One-side boundary controllability of the 1-D compressible Euler equation

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Partial differential equations, optimal design and numerics X
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I. Introduction

One-dimensional isentropic Euler equations :

- Compressible Euler equation (in standard Eulerian coordinates) :

$$\begin{cases} \partial_t \rho + \partial_x(\rho v) = 0, \\ \partial_t(\rho v) + \partial_x \left(\rho v^2 + \kappa \rho^\gamma\right) = 0. \end{cases}$$

- The p-system (compressible Euler equation in Lagrangian coordinates) :

$$\begin{cases} \partial_t \tau - \partial_x v = 0, \\ \partial_t v + \partial_x(\kappa \tau^{-\gamma}) = 0. \end{cases} \tag{P}$$

where

- $\rho = \rho(t, x) \geq 0$ is the density of the fluid,
- $v = v(t, x)$ is the velocity of the fluid, so that $m := \rho v$ is the local momentum,
- $\tau := 1/\rho$ is the specific volume,
- the pressure law is $p(\rho) = \kappa \rho^\gamma$, $\gamma \in (1, 3]$.

Controllability problem

- Domain : $(t, x) \in [0, T] \times [0, L]$.
- State of the system : $u = (\tau, v)$.
- Control : the "boundary data" : here, on one side, say $x = 0$, while there is a fixed boundary law at $x = L$.
- Controllability problem : given u_0 and u_1, can we find boundary data $x = 0$ driving the state from u_0 to u_1 ?
- Equivalently : given u_0 and u_1, can we find a solution of the system satisfying the boundary condition and driving u_0 to u_1 ?

Systems of conservation laws

- Both systems enter the class of hyperbolic systems of conservation laws :

$$U_t + f(U)_x = 0, \quad f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n, \qquad \text{(SCL)}$$

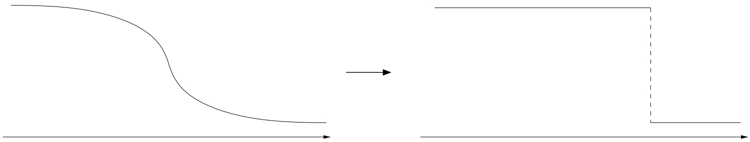
  satisfying the (strict) hyperbolicity condition that at each point

$$df \text{ has } n \text{ distinct real eigenvalues } \lambda_1 < \cdots < \lambda_n.$$

- Hyperbolic systems of conservation laws develop singularities in finite time.

- This easy to see for instance for the Burgers equation :

$$u_t + (u^2)_x = 0.$$



Class of solutions

- One can either work with regular solutions (C^1) with small C^1-norm (for small time), or with discontinuous (weak) solutions.
- For the latter case, is natural for the sake of uniqueness to consider weak solutions which satisfy entropy conditions (entropy solutions).
- This is not a mere regularity issue : in the C^1 case, the system is reversible, but it is irreversible in the context of entropy solutions.
- More precisely, the solutions will be of bounded variation, with small total variation in x ("à la Glimm") :

$$TV(u) := \sup_N \sup_{x_1 < \cdots < x_N} \sum_{k=0}^{N-1} |u(x_{k+1}) - u(x_k)| \ll 1.$$

- Note that there exist weaker solutions (Glimm-Lax, DiPerna, Lions-Perthame-Souganidis-Tadmor, etc.)

Entropy conditions

Definition

An entropy/entropy flux couple for a hyperbolic system of conservation laws (SCL) is defined as a couple of regular functions $(\eta, q) : \Omega \to \mathbb{R}$ satisfying :

$$\forall U \in \Omega, \quad D\eta(U) \cdot Df(U) = Dq(U).$$

Definition

A function $U \in L^\infty(0, T; BV(0, L)) \cap \mathcal{L}ip(0, T; L^1(0, L))$ is called an entropy solution of (SCL) when, for any entropy/entropy flux couple (η, q), with η convex, one has in the sense of measures

$$\eta(U)_t + q(U)_x \leq 0,$$

that is, for all $\varphi \in \mathcal{D}((0, T) \times (0, L))$ with $\varphi \geq 0$,

$$\int_{(0,T)\times(0,L)} \big(\eta(U(t,x))\varphi_t(t,x) + q(U(t,x))\varphi_x(t,x)\big)\, dx\, dt \geq 0.$$

Boundary condition

- Our boundary condition will take the following form at $x = L$:

$$b(u(t, L)) = 0 \text{ for a.e. } t,$$

  where $b = b(\rho, v) : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ is a function satisfying some non-degeneracy conditions (to be specified later).

