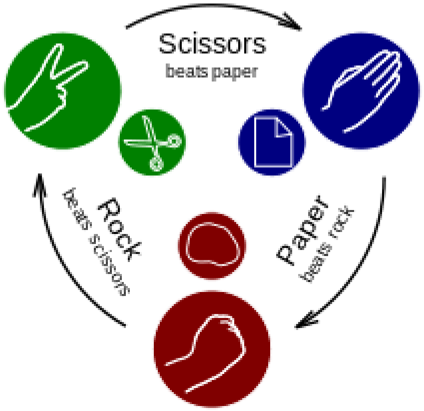
The rock-paper-scissors game as a nonlinear nonlocal diffusion PDE

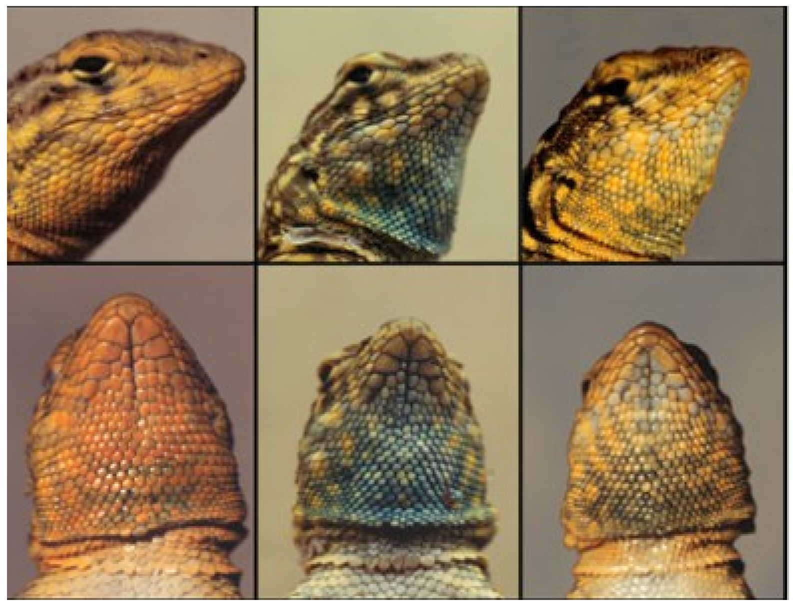
Marco A. Fontelos
(joint with F. Salvarani and N. Pouradier Duteil)

