the nonlocal term W_η converges to q.

Literature

[16] A. Bressan and W. Shen Entropy admissibility of the limit solution for a nonlocal model of traffic flow ***Comm. Math. Sci.*** , (2021).

[17] C. De Lellis, F. Otto, and M. Westdickenberg Minimal Entropy conditions for Burgers equation ***Quaterly of Applied Mathematics***, (2004).

[18] A. Keimer and L. Pflug On approximation of local conservation laws by nonlocal conservation laws ***Journal of Mathematical Analysis and Applications (JMAA)***, (2019).

[19] G. M. Coclite, J. M. Coron, N. De Nitti, A. Keimer, and L. Pflug A general result on the approximation of local conservation laws by nonlocal conservation laws: The singular limit problem for exponential kernels ***Annales de l'Institut Henri Poincaré C, Analyse Non Linéaire***, (2022).

Illustration (I)

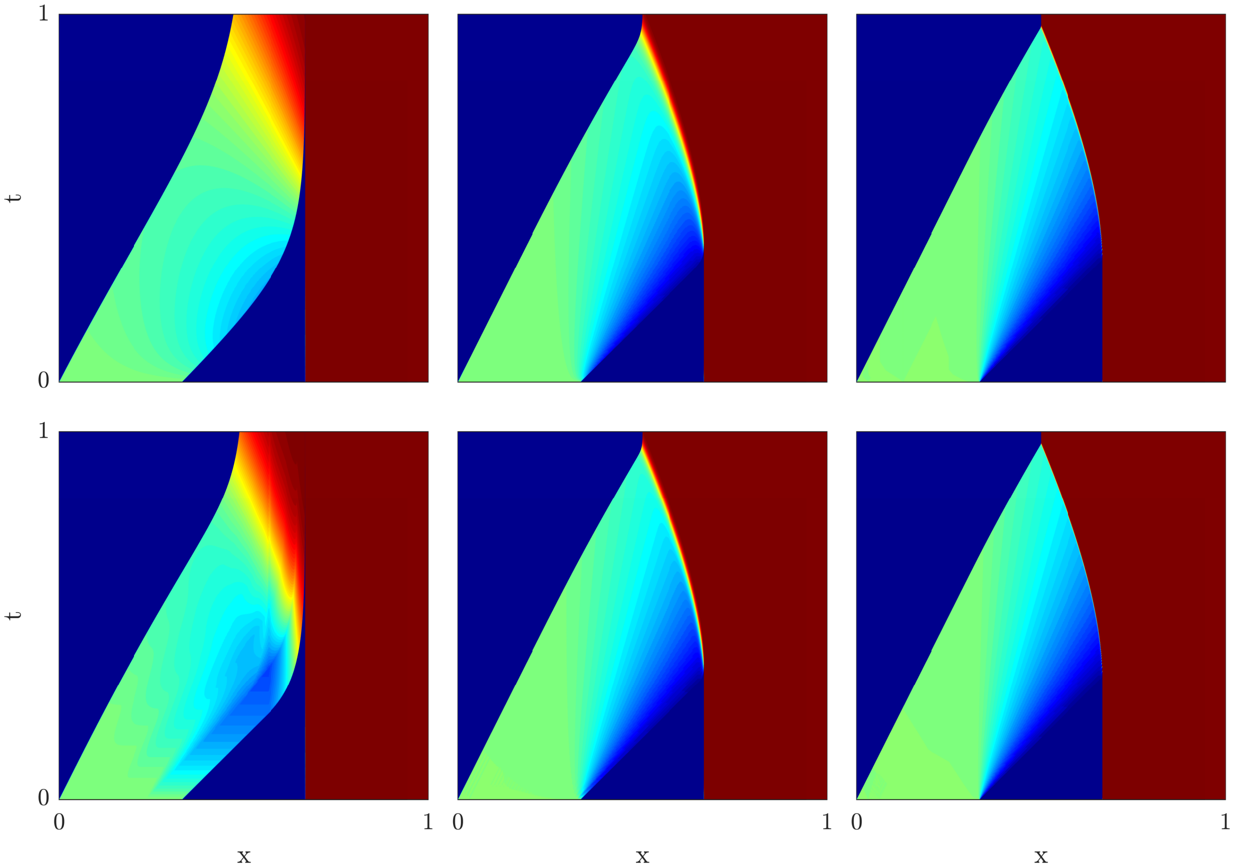

Figure: Solution of the nonlocal balance law with exponential kernel (**top**) and constant kernel (**bottom**). From left to right η is decreasing, $\eta \in \{10^{-1},\ 10^{-2}, 10^{-3}\}$. **Colorbar:** 0  1