- Examples :
  - $v = 0$: zero-speed on the right boundary,
  - $\rho = \overline{\rho}$: constant density (or constant pressure) at $x = L$.

Main result

Theorem

Let b satisfy the non-degeneracy condition.

Let $\overline{u}_0 := (\overline{\tau}_0, \overline{v}_0) \in \mathbb{R}^2$ with $\overline{\tau}_0 > 0$ and $b(\overline{u}_0) = 0$ and let $\overline{u}_1 = (\overline{\tau}_1, \overline{v}_1)$ with $\overline{\tau}_1 > 0$ and $b(\overline{u}_1) = 0$.

There exist $\varepsilon > 0$ and $T > 0$ such that for any $u_0 = (\tau_0, v_0)$ in $BV(0, L; \mathbb{R}^2)$ such that

$$\|u_0 - \overline{u}_0\|_{L^\infty(0,L)} + TV(u_0) \leq \varepsilon,$$

and $b(u_0(L^-)) = 0$, there is

$$u \in L^\infty(0, T; BV(0, L)) \cap \mathcal{L}ip([0, T]; L^1(0, L)),$$

a weak entropy solution of the p-system such that

$$u_{|t=0} = u_0 \quad \text{and} \quad u_{|t=T} = \overline{u}_1.$$

Refined variant

Theorem

Let b satisfy the non-degeneracy condition.

Let $\overline{u}_0 := (\overline{\tau}_0, \overline{v}_0) \in \mathbb{R}^2$ with $\overline{\tau}_0 > 0$ and $b(\overline{u}_0) = 0$ and let $\overline{u}_1 = (\overline{\tau}_1, \overline{v}_1)$ with $\overline{\tau}_1 > 0$ and $b(\overline{u}_1) = 0$.

Let $\eta > 0$. There exist $\varepsilon > 0$ and $T > 0$ such that for any $u_0 = (\tau_0, v_0)$ in $BV(0, L; \mathbb{R}^2)$ such that

$$\|u_0 - \overline{u}_0\|_{L^\infty(0,L)} + TV(u_0) \leq \varepsilon,$$

and $b(u_0(L^-)) = 0$, there is

$$u \in L^\infty(0, T; BV(0, L)) \cap \mathcal{L}ip([0, T]; L^1(0, L)),$$

a weak entropy solution of the p-system such that

$$u_{|t=0} = u_0 \quad \text{and} \quad u_{|t=T} = \overline{u}_1,$$

and

$$TV(u(t, \cdot)) \leq \eta, \quad \forall t \in (0, T).$$

II. Previous results : control problems in the context of entropy solutions

- There are very general results in the C^1 case : Li-Rao (2002), ...
- Several works on the scalar case :
  Ancona and Marson (1998), Horsin (1998), Perrollaz (2011), Adimurthi-Gowda-Goshal (2013), Andreianov-Donadello-Marson (2015), Adimurthi-Goshal-Marcati (2016), ...
- Several works on the system case :
  - Bressan-Coclite (asymptotic result and a counterexample, 2002),
  - Ancona-Coclite (Temple systems, 2002),
  - Ancona-Marson (one-side open loop stabilization, 2007),
  - G. (Isentropic and non-isentropic Euler, two-sided control 2007, 2014),
  - Andreianov-Donadello-Ghoshal-Razafison (2015, triangular system),
  - T. Li-L. Yu (2015, partially LD systems),
  - Coron-Ervedoza-G.-Goshal-Perrollaz (Feedback stabilization, 2015),
  - see Nicola De Nitti's talk last week !
  - ...