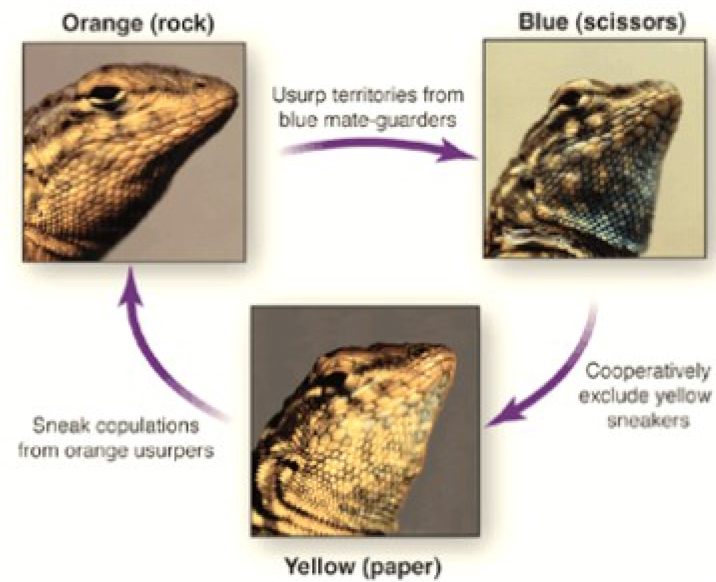
CSIC



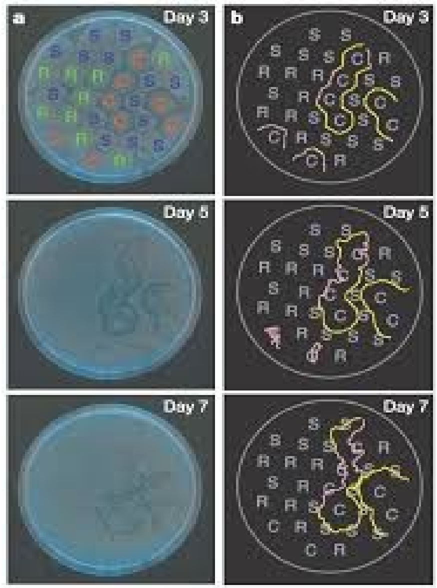
Benjamin Kerr, Margaret A. Riley, Marcus W. Feldman, and Brendan J. M. Bohannan. Local dispersal promotes biodiversity in a real-life game of rock-paper-scissors. *Nature*, 418:171–174, 2002.
Benjamin C. Kirkup and Margaret A. Riley. Antibiotic-mediated antagonism leads to a bacterial game of rock-paper-scissors in vivo. *Nature*, 428:412–414, 2004.

Hongjing Shi, Wen-Xu Wang, Rui Yang, and Ying-Cheng Lai. Basins of attraction for species extinction and coexistence in spatial rock-paper-scissors games. *Physical review. E, Statistical, nonlinear, and soft matter physics*, 81 3 Pt 1:030901, 2010.
Barry Sinervo and Curt M. Lively. The rock–paper–scissors game and the evolution of alternative male strategies. *Nature*, 380, 1996.





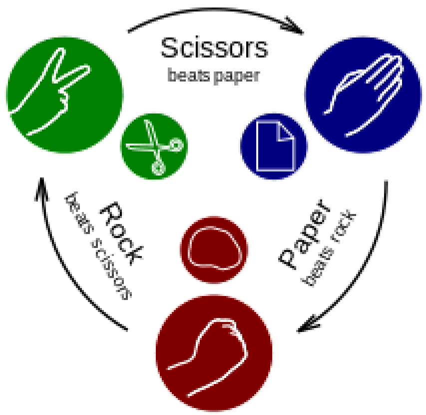
An unusual game is being played out in the Coast Range of California. Three alternative male strategies are locked in an ecological "perpetual motion machine" from which there appears little escape. As in the RPS game, three morphs of lizards cycle from the ultra-dominant polygynous orange-throated males, which best the more monogamous mate gaurding blues; the oranges are in turn bested by the sneaker strategy of yellow-throated males, and the sneaker strategy of yellows is in turn bested by the mate guarding strategy of blue-throated males. Each strategy in this game has a strength and a weakness, and there is the evolutionary rub that keeps the wheels spinning.



Three different types of bacteria E. Coli coexist thanks to RPS game strategies.

Modelling strategy:
- Three population densities.
- A certain quantity x is modified as a result of the interactions.
- Quadratic coupling (binary interactions) of densities.
- Result:. A kinetic model or system.

We consider the rock-paper-scissors game played in an interconnected population.



	0	1	2
0	$(0,0)$	$(h,-h)$	$(-h,h)$
1	$(-h,h)$	$(0,0)$	$(h,-h)$
2	$(h,-h)$	$(-h,h)$	$(0,0)$

Table 1: Payoff table of the rock-paper-scissors game

Let $(x, x_*) \in \mathbb{R}^2$ denote the values of the exchange variables of two agents after the interaction, and $(x', x'_*) \in \mathbb{R}^2$ their values just before the interaction. If the player with post-interaction exchange variable x wins, x and x_* satisfy:

$$\begin{cases} x = x' + h \\ x_* = x'_* - h. \end{cases}$$

If the players play the same, there is no winner and we have

$$\begin{cases} x = x' \\ x_* = x'_*. \end{cases}$$

$$\begin{cases} \dfrac{1}{\eta}\dfrac{\partial f}{\partial t}(t,x) = \dfrac{1}{3}\displaystyle\int_{\mathbb{R}} f(t,x-h)f(t,x_*)\,\mathrm{d}x_* + \dfrac{1}{3}\int_{\mathbb{R}} f(t,x+h)f(t,x_*)\,\mathrm{d}x_* - \dfrac{2}{3}\int_{\mathbb{R}} f(t,x)f(t,x_*)\,\mathrm{d}x_* \\ f(0,x) = f^{\mathrm{in}}(x) \end{cases}$$