Illustration (II)

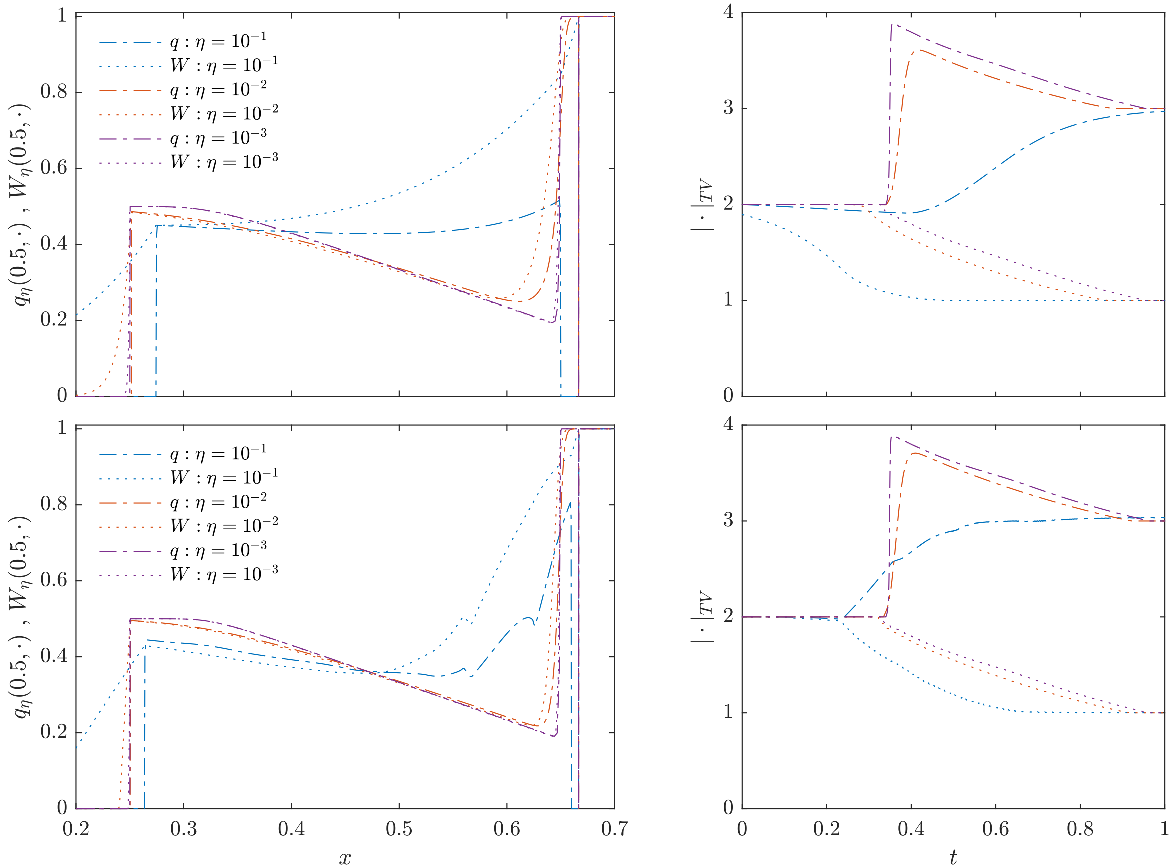

Figure: **Left:** Solution of the nonlocal balance law with exponential kernel (**top**), and constant kernel (**bottom**) its corresponding nonlocal term plotted for $t = 0.5$ and $\eta \in \{10^{-1}, 10^{-2}, 10^{-3}\}$. **Right:** Evolution of the corresponding total variations showing a monotone decreasing nature in terms of the nonlocal term (dotted lines) which is also the case for the local counterpart.