Two connected results

- Bressan and Coclite (2002) : for a class of systems containing Di Perna's system :

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t u + \partial_x \left(\frac{u^2}{2} + \frac{K^2}{\gamma - 1} \rho^{\gamma - 1} \right) = 0, \end{cases}$$

  there are initial conditions $\varphi \in BV([0,1])$ of arbitrary small total variation such that any entropy solution u remaining of small total variation satisfies : for any t, $u(t, \cdot)$ is not constant. $\neq C^1$ case !

- G. (2007) : A sufficient condition concerning the isentropic Euler equation

$$(E) : \begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2 + \kappa \rho^\gamma) = 0, \end{cases} \qquad (P) : \begin{cases} \partial_t \tau - \partial_x v = 0, \\ \partial_t v + \partial_x(\kappa \tau^{-\gamma}) = 0, \end{cases}$$

  for final states to be reachable by acting on both sides. For instance, all constant states are reachable.

III. Basic facts on systems of conservation laws

- Systems of conservations laws :

$$u_t + f(u)_x = 0, \quad f : \mathbb{R}^n \to \mathbb{R}^n,$$

$A(u) := df(u)$ has n real distinct eigenvalues $\lambda_1 < \cdots < \lambda_n$, which are characteristic speeds of the system with corresponding eigenvectors $r_i(u)$.

- Genuinely non-linear fields in the sense of Lax :

$$\nabla\lambda_i.r_i \neq 0 \quad \text{for all } u.$$

$\Rightarrow$ we normalize $\nabla\lambda_i \cdot r_i = 1$.

- In the case of (P) we have

$$\lambda_1 = -\sqrt{\kappa\gamma\tau^{-\gamma-1}} \text{ and } \lambda_2 = \sqrt{\kappa\gamma\tau^{-\gamma-1}}.$$

Boundary conditions

- We can now express our non-degeneracy condition on the boundary law $b : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$.

  We ask that b satisfies the two following conditions :

  - Standard condition for the Cauchy problem :

    $$r_1 \cdot \nabla b \neq 0 \text{ on } \Omega,$$

  - Condition for the backward in time Cauchy problem :

    $$r_2 \cdot \nabla b \neq 0 \text{ on } \Omega,$$

  - Example : $b(u) = v$ (control by the velocity)

The Riemann problem

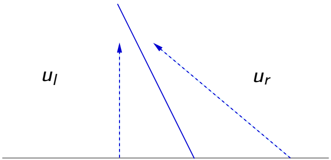
- Find autosimilar solutions $u = \overline{u}(x/t)$ to

$$\begin{cases} u_t + (f(u))_x = 0 \\ u_{|\mathbb{R}^-} = u_l \text{ and } u_{|\mathbb{R}^+} = u_r. \end{cases}$$

- Solved by introducing Lax's curves which consist of points that can be joined starting from u_l either by a shock or a rarefaction wave.

Shocks and rarefaction waves

Shocks



Discontinuities satisfying :

- Rankine-Hugoniot (jump) relations

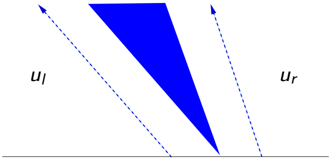
$$\big[\, f(u) \,\big] = s \big[\, u \,\big],$$

- Lax's inequalities :

$$\lambda_i(u_r) < s < \lambda_i(u_l)$$

Propagates at speed $s \sim \fint_{u_l}^{u_r} \lambda_i$

Rarefaction waves



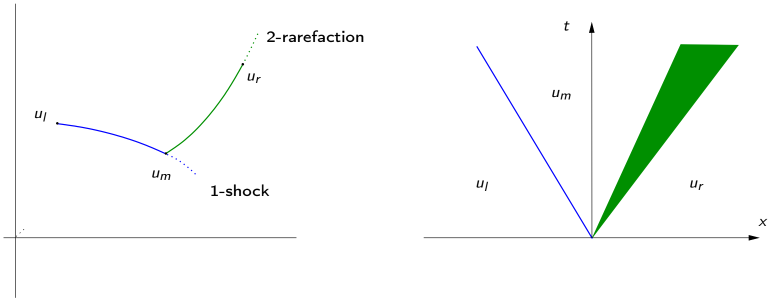
Regular solutions,
obtained with integral curves of r_i :

$$\begin{cases} \dfrac{d}{d\sigma} R_i(\sigma) = r_i(R_i(\sigma)), \\ R_i(0) = u_l, \end{cases}$$

with $\sigma \geq 0$.