$$\int_{\mathbb{R}} f(t,x)\,\mathrm{d}x = \|f(t,\cdot)\|_{L^1(\mathbb{R})} = \|f^{\mathrm{in}}\|_{L^1(\mathbb{R})} := \rho \quad \text{for all } t \in [0,T].$$

$$\int_{\mathbb{R}} x f(t,x)\,\mathrm{d}x = \int_{\mathbb{R}} x f^{\mathrm{in}}(x)\,\mathrm{d}x \quad \text{for all } t \in [0,T].$$

$$\begin{cases} \dfrac{1}{\eta}\dfrac{\partial f}{\partial t}(t,x) = \dfrac{1}{3}\rho\,[f(t,x+h) + f(t,x-h) - 2f(t,x)] \\ f(0,\cdot) = f^{\mathrm{in}}(\cdot) \end{cases}$$

$$\begin{cases} \partial_t f_\varepsilon(t,x) = \dfrac{\eta}{3\varepsilon^2}\|f^{\mathrm{in}}\|_{L^1(\mathbb{R})}\,(f_\varepsilon(t,x+\varepsilon) + f_\varepsilon(t,x-\varepsilon) - 2f_\varepsilon(t,x)) \\ f(0,x) = f^{\mathrm{in}}(x) \end{cases}$$

Limit equation: heat equation

[8] Nastassia Pouradier Duteil and Francesco Salvarani. Kinetic approach to the collective dynamics of the rock-paper-scissors binary game. *Appl. Math. Comput.*, 388:Paper No. 125496, 13, 2021.

The constrained model

$$\begin{cases} \frac{1}{\eta}\frac{\partial f}{\partial t} = \frac{1}{3}\int_h^{+\infty} f(t,x_*)\,\mathrm{d}x_*\,\left[\mathbb{1}_{x\geq 2h} f(t,x-h) + \mathbb{1}_{x\geq 0} f(t,x+h) - \mathbb{1}_{x\geq h} 2f(t,x)\right] \\ f(0,x)| = f^{\mathrm{in}}(x) \end{cases}$$

$$\begin{cases} \partial_t f_\varepsilon(t,x) = \frac{\eta}{3\varepsilon^2}\|f^{\mathrm{in}}\|_{L^1(\mathbb{R})}\,(f_\varepsilon(t,x+\varepsilon) + f_\varepsilon(t,x-\varepsilon) - 2f_\varepsilon(t,x) \\ f(0,x) = f^{\mathrm{in}}(x) \end{cases}$$

$$\begin{cases} \partial_t f(t,x) = \frac{\eta}{3}\left(\int_{\mathbb{R}+} f(t,x_*)\,\mathrm{d}x_*\right)\partial_x^2 f(t,x) & \\ f(t,0) = 0 & \text{for a.e. } t \in \mathbb{R}^+ \\ f(0,x) = f^{\mathrm{in}}(x) & \text{for a.e. } x \in \mathbb{R}^+. \end{cases}$$

$$\partial_t u(t,x) = \left(\int_{\mathbb{R}_+} u(t,z)\mathrm{d}z\right)\partial_x^2 u(t,x) \qquad \text{for a.e. } (t,x)\in\mathbb{R}_+^*\times\mathbb{R}_+^*$$

$$u(t,0) = 0 \qquad \text{for a.e. } t\in\mathbb{R}_+^*$$

$$u(0,x) = u^{\mathrm{in}}(x) \qquad \text{for a.e. } x\in\mathbb{R}_+,$$

Theorem 1. *Consider the initial-boundary value problem (1)-(3), with initial condition $u^{\mathrm{in}} \in L^1(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$ and such that $u^{\mathrm{in}} \geq 0$ for a.e. $x \in \mathbb{R}_+$. Let $T > 0$. Then, it has a unique very weak solution, which belongs to $L^1((0,T)\times\mathbb{R}_+) \cap L^\infty((0,T)\times\mathbb{R}_+)$. Moreover, $\|u(t,\cdot)\|_{L^\infty(\mathbb{R}_+)} \leq \|u^{\mathrm{in}}\|_{L^\infty(\mathbb{R}_+)}$ for a.e. $t \in (0,T)$. Lastly, the solution is non-negative, i.e. $u(t,x) \geq 0$ for a.e. $t \in (0,T)$ and for a.e. $x \in \mathbb{R}_+$.*