Generalizations (I)

Theorem (General convergence in the singular limit case)

The singular limit convergence holds in the case that $V' \leqq 0$ (no concavity of the flux required) and for kernels which are convex.

Literature

[20] M. Colombo, G. Crippa, E. Marconi, and L. V. Spinolo, Nonlocal Traffic Models with General Kernels: Singular Limit, Entropy Admissibility, and Convergence Rate. ***Archive for Rational Mechanics and Analysis*** , (2023).

Generalizations (II) – kernels with fixed support

Problem setup – kernels with fixed support

Consider again (under suitable assumptions)

$$\begin{aligned} q_t(t,x) &= -\partial_x\Big(V\big(W[q,\gamma_\eta](t,x)\big)q(t,x)\Big) && (t,x)\in(0,T)\times\mathbb{R}\\ q(0,x) &= q_0(x) && x\in\mathbb{R}\\ W[q,\gamma_\eta](t,x) &:= \int_x^\infty \gamma_\eta(y-x)q(t,y)\,\mathrm{d}y && (t,x)\in(0,T)\times\mathbb{R}. \end{aligned}$$

with

$$\gamma_\eta(x) = c_\gamma(\eta)\gamma(x)^{\frac{1}{\eta}}, \qquad c_\gamma(\eta) := \Big(\int_0^\infty \gamma(y)^{\frac{1}{\eta}}\,\mathrm{d}y\Big)^{-1}.$$

Assumptions on γ

We assume that $\exists\delta\in\mathbb{R}_{>0}$ s.t:

1) Integrability and total variation bound: $\gamma\in BV(\mathbb{R}_{>0};\mathbb{R}_{\geqq 0})$
2) Bounded second derivative in arbitrary small neighborhood: $\gamma|_{(0,\delta)}\in W^{2,\infty}((0,\delta))$
3) Negative derivative in zero: $\gamma'(0)<0$
4) Upper bound on $\mathbb{R}_{>\delta}$: $\gamma(x)\geq\gamma(y)\ \forall(x,y)\in(0,\delta)\times\mathbb{R}_{>\delta}$

Interpretation of the kernel and a weakened maximum principle

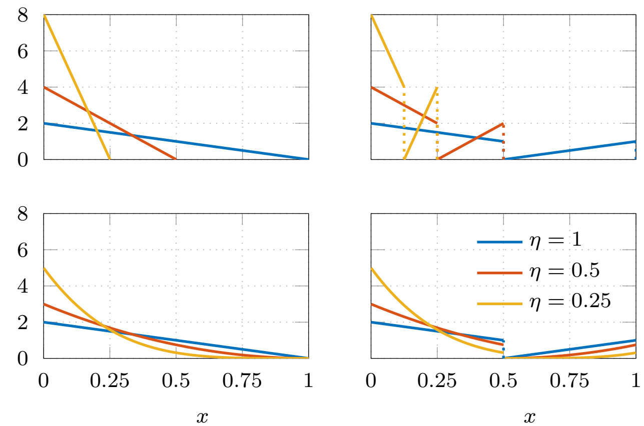

Figure: **Left:** In the **top** row, the spatial scaling, in the **bottom** row, the power scaling is visualized for two different values of γ, viz. a linear kernel ($\gamma \equiv 2(1-\cdot)$, **left**) and a non-monotone, piecewise-linear kernel ($\gamma \equiv 2(1-\cdot)\chi_{(0,0.5)}(\cdot) + (2\cdot -1)\chi_{(0.5,1)}(\cdot)$, **right**). In all cases, the kernels γ_η for $\eta \in \{1, 0.5, 0.25\}$ are shown.

Theorem (Weakened maximum principle)

For $\eta \in \mathbb{R}_{>0}$, there exists a unique solution q_η until a given time $T \in \mathbb{R}_{>0}$. Even more, for any given $\kappa \in \mathbb{R}_{>0}$ and a time horizon $T \in \mathbb{R}_{>0}$, there exists $\eta_{\kappa,T} \in \mathbb{R}_{>0}$ s.t.

$$\forall (t,\eta) \in (0,T) \times \big(0, \eta_{\kappa,T}\big) \ : \quad \|q_\eta(t,\cdot)\|_{L^\infty(\mathbb{R})} \leq (1+\kappa)\|q_0\|_{L^\infty(\mathbb{R})}.$$

