Local Energy Decay for the Damped Wave Equation

Julien Royer

Work in progress with Rayan Fahs

Institut de Mathématiques de Toulouse

Benasque

X Partial differential equations, optimal design and numerics

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The Model Case

We consider on $\mathbb{R}^d$ the free wave equation

$$\partial_t^2 u - \Delta u = 0,$$

with $(u, \partial_t u)_{|t=0} = (u_0, u_1)$ supported in the ball $B(R)$.

The usual energy is a constant of the motion

$$E(u;t) = \int_{\mathbb{R}^d} \left(|\nabla u(t,x)|^2 + |\partial_t u(t,x)|^2 \right) \mathrm{d}x.$$

- In odd dimension, the wave propagates exactly at speed 1 (strong Huyghens principle). In particular, it goes to infinity and

$$\forall t > 2R, \quad \|\nabla u(t)\|^2_{L^2(B(R))} + \|\partial_t u(t)\|^2_{L^2(B(R))} = 0.$$

- In even dimension, the wave propagates at speed at most 1. We still have local energy decay:

$$\|\nabla u(t)\|^2_{L^2(B(R))} + \|\partial_t u(t)\|^2_{L^2(B(R))} \leqslant C_R\, t^{-2d} E(u;0).$$

In general settings, we are interested in the decay of the energy which remains in a bounded region for localized initial condition.

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The asymptotically free setting

We consider the (possibly damped) wave equation

$$\begin{cases} \partial_t^2 u + Pu + a(x)\partial_t u = 0, \\ (u, \partial_t u)_{|t=0} = (f, g). \end{cases}$$

- P is the Laplace-Beltrami operator associated to a long-range perturbation of the flat metric:

$$P = -\frac{1}{w(x)}\Delta_G, \qquad \Delta_G = \operatorname{div} G(x)\nabla,$$

  with $w(x) \geqslant c_0 > 0$, $G(x) \geqslant c_0 \,\mathrm{Id}$ and

$$|\partial^\alpha (G(x) - \mathrm{Id})| + |\partial^\alpha (w(x) - 1)| \lesssim \langle x \rangle^{-\rho_0 - |\alpha|}, \quad \rho_0 > 0.$$

- The absorption index $a(x) \geqslant 0$ is of short range

$$|\partial^\alpha a(x)| \lesssim \langle x \rangle^{-1-\rho_0-|\alpha|}.$$

- The (non-decreasing) energy is

$$E(u;t) = \int_{\mathbb{R}^d} \left(\langle G(x)\nabla u(t,x), \nabla u(t,x)\rangle + w(x)\,|\partial_t u(t,x)|^2 \right) dx.$$

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From resolvent estimates to time decay

We study the time-dependent solution from a spectral point of view:

$$u(t) = \frac{1}{2\pi} \int_{\mathrm{Im}(z)=\mu} e^{-itz} R(z) F_z \,\mathrm{d}z, \quad (\mu > 0)$$

where

$$R(z) = \big(- \Delta_G - iawz - wz^2 \big)^{-1}, \quad F_z = (aw - izw)f + wg.$$

- We prove resolvent estimates in weighted spaces.
- We have the limiting absorption principle: the weighted resolvent is uniform in $\mathrm{Im}(z) > 0$.
- We have control the dependence in $\mathrm{Re}(z)$, in particular for $|\mathrm{Re}(z)| \gg 1$ and $|\mathrm{Re}(z)| \ll 1$.
- We estimate the resolvent $R(z)$ and its derivatives.

We can do the same for $\nabla u(t)$ and $\partial_t u(t)$.

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We study the time-dependent solution from a spectral point of view:

$$\chi(x)u(t) = \frac{1}{2\pi}\int_{\mathrm{Im}(z)=\mu} e^{-itz}\chi(x)R(z)\chi(x)F_z\,\mathrm{d}z, \quad (\mu > 0)$$

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$$t^m \chi(x) u(t) \simeq \frac{1}{2\pi} \int_{\mathrm{Im}(z)=\mu} e^{-itz} \chi(x) R^{(m)}(z) \chi(x) F_z \, \mathrm{d}z, \quad (\mu > 0)$$

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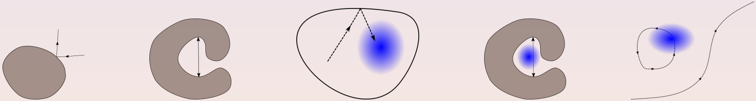
Contribution of high frequencies

The propagation of high frequency waves is well approximated by the corresponding classical rays of light.

For the damped wave equation on an unbounded domain, the local energy decays uniformly if and only if all rays of light escape to infinity or go through the damping region.

- Without damping, we recover the non-trapping condition.
- For the damped wave equation on a compact domain, we recover the geometric control condition.



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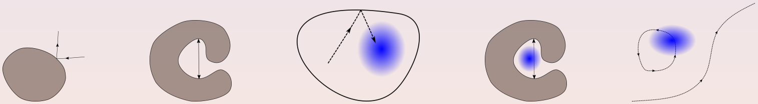

The decay of the local energy is actually governed by the contribution of low frequencies.

State of the art

Theorem (Bouclet-R. '14, R. '18)

Assume that the geometric damping condition holds. Let $\varepsilon > 0$. Then for $(f, g) \in H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$ supported in $B(R)$ and $t \geqslant 0$ we have

$$\|\nabla u(t)\|^2_{L^2(B(R)))} + \|\partial_t u(t)\|^2_{L^2(B(R))} \lesssim \langle t\rangle^{-(2d-\varepsilon)}\big(\|\nabla f\|^2_{L^2(\mathbb{R}^d)} + \|g\|^2_{L^2(\mathbb{R}^d)}\big).$$

This is almost optimal in even dimensions.

Theorem (Bouclet-Burq '21)

Assume that $a = 0$ and the non-trapping condition holds. Then for $(f, g) \in H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$ supported in $B(R)$ and $t \geqslant 0$ we have

$$\|u(t)\|^2_{L^2(B(R)))} \lesssim \langle t\rangle^{-2d}\|f\|^2_{L^2(\mathbb{R}^d)} + \langle t\rangle^{2-2d}\|g\|^2_{L^2(\mathbb{R}^d)}.$$

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Main result

Let $u_0(t)$ be the solution of

$$\begin{cases} \partial_t^2 u_0 - \Delta u_0 = 0, \\ (u_0, \partial_t u_0)_{|t=0} = (wf, awf + wg). \end{cases}$$

Theorem (Fahs-R., in progress)

Assume that the geometric damping condition holds. Then for $(f, g) \in H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$ *supported in* $B(R)$ *and* $t \geqslant 0$ *we have*

$$\|u(t) - u_0(t)\|^2_{L^2(B(R)))} \lesssim \langle t \rangle^{-2d-2\rho} \|f\|^2_{L^2(\mathbb{R}^d)} + \langle t \rangle^{2-2d-2\rho} \|af + g\|^2_{L^2(\mathbb{R}^d)}.$$

- In even dimension, this gives the asymptotic profile for $u(t)$ and in particular the optimal decay.
- In odd dimensions, this gives a rate of decay which is better than what would be optimal in even dimensions !

Open problem: Prove a similar result for $\nabla u(t)$ (our proof works for $\partial_t u(t)$).

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