$$v(t,x) = u(t,x)\mathbb{1}_{x\geq 0} - u(t,-x)\mathbb{1}_{x\leq 0},$$

$$\begin{cases} \partial_t v(t,x) = \left(\int_{\mathbb{R}+} v(t,z)\,\mathrm{d}z\right)\partial_x^2 v(t,x) & \text{for a.e. } (t,x)\in\mathbb{R}_+\times \\ v(0,x) = v^{\mathrm{in}}(x) & \text{for a.e. } x\in\mathbb{R}, \end{cases}$$

$$u(t,x) = \left(4\pi \int_0^t \int_{\mathbb{R}_+} u(\theta,z)\,\mathrm{d}z\,\mathrm{d}\theta\right)^{-1/2} \times$$
$$\int_{\mathbb{R}_+} u^{\mathrm{in}}(y) \left\{ \exp\left[-(x-y)^2 \left(4\int_0^t\int_{\mathbb{R}_+} u(\theta,z)\,\mathrm{d}z\,\mathrm{d}\theta\right)^{-1}\right] - \exp\left[-(x+y)^2 \left(4\int_0^t\int_{\mathbb{R}_+} u(\theta,z)\,\mathrm{d}z\,\mathrm{d}\theta\right)^{-1}\right]\right\} \mathrm{d}y.$$

Proposition 2. *Let u be the solution of the initial-boundary value problem (1)-(3) and let*

$$M : t \mapsto \int_{\mathbb{R}_+} u(t,x)\,\mathrm{d}x.$$

Then M is a decreasing function of time. In particular, $M \in C^\infty((0,T))$ and, for all $t \in \mathbb{R}_+$,

$$M(t) \leq M(0) = \int_{\mathbb{R}_+} u^{\mathrm{in}}(x)\,\mathrm{d}x.$$

$$M'(t) = \left(\int_0^{+\infty} \partial_x^2 u(t,z)\mathrm{d}z\right) M(t) = \left(\lim_{x\to+\infty} \partial_x u(t,x) - \partial_x u(t,0)\right) M(t).$$

$$a(t) := \int_0^t M(s)\,\mathrm{d}s.$$

$$u(t,x) = \frac{1}{2\sqrt{\pi a(t)}} \int_0^{+\infty} \left(e^{-\frac{(x-s)^2}{4a(t)}} - e^{-\frac{(x+s)^2}{4a(t)}}\right) u^{\mathrm{in}}(s)\,\mathrm{d}s$$

$$\partial_x u(t,0) = \frac{1}{2\sqrt{\pi}} a^{-\frac{3}{2}}(t) \int_0^{+\infty} s e^{-\frac{s^2}{4a(t)}} u^{\mathrm{in}}(s)\,\mathrm{d}s.$$

$$\frac{\mathrm{d}M(t)}{\mathrm{d}t} = -M(t)\partial_x u(t,0);$$

$$a''(t) = -a'(t)\frac{1}{2\sqrt{\pi}}a^{-\frac{3}{2}}(t)\int_0^{+\infty} se^{-\frac{s^2}{4a(t)}}u^{\mathrm{in}}(s)\,\mathrm{d}s.$$

$$G(a) := \frac{1}{2\sqrt{\pi}}a^{-\frac{3}{2}}\int_0^{+\infty} se^{-\frac{s^2}{4a}}u^{\mathrm{in}}(s)\,\mathrm{d}s.$$

Lemma 1. *Let $u^{\mathrm{in}} \in L^1(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$ a positive and admissible initial condition. The integro-differential equation (13), with initial condition $a(0) = 0$ and $a'(0) = M(0)$, has a solution $a : \mathbb{R}_+ \to \mathbb{R}$ such that*

$$a(t) \sim \left(\frac{3}{2\sqrt{\pi}}M_1\right)^{\frac{2}{3}} t^{\frac{2}{3}} \text{ as } t \to \infty$$

and

$$a'(t) \sim \frac{2}{3}\left(\frac{3}{2\sqrt{\pi}}M_1\right)^{\frac{2}{3}} t^{-\frac{1}{3}} \text{ as } t \to \infty.$$