Singular limit convergence for kernels with fixed support

Theorem (Convergence to the local Entropy solution)

The solution $q_\eta \in C\big([0,T]; L^1_{\text{loc}}(\mathbb{R})\big)$ *converges to the local entropy solution* q^* *for* $\eta \to 0$ *and so does the nonlocal term*

$$\lim_{\eta\to 0} \left\| q_\eta - q^* \right\|_X = 0 \ \wedge\ \lim_{\eta \to 0} \left\| W[q_\eta, \gamma_\eta] - q^* \right\|_X = 0$$

with $X \coloneqq C\big([0,T]; L^1_{\text{loc}}(\mathbb{R})\big)$.

Idea of the proof

Use again the exponential kernel $E_\eta(t,x) = \frac{1}{\eta} \int_x^\infty \exp\left(\frac{x-y}{\eta}\right) q(t,y)\,\mathrm{d}y,\ (t,x) \in \Omega_T$ as well as lower and upper estimations for the kernel γ.

Literature

[21] A. Keimer and L. Pflug, On the singular limit problem for nonlocal conservation laws: A general approximation result for kernels with fixed support *arXiv*, (2023).

Consequences

Consequences

- Justifies nonlocal conservation laws also by the corresponding local equations.
- Enables another approach on how to tackle local problems: Define solutions of local conservation laws as limit of nonlocal ones.
- Hyperbolic nature of the dynamics is conserved (not true for viscosity approximations), only infinite speed of information.
- Many results – see also open problems – can be semi-explicitly stated in the nonlocal setup (solution, control, optimal control, etc....).
- Limits for related problems of control and optimal control, of <u>local</u> conservation laws

1. Nonlocal conservation laws in applications and theory

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Oleinik's Entropy condition for the nonlocal term

Theorem (Oleĭnik-type inequality for W_η)

Consider again the ***exponential kernel*** *for the nonlocal conservation law. Let $\delta > 0$, $0 < \kappa_1 < \kappa_2$, and $q_0 \in L^\infty(\mathbb{R}; \mathbb{R}_{\geq 0})$ and let $V \in W^{2,\infty}_{\text{loc}}(\mathbb{R})$ be a non-increasing velocity function such that at least one of the following conditions is satisfied:*

$$V'(\xi) = -\delta < 0, \qquad \forall \xi \in [\operatorname{ess\,inf} q_0, \operatorname{ess\,sup} q_0]; \tag{1}$$

$$0 \leq V'(\xi) + V''(\xi)\xi \leq \kappa_1, \quad V'(\xi) \leq -\kappa_2, \qquad \forall \xi \in [\operatorname{ess\,inf} q_0, \operatorname{ess\,sup} q_0]. \tag{2}$$

Then the nonlocal term W_η satisfies the following inequality:

$$\frac{W_\eta(t,x) - W_\eta(t,y)}{x-y} \geq -\frac{1}{\kappa t}, \qquad \text{for all } t > 0 \text{ and } x, y \in \mathbb{R} \text{ with } x \neq y, \tag{3}$$

with $\kappa := \delta$ (in case of assumption (1)) or $\kappa := \kappa_2 - \kappa_1$ (in case of assumption (2)) and W_η converges to the local entropy solution for $\eta \to 0$.

Remark (Further estimates)

When looking at $V'(W_\eta)W_\eta$, one can get further Oleinik-type estimates.

Literature

[22] G. M. Coclite, M. Colombo, G. Crippa, N. De Nitti, A. Keimer, E. Marconi, L. Pflug, L. V. Spinolo, Oleinik-type estimates for nonlocal conservation laws and applications to the nonlocal-to-local limit ***JHDE***, (2024).