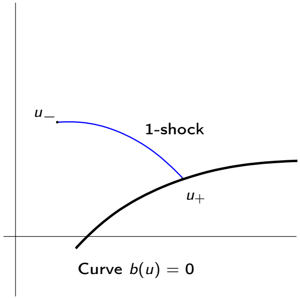
Propagates at speed $\lambda_i(R_i(\sigma))$

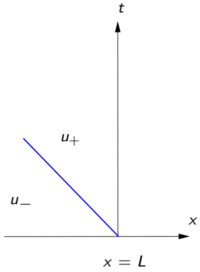
Solving the Riemann problem



- Lax's Theorem proves that one can solve (at least locally) the Riemann problem by first following the 1-curve (gathering states connected to u_l by a 1-rarefaction/1-shock), then the 2-curve.

Boundary Riemann problem

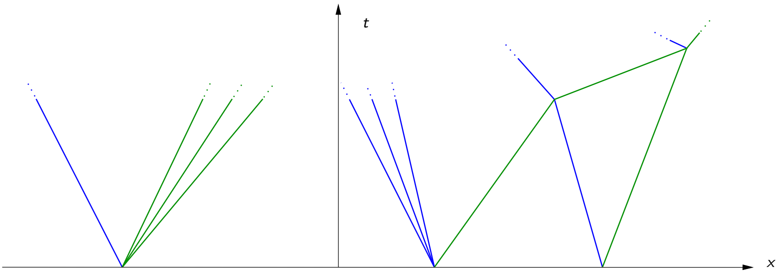




- The same principle applies on the boundary (both forward and backward in time)

Front-tracking algorithm (Dafermos, Di Perna, Bressan, Risebro, . . .)

- Approximate initial condition by piecewise constant functions
- Solve the Riemann problems and replace rarefaction waves by rarefaction fans



- One obtain a piecewise constant function, with straight discontinuities (fronts)
- iterate the process at each interaction point (points where fronts meet)

Estimates, convergence, etc.

- One shows than this defines a piecewise constant function, with a finite number of fronts and discrete interaction points.
- A central argument is due to Glimm : analyzing interactions of fronts $\alpha + \beta \to \alpha' + \beta' + \gamma'$ and the evolution of the strength of waves across an interaction, one proves that if $TV(u_0)$ is small enough :

  $$\Upsilon(t) := \sum_{\alpha \text{ waves}} |\sigma_\alpha| + C \sum_{\alpha,\beta \text{ approaching waves}} |\sigma_\alpha||\sigma_\beta| \text{ is non-increasing,}$$

  (σ_α the size of the front α) and then

  $$TV(u(t)) \leq C\, TV(u_0) \text{ for some } C > 0.$$

- One deduces bounds in $L^\infty_t BV_x$, then in $\mathrm{Lip}_t L^1_x$, so we have compactness (Helly's theorem)...

IV. Some ideas of the construction. Main difficulty.

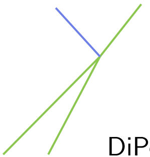
- Bressan & Coclite's counterexample. DiPerna's system is a 2×2 hyperbolic system with GNL fields, and which satisfies

  the interaction of two shocks of the same family generates a shock in this family (normal) and a shock in the other family.

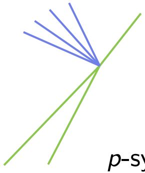
  Starting from an initial date with a dense set of shocks, this propagates over time, even with control on both sides.

- A basic idea (even to control on both sides) is to use the fact that for the p-system :

  the interaction of two shocks of the same family generates a shock in this family (normal) and a rarefaction in the other family.