Proof. Since

$$a''(t) = -\frac{d}{dt}\int_0^{a(t)} \left(\frac{1}{2\sqrt{\pi}}(a^*)^{-3/2}\int_0^{+\infty} se^{-\frac{s^2}{4a^*}}u^{\text{in}}(s)\,\mathrm{d}s\right)\mathrm{d}a^*$$

Integrating once in time and using that $a'(0) = M(0)$, it holds

$$a'(t) + \int_0^{a(t)} \left(\frac{1}{2\sqrt{\pi}}(a^*)^{-3/2}\int_0^{+\infty} se^{-\frac{s^2}{4a^*}}u^{\text{in}}(s)\,\mathrm{d}s\right)\mathrm{d}a^* = M(0),$$

which we rewrite as

$$a'(t) + F(a(t)) = M(0), \tag{14}$$

$$F(a) = \int_0^a G(a^*)\,\mathrm{d}a^* = \int_0^a \left(\frac{1}{2\sqrt{\pi}}(a^*)^{-3/2}\int_0^{+\infty} se^{-\frac{s^2}{4a^*}}u^{\text{in}}(s)\,\mathrm{d}s\right)\mathrm{d}a^*$$
$$= \frac{1}{2\sqrt{\pi}}\int_0^{+\infty}\left(\int_0^a (a^*)^{-3/2}e^{-\frac{s^2}{4a^*}}\,\mathrm{d}a^*\right)su^{\text{in}}(s)\,\mathrm{d}s = \frac{1}{2\sqrt{\pi}}\int_0^{+\infty}\left(\int_0^{as^{-2}} (a^*)^{-3/2}e^{-\frac{1}{4a^*}}\,\mathrm{d}a^*\right)u^{\text{in}}(s)$$

and, since $\int_0^{+\infty} a^{-\frac{3}{2}}e^{-\frac{1}{4a}}da = 2\sqrt{\pi}$, we obtain

$$\lim_{a^\star\to+\infty} F(a^\star) = M(0).$$

We conclude that as t tends to infinity, if $a(t)\to+\infty$, then $a'(t)\to 0$.

Notice that from its definition, a is an increasing function, hence it has a limit when t goes to infinit Let $a_\infty := \lim_{t\to+\infty} a(t)$, and suppose that $a_\infty < +\infty$. Then $\lim_{t\to+\infty} a'(t) = 0$, from which we ge $F(a_\infty) = M(0)$. However, $F(a)$ is the primitive of a strictly positive function and hence is strictl growing as a function of a, which contradicts $\lim_{a^\star\to+\infty} F(a^\star) = M(0)$.

$a(t)\to+\infty$, $G(a(t)) \sim \frac{1}{2\sqrt{\pi}}a(t)^{-\frac{3}{2}}M_1$

$$a''(t) \sim -a'(t)a^{-\frac{3}{2}}(t)\frac{1}{2\sqrt{\pi}}M_1,$$

Similarity solutions:

$$u(t,x) = t^{\mu-1} g_\mu(x/t^\mu),$$

so that the mass of the solution u satisfies for all $t \in \mathbb{R}_+$:

$$\int_{\mathbb{R}_+} u(t,z)\mathrm{d}z = t^{2\mu-1} \int_{\mathbb{R}_+} g_\mu(\xi)\mathrm{d}\xi.$$

$$(\mu-1)g_\mu(\eta) - \mu\eta g'_\mu(\eta) = \left(\int_0^\infty g_\mu(s)\,\mathrm{d}s\right) g''_\mu(\eta).$$

$$\eta : \xi \mapsto \left(\int_0^{+\infty} g_\mu(s)\,\mathrm{d}s\right)^{1/2} \xi,$$

$$f_\mu : \xi \mapsto g_\mu\left(\left(\int_0^{+\infty} g_\mu(s)\,\mathrm{d}s\right)^{1/2} \xi\right)$$

$$(\mu-1)f_\mu(\xi) - \mu\xi f'_\mu(\xi) = f''_\mu(\xi).$$

$$f_\mu(\xi) = \xi\, {}_1F_1\left(\frac{1}{2\mu}, \frac{3}{2}; -\frac{\mu}{2}\xi^2\right)$$