The singular limit for weakly coupled systems of nonlocal balance laws (I)

The weakly coupled nonlocal system of balance laws – traffic flow modelling and lane-changing

Consider the "weakly" coupled (via r.h.s.) system as Cauchy problem on $\mathbb{R}$ with exponential kernel

$$\begin{aligned}\partial_t \boldsymbol{q}_\eta^1 + \partial_x\Big(V_1(\mathcal{W}_\eta[\boldsymbol{q}_\eta^1])\boldsymbol{q}_\eta^1\Big) &= S\big(\boldsymbol{q}_\eta, \mathcal{W}_\eta[\boldsymbol{q}_\eta], x\big),\\ \partial_t \boldsymbol{q}_\eta^2 + \partial_x\Big(V_2(\mathcal{W}_\eta[\boldsymbol{q}_\eta^2])\boldsymbol{q}_\eta^2\Big) &= -S\big(\boldsymbol{q}_\eta, \mathcal{W}_\eta[\boldsymbol{q}_\eta], x\big),\\ \boldsymbol{q}_\eta(0,\cdot) &\equiv \boldsymbol{q}_0,\end{aligned}$$

for $i \in \{1,2\}$ $\quad \mathcal{W}_\eta[\boldsymbol{q}_\eta^i](t,x) = \frac{1}{\eta}\int_x^\infty \exp\left(\frac{x-y}{\eta}\right)\boldsymbol{q}_\eta^i(t,y)\,\mathrm{d}y, \qquad (t,x)\in(0,T)\times\mathbb{R}.$

Problem setup

Do we converge for $\eta \to 0$ towards the entropy solution of the system of balance laws

$$\begin{aligned}\partial_t \boldsymbol{q}^1 + \partial_x\Big(V_1(\boldsymbol{q}^1)\boldsymbol{q}^1\Big) &= S\big(\boldsymbol{q}, \boldsymbol{q}, x\big),\\ \partial_t \boldsymbol{q}^2 + \partial_x\Big(V_2(\boldsymbol{q}^2)\boldsymbol{q}^2\Big) &= -S\big(\boldsymbol{q}, \boldsymbol{q}, x\big),\\ \boldsymbol{q}(0,\cdot) &\equiv \boldsymbol{q}_0?\end{aligned}$$

The singular limit for weakly coupled systems of nonlocal balance laws (II)

Theorem (Exponential (in time) but uniform in η TV bounds)

Assume that $\boldsymbol{q}_0 \in L^\infty(\mathbb{R};\mathbb{R}^2_{\geq 0}) \cap TV(\mathbb{R};\mathbb{R}^2),\ V_i \in W^{1,\infty}_{\text{loc}}(\mathbb{R}) : V_i' \leqq 0,\ i \in \{1,2\}$ *and that the source term has the structure*

$$S\big(\boldsymbol{q}_\eta, \mathcal{W}_\eta[\boldsymbol{q}_\eta], x\big) = \Big(\frac{\boldsymbol{q}_\eta^2}{\boldsymbol{q}_{\max}^2} - \frac{\boldsymbol{q}_\eta^1}{\boldsymbol{q}_{\max}^1}\Big) H(\mathcal{W}_\eta[\boldsymbol{q}_\eta], x), \qquad x \in \mathbb{R}$$

with $H : \mathbb{R}^3 \to \mathbb{R}_{\geq 0}$ *smooth enough. Then, we obtain uniformly in* η

$$\exists C \in \mathbb{R}_{>0}\ \forall \eta \in \mathbb{R}_{>0} :\ |\mathcal{W}_\eta[\boldsymbol{q}_\eta](t,\cdot)|_{TV(\mathbb{R};\mathbb{R}^2)} \leq \exp(Ct)\ \forall t \in [0,T].$$

Theorem (Convergence towards the entropy solution)

For $\eta \to 0$, *the nonlocal term* $W_\eta[\boldsymbol{q}_\eta]$ *as well as* $\boldsymbol{q}_\eta$ *converge in* $C[0,T];L^1_{loc}(\mathbb{R};\mathbb{R}^2))$ *to the entropy solution of the corresponding system of weakly coupled local balance laws.*

Remark (Generalization)

Weakly coupled systems $(N > 2)$, *source term coupling, kernels.*

Literature

[23] F. Chiarello and A. Keimer, On the singular limit problem in nonlocal balance laws: Applications to nonlocal lane-changing traffic flow models ***JMAA***, (2024).