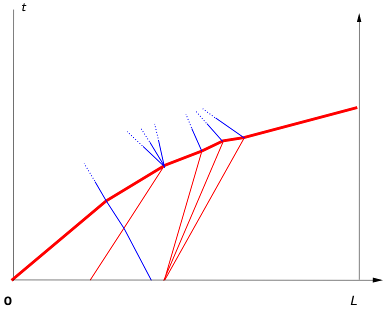
DiPerna's system



p-system

Some ideas, control from both boundaries, 1

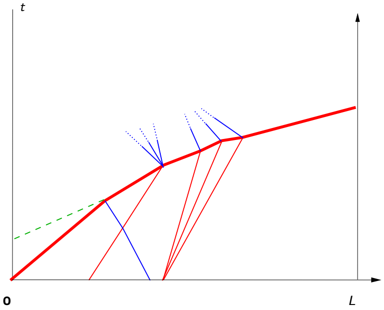
- To begin with, one would like to absorb the waves of a family 2 in the solution by sending a strong (large) shock of this family from the boundary.



- This is connected to Coron's return method.
- Such a strong shock absorbs waves of its own family in a first time, but waves that cross may create interact again above this shock...

Some ideas, control from both boundaries, 2

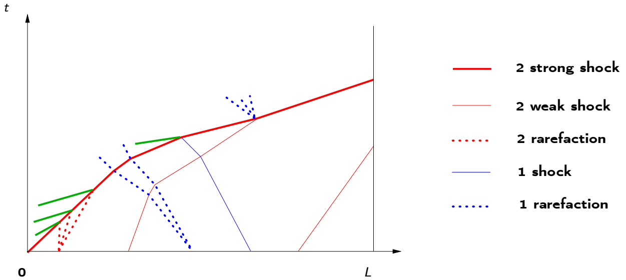
- An idea is then to send additional 2-shocks from the boundary to improve the situation.



- In particular, we want to prevent 1-shocks to cross.
- Indeed, if only 1-rarefactions cross, since they do not interact, the system reaches a constant state.

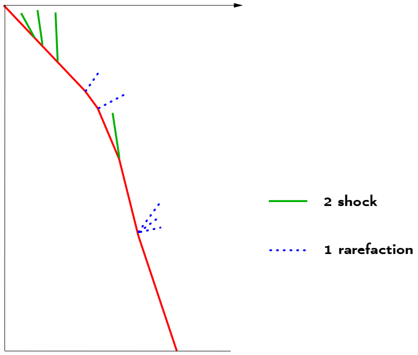
The construction

- First we construct the solution under the 2-strong shock, taking the additional 2-shocks described above in to account :



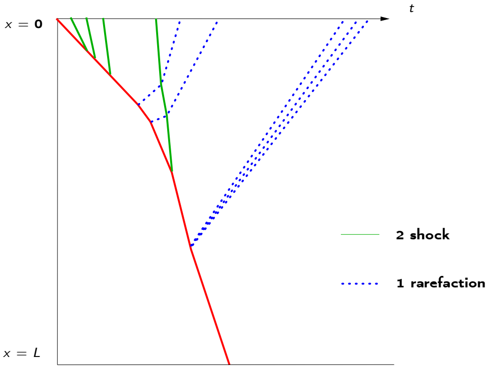
- It remains to construct the approximations beyond the strong 2-shock, that is, we have to extend :
  - the 1-rarefaction waves forward in time
  - the 2-shocks backward in time

- We construct this approximation by using $1 - x$ as the time variable.



- we have to solve the interactions.

- Finally we get an approximation like :



- This solves the controllability problem when one controls on both sides.

One-side controls

- When one controls only from one side (say, from the left), there are two differences :
  - One has to take into account the reflections at $x = L$ below the strong shock. Not an issue.
  - One has to take into account the reflections at $x = L$ of the strong shock. There are two situations, one of which changes everything.
- Situation 1. The strong 2-shock is reflected as a 1-rarefaction when

$$(r_1 \cdot \nabla b)(r_2 \cdot \nabla b) < 0.$$

  In this case, since this adds a rarefaction to the picture, the above construction still works.