In the particular case $\mu = \frac{1}{3}$ one has

$$f_{\frac{1}{3}}(\xi) = \xi e^{-\frac{1}{6}\xi^2}$$

while, for $\mu \in (\frac{1}{3}, 1)$ (cf. [1] formula 13.7.1)

$${}_1F_1\left(\frac{1}{2\mu}, \frac{3}{2}; -\frac{\mu}{2}\xi^2\right) \sim \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2} - \frac{1}{2\mu}\right)} \left(\frac{\mu}{2}\xi^2\right)^{-\frac{1}{2\mu}} \quad \text{as } \xi \to \infty,$$

Theorem 2. *If the initial data u^{in} has a bounded second moment $M_2(0)$, then there exists $C>0$ such that*

$$\left|u(t,x)-M_1\frac{x}{2\sqrt{\pi}a^{\frac{3}{2}}(t)}e^{-\frac{x^2}{4a(t)}}\right|\le\frac{CM_2(0)}{t}$$

for $t>1$.

Proof. Since

$$u(t,x)=\frac{1}{2\sqrt{\pi a(t)}}\int_0^{+\infty}\left(e^{-\frac{(x-y)^2}{4a(t)}}-e^{-\frac{(x+y)^2}{4a(t)}}\right)u^{\text{in}}(y)\,\mathrm{d}y,$$

denoting

$$v(x)=\frac{1}{2\sqrt{\pi a}}\int_0^{+\infty}\left(\frac{e^{-\frac{(x-y)^2}{4a}}-e^{-\frac{(x+y)^2}{4a}}}{y}\right)yv^{\text{in}}(y)\,\mathrm{d}y,$$

we have

$$\begin{aligned}&v(x)-M_1\frac{x}{2\sqrt{\pi}a^{\frac{3}{2}}}e^{-\frac{x^2}{4a}}\\=&\frac{1}{2\sqrt{\pi a}}\int_0^{+\infty}\left(\frac{e^{-\frac{(x-y)^2}{4a}}-e^{-\frac{(x+y)^2}{4a}}}{y}-\frac{x}{a}e^{-\frac{x^2}{4a}}\right)yv^{\text{in}}(y)\,\mathrm{d}y.\end{aligned}$$

We write now

$$\frac{e^{-\frac{(x-y)^2}{4a}}-e^{-\frac{(x+y)^2}{4a}}}{y}-\frac{x}{a}e^{-\frac{x^2}{4a}}\equiv\frac{1}{a^{\frac{1}{2}}}\Phi\left(\frac{x}{a^{\frac{1}{2}}},\frac{y}{a^{\frac{1}{2}}}\right)$$

with

$$\Phi\left(X,Y\right)=\frac{e^{-\frac{(X-Y)^2}{4}}-e^{-\frac{(X+Y)^2}{4}}}{Y}-Xe^{-\frac{X^2}{4}}.$$

It is simple to show that there exists a constant C such that

$$\left|\Phi\left(X,Y\right)\right|\le CY$$

so that

$$\left|v(x) - M_1 \frac{x}{2\sqrt{\pi} a^{\frac{3}{2}}} e^{-\frac{x^2}{4a}}\right| \leq \frac{C}{a^{\frac{3}{2}}} \int_0^{+\infty} y^2 u^{\text{in}}(y) dy.$$

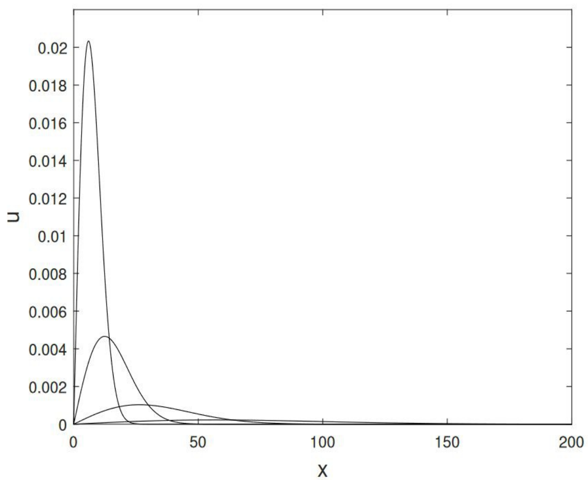


FIGURE 1. Numerical profiles of the solution of (1)-(3) at times $t = 50, 500, 5000, 50000$, with initial condition given in (17).