The p-norm singular limit

Definition (Problem setup)

Let $p \in \mathbb{R}_{>0}$, we call the following Cauchy problem

$$q_t(t,x) + \partial_x\Big(V(W_p[q,\gamma](t,x))q(t,x)\Big) = 0 \qquad (t,x) \in \Omega_T$$
$$q(0,x) = q_0(x) \qquad x \in \mathbb{R}$$

with the nonlocal term in p

$$W_p[q,\gamma](t,x) = \left(\frac{1}{\eta}\int_x^\infty \left(\gamma\left(\tfrac{x-y}{\eta}\right)q(t,y)\right)^p \mathrm{d}y\right)^{\frac{1}{p}} \qquad (t,x) \in \Omega_T$$

the nonlocal $p-$norm problem.

Remark (Challenges)

- *$p-$norm not "differentiable" at 0. Banach's fixed-point approach does not work* ☹.
- *In the case of well-posedness, do we obtain the singular limit convergence towards the local entropy solution?*

Well-posedness (p-norm)

Assumptions

- $q_0 \in L^\infty(\mathbb{R};\mathbb{R}_{>q_{\min}}),\ q_{\min} \in \mathbb{R}_{>0}$
- $q_0 \in TV(\mathbb{R})$
- $V \in W^{2,\infty}_{\text{loc}}(\mathbb{R};\mathbb{R}),\ V' \leqq 0$ on $\mathbb{R}$
- $\gamma \in W^{2,1}$ on its support
- $\|\gamma\|_{L^p(\mathbb{R}_{<0})} = 1,\ \gamma' \geq 0$
- $\eta \in \mathbb{R}_{>0}$.

Theorem (Existence, uniqueness and maximum principle)

For each $T \in \mathbb{R}_{>0}$, there is a unique weak solution

$$q_\eta \in C([0,T];L^1_{\text{loc}}(\mathbb{R})) \cap L^\infty((0,T);L^\infty(\mathbb{R}) \cap TV(\mathbb{R}))$$

and the maximum principle is satisfied, i.e.

$$\forall (t,x) \in (0,T)\times\mathbb{R} \text{ a.e.:}\quad q_{\min} \leq \operatorname*{ess\,inf}_{\tilde{x}\in\mathbb{R}} q_0(\tilde{x}) \leq q(t,x) \leq \|q_0\|_{L^\infty(\mathbb{R})}.$$

Convergence to the local Entropy solution (p-norm)

Theorem (Convergence to the Entropy solution)

Given the previously stated assumptions and use again the exponential kernel. Then, the solution and the nonlocal term converge towards the local entropy solutions for $\eta \to 0$.

Remark (Generalizations and related questions)

What about

- q_0 *being not bounded away from zero? Compactness in space holds for the nonlocal operator.*
- *Do we converge (in some sense) for* $p \to \infty$ *to the solution of*

$$q_t + \partial_x\Big(V\Big(\|\tfrac{1}{\eta}\mathrm{e}^{\frac{x-\cdot}{\eta}} q(t,\cdot)\|_{L^\infty(x,\infty)}\Big)q(t,x)\Big) = 0$$

  and what can be said about solutions to this conservation law?
- *Better singular limit convergence for certain* $p \neq 1$*?*

Work in progress

[24] D. Amadori, F. Chiarello, A. Keimer and L. Pflug, Nonlocal conservation laws with p-norm, the singular limit problem and applications to traffic flow, (2024?).

1. Nonlocal conservation laws in applications and theory

2. The singular limit problem

3. More on the singular limit problem

4. **Open problems – future research**

Open Problems

Open problems

- General kernels for the singular limit problem, particularly symmetric and **constant** kernels
- Systems of nonlocal conservation laws and the singular limit problem

  Example: The nonlocal GARZ (Generalized Aw-Rascle-Zhang) model

$$\begin{aligned} \partial_t \rho + \partial_x\Big((\gamma_\eta * V(\rho,\omega))\rho\Big) &= 0 \\ \partial_t \omega + \Big(\gamma_\eta * V(\rho,\omega)\Big)\partial_x \omega &= 0 \end{aligned}$$

- Optimal control of nonlocal conservation laws (control to state mapping differentiable)
- The singular limit problem for optimal control
- Networks and how to handle the nonlocality close to the junctions

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The field of nonlocal conservation laws and more general nonlocal PDE models is barely studied, in many cases nonlocal modelling is more reasonable and there is (still) a lot to work on!

The End?

Thank you very much!