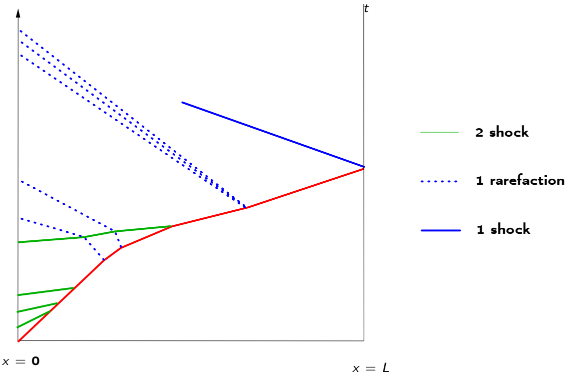
- Situation 2. The strong 2-shock is reflected as a 1-shock when

$$(r_1 \cdot \nabla b)(r_2 \cdot \nabla b) > 0.$$

  In this case, one needs an additional construction.
  Example : $v = 0$ at $x = L$.

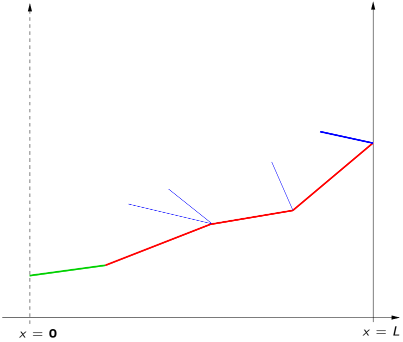
A reflection as a shock

- When the strong 2-shock is reflected as a 1-shock, it can then interact with 1-rarefactions, and one does not reach a constant state.



Ideas of the construction, 1

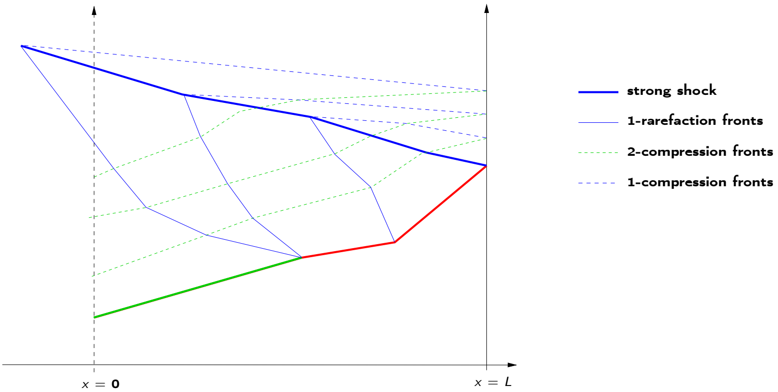
- We first consider the same construction as in the two-sided case. We can construct everything that is below the strong shock and the backward additional 2-shocks.



- One has to extend the 1-rarefactions and the strong reflected 1-shock.

Ideas of the construction, 2

- The idea is again to send additional 2-shocks from the boundary to treat the interactions between the 1-rarefactions and the reflected 1-shock.
- More precisely, we will use their reflection at $x = L$ to interact appropriately with the 1-strong shock.
- The idea is to reach this situation :



Ideas of the construction, 3

- However, here there is no “”’privileged direction of time”, the result always depends on the future.
- Hence we use a fixed-point scheme.
- A difficulty is that the map is discontinuous, and one uses an “almost-fixed point theorem” for discontinuous mappings.
- Precisely, we use Klee's theorem

Theorem (Klee, 1961)

A mapping from a closed convex in $\mathbb{R}^n$ into itself with discontinuities of size less than ε, has an almost fixed point :

$$\|f(x^\star) - x^\star\| \leq \varepsilon.$$

Open problems

- General controllability problem. Is there a good general condition to distinguish controllable systems (e.g. p-system) from uncontrollable ones (e.g. DiPerna's system) ?

- Control from one side. What about the 3×3 full Euler system ?

- Other possible approaches ?
  Vanishing viscosity (cf. Bianchini-Bressan) ? Glimm scheme ? Kinetic approaches ?

- Asymptotic stabilization. In the BV case with a closed-loop feedback, much is yet to be done...

Thank you for your attention !