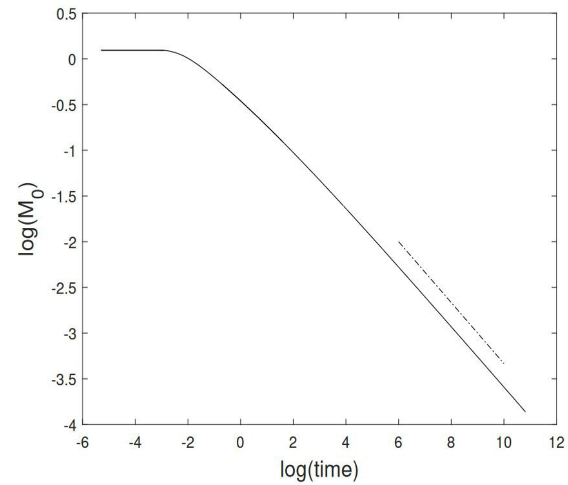


FIGURE 2. Logarithm of the mass vs logarithm of time and comparison with a $-1/3$ slope (dotted-dashed line).

$$u_0(x) = \chi_{[1,2]} = \begin{cases} 1 & x \in [1,2] \\ 0 & \text{otherwise.} \end{cases}$$

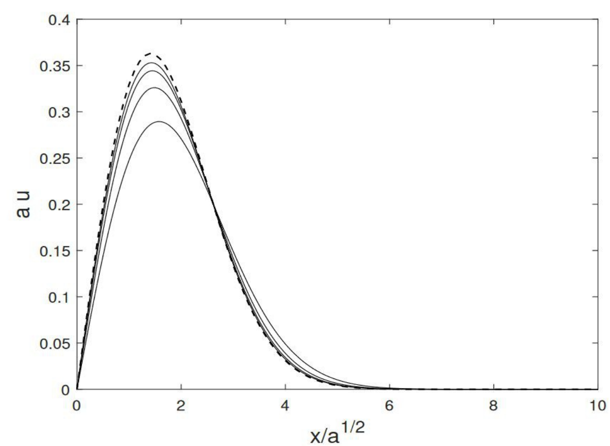


FIGURE 3. Rescaled numerical profiles and comparison with the self-similar profile (dashed line).

The Dirichlet problem

$$\Omega = (0,\pi)$$

$$w_t = \left[\int_0^\pi w(t,\xi)\,\mathrm{d}\xi\right] w_{xx}, \qquad (t,x) \in \mathbb{R}_+ \times \Omega$$

$$w(0,x) = w^{\mathrm{in}} \in L^2(0,\pi) \qquad x \in \Omega$$

$$w(t,0) = w(t,\pi) = 0, \qquad t \in \mathbb{R}_+,$$

$$w(t,x) = \sum_{n=1}^{+\infty} w_n(t)\sin(nx).$$

$$a'(t) = \sum_{n \text{ odd}} \frac{2w_n(0)}{n} e^{-n^2 a(t)} \qquad G(a) \equiv \int_0^a \frac{e^{a'}}{\sum_{n \text{ odd}} \frac{2w_n(0)}{n} e^{(1-n^2)a'}} da' = t.$$

Lemma 2. *Let w^{in} be the initial condition of the initial-boundary value problem (18)-(20) and suppose that $w^{in} \in L^1(\Omega) \cap L^\infty(\Omega)$. Then there exists a constant C, depending on w^{in}, and a time $T > 0$ such that, for any $t > T$,*

$$\left| w(t,x) - \frac{M}{2}\frac{\sin(x)}{1+Mt} \right| \le \frac{C}{t^2}.$$

Note that

$$
\begin{aligned}
G(a) &= \frac{1}{2w_1(0)}(e^a - 1) - \frac{1}{2w_1(0)} \int_0^a \frac{\sum_{n=3,5,\ldots} \frac{w_n(0)}{nw_1(0)} e^{(2-n^2)a'}}{1 + \sum_{n=3,5,\ldots} \frac{w_n(0)}{nw_1(0)} e^{(1-n^2)a'}} da' \\
&= \frac{1}{2w_1(0)} e^a - K + O(e^{-7a}), \text{ as } a \to +\infty
\end{aligned}
$$

with

$$
K = \frac{1}{2w_1(0)} + \frac{1}{2w_1(0)} \int_0^\infty \frac{\sum_{n=3,5,\ldots} \frac{w_n(0)}{nw_1(0)} e^{(2-n^2)a'}}{1 + \sum_{n=3,5,\ldots} \frac{w_n(0)}{nw_1(0)} e^{(1-n^2)a'}} da'
$$