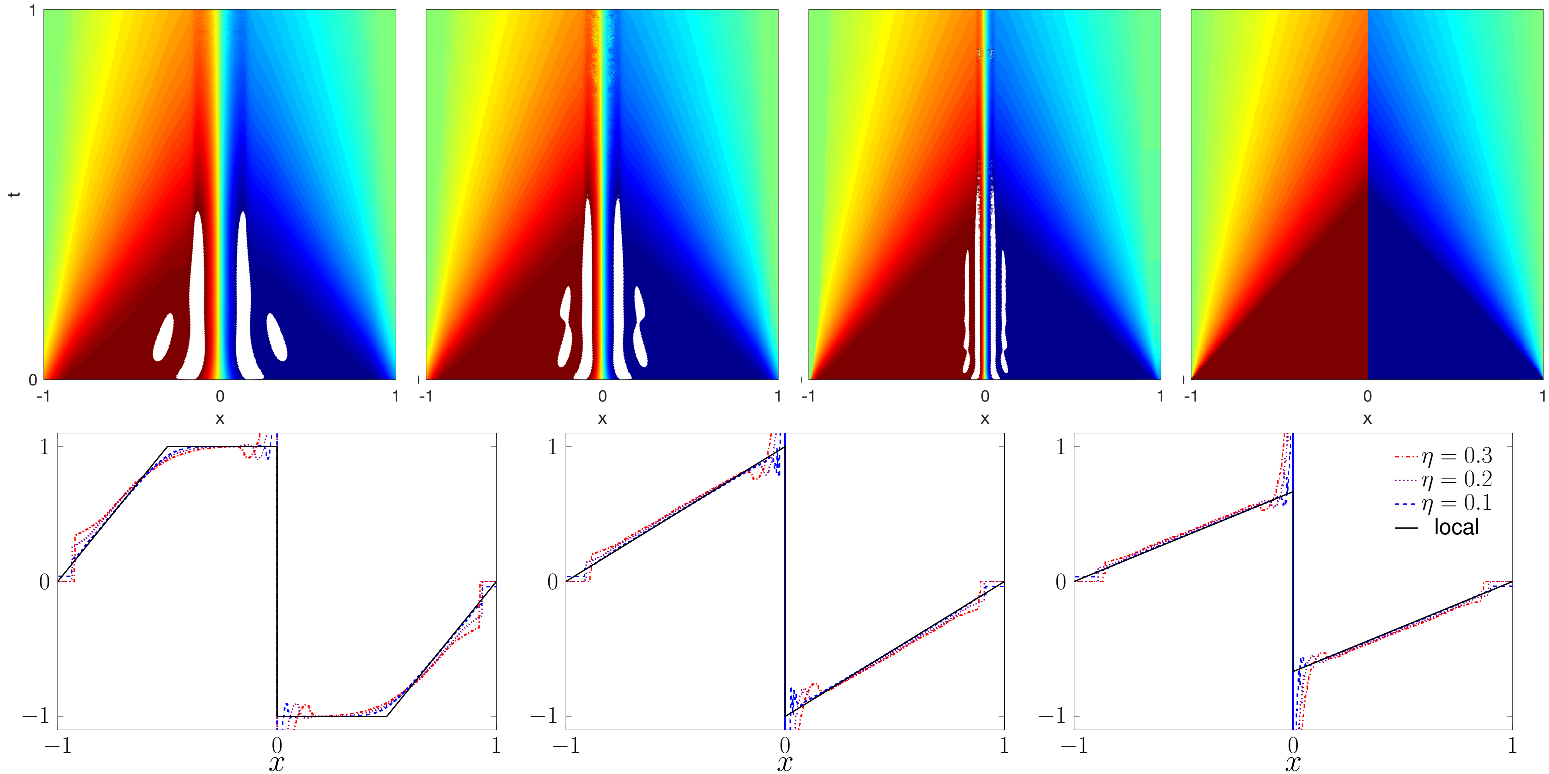

Figure: **Top**: Nonlocal term W for Burgers' equation and $\eta \in \{0.3, 0.2, 0.1\}$ with symmetric kernel and sign changing initial datum **Bottom**: Nonlocal term W at time $t \in \{0.25, 0.5, 0.75\}$ from left to right. **Colorbar:** -1 [colorbar] $+1$