We have then

$$
a \sim \log(2w_1(0)(t+K) + O(t^{-7})), \text{ as } t \to +\infty,
$$

and hence

$$
M(t) = a' \sim \frac{1}{t+K}, \text{ as } t \to +\infty.
$$

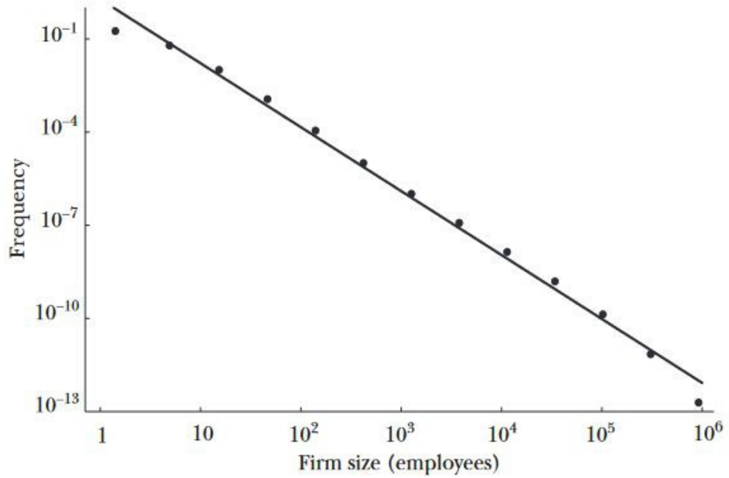
Therefore

(22)
$$
w_n(t) \sim \frac{w_n(0)}{(2w_1(0)(t+K))^{n^2}} \text{ as } t \to +\infty.
$$

Perspectives

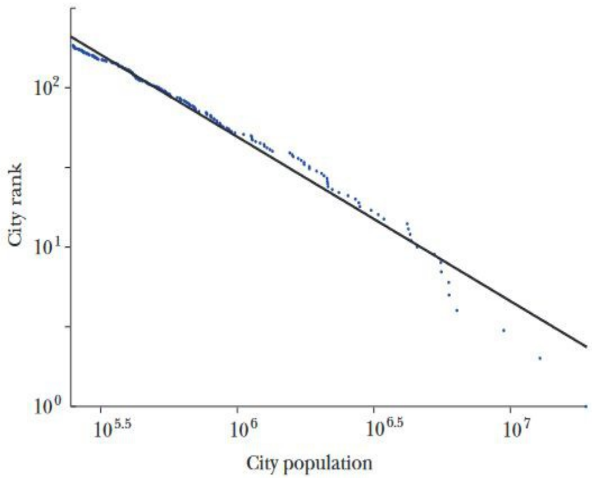
General nonlinear kinetic models. Explanation of the prevalence of Zipf's law in social sciences.

Log *Frequency* versus log *Size* of US firms (by Number of Employees) for 1997



Source: Axtell (2001).
Notes: Ordinary least squares (OLS) fit gives a slope of 2.06 (s.e. = 0.054; $R^2 = 0.99$). This corresponds to a frequency $f(S) \sim S^{-2.059}$, which is a power law distribution with exponent 1.059. This is very close to an ideal Zipf's law, which would have an exponent $\zeta = 1$.

A Plot of City Rank versus Size for all US Cities with Population over 250,000 in 2010



Source: Author, using data from the *Statistical Abstract of the United States* (2012).
Notes: The dots plot the empirical data. The line is a power law fit ($R^2 = 0.98$), regressing ln *Rank* on ln *Size*. The slope is −1.03, close to the ideal Zipf's law, which would have a slope of −1.