Discrete controllability of a parabolic system*

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X Partial differential equations, optimal design and numerics

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Contents

- **Control System:** Let $Q := (0, T) \times (0, 1)$.

$$
\begin{cases}
\partial_t u_1 - \partial_x(\gamma_1 \partial_x u_1) = 0, & (t, x) \in Q, \\
\partial_t u_2 - \partial_x(\gamma_2 \partial_x u_2) = 0, & (t, x) \in Q, \\
u_1(t, 0) = 0, u_2(t, 0) = v(t), & t \in (0, T), \\
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u_1(0, x) = u_{1,0}(x), u_2(0, x) = u_{2,0}(x), & x \in (0, 1),
\end{cases}
\tag{1}
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where $\alpha \geq 0$, $\gamma_i \in C^1([0, 1])$ with $\gamma_i > 0$, and v is the control.

- **Control System:** Let $Q := (0,T) \times (0,1)$.

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- **Relevant Spaces:**

$$\mathcal{H} := \left\{(v_1, v_2) \in (H^1(0,1))^2 : v_1(0) = v_2(0) = 0, v_1(1) = v_2(1)\right\},$$

$$\|v\|_{\mathcal{H}} = \left(\sum_{i=1}^{2} \int_0^1 \gamma_i(1)|v_i'(x)|^2\,dx + \alpha|v_1(1)|^2\right)^{1/2}.$$

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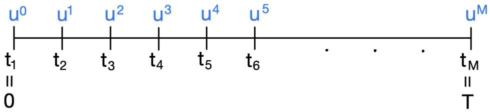
- **Existing Results:**
  1. For $T > 0$, $u_0 \in \mathcal{H}'$, $v \in L^2(0,T)$, $\exists !\, u \in C([0,T]; \mathcal{H}') \cap L^2(0,T;E)$.
  2. For $(u_{1,0}, u_{2,0}) \in \mathcal{H}', \exists\, v \in L^2(0,T) \ni (u_1(T), u_2(T)) = 0$.

K. Bhandari, F. Boyer, and V. Hernández-Santamaría. *Boundary null-controllability of 1-D coupled parabolic systems with Kirchhoff-type conditions*. Math. Control Signals Systems, vol. 33, no. 3, pp. 413–471, 2021.

Time-discrete control system

For $M \in \mathbb{N}$, define $\Delta t = T/M$.

$$\begin{cases} \left(\frac{u_1^{n+1} - u_1^n}{\Delta t}\right) - \partial_x(\gamma_1 \, \partial_x u_1^{n+1}) = 0, \\ \left(\frac{u_2^{n+1} - u_2^n}{\Delta t}\right) - \partial_x(\gamma_2 \, \partial_x u_2^{n+1}) = 0, \\ u_1^{n+1}(0) = 0, \, u_2^{n+1}(0) = v^{n+1}, \\ u_1^{n+1}(1) = u_2^{n+1}(1), \\ \gamma_1(1)\partial_x u_1^{n+1}(1) + \gamma_2(1)\partial_x u_2^{n+1}(1) + \alpha u_1^{n+1}(1) = 0, \\ u_1^0 = u_{1,0}, \, u_2^0 = u_{2,0}, \end{cases} \qquad n \in \{0, 1, \ldots, M-1\}. \quad (2)$$



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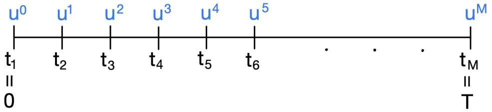

Analogous null controllability notion:

For any $(u_{1,0}, u_{2,0}) \in \mathcal{H}', \exists \{v^{n+1}\}$ with $\sum_{n=0}^{M-1} |v^{n+1}|^2 \lesssim \|u_0\|^2_{\mathcal{H}'} h \ni (u_1^M, u_2^M) = 0.$

Introducing the Problem

(a) The above controllability notion is not achievable.

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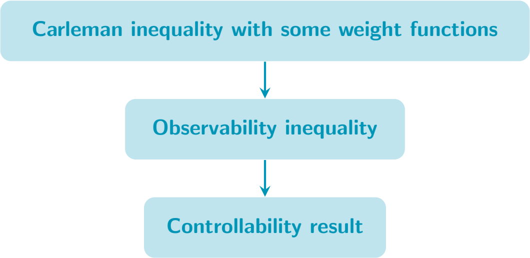
(c) We further try to check whether this discrete control approximates any control of the associated continuous system.

The duality approach (For continuous system)

Consider the following adjoint system

$$\begin{cases} -\partial_t \varphi_1 - \partial_x \left(\gamma_1 \, \partial_x \varphi_1 \right) = 0, \\ -\partial_t \varphi_2 - \partial_x \left(\gamma_2 \, \partial_x \varphi_2 \right) = 0, \\ \varphi_1(t,0) = 0, \varphi_2(t,0) = 0, \\ \varphi_1(t,1) = \varphi_2(t,1), \\ \gamma_1(1)\partial_x \varphi_1(t,1) + \gamma_2(1)\partial_x \varphi_2(t,1) + \alpha \varphi_1(t,1) = 0, \\ \varphi_1(T,\cdot) = \varphi_{1,T}, \varphi_2(T,\cdot) = \varphi_{2,T}. \end{cases} \tag{3}$$



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The duality approach (For discrete system)

Consider adjoint system for the discrete control system (2)

$$\begin{cases} -\left(\frac{\varphi_1^{n+1}-\varphi_1^n}{\Delta t}\right) - \partial_x(\gamma_1\,\partial_x \varphi_1^n) = 0, \\ -\left(\frac{\varphi_2^{n+1}-\varphi_2^n}{\Delta t}\right) - \partial_x(\gamma_2\,\partial_x \varphi_2^n) = 0, \\ \varphi_1^n(0) = \varphi_2^n(0) = 0,\ \ \varphi_1^n(1) = \varphi_2^n(1), \\ \gamma_1(1)\partial_x\varphi_1^n(1) + \gamma_2(1)\partial_x\varphi_2^n(1) + \alpha\varphi_1^n(1) = 0, \\ \varphi_1^M(x) = \varphi_{1,M}(x), \varphi_2^M(x) = \varphi_{2,M}(x), \end{cases} \quad n \in \{0, 1, \ldots, M-1\} \qquad (4)$$

Discrete Carleman type inequality with similar weight functions

↓

Relaxed Observability inequality

↓

$\Phi(\Delta t)$-controllability result

Weight function

- Let $0 < \nu_0 < 1$ be a constant close to 1 satisfying

$$\left(\frac{216\nu_0}{(1-\nu_0)^3}\gamma_2^2(1) - 7\gamma_1^2(1)\right) \geq 1 \tag{5}$$

- For $i \in \{1,2\}$, consider the function

$$\beta_i(x) = 2 + c_i(x-1),\ x \in [0,1],$$

with $c_1 = 1$ and $c_2 = c_2(\gamma_1, \gamma_2) := -\frac{6}{(1-\nu_0)} < 0$.

- Let $K := 2\max\{\|\beta_1\|_\infty, \|\beta_2\|_\infty\}$ and let $\lambda > 1$. Then for $i \in \{1,2\}$, define

$$\eta_i(x) = e^{\lambda K} - e^{\lambda\beta_i(x)}$$
$$\mu_i(x) = e^{\lambda\beta_i(x)}$$

- For $\tau > 0$, let $s(t) = \tau\,\theta(t)$, where

$$\theta(t) = \frac{1}{(t+\delta T)(T+\delta T - t)},\quad \delta > 0.$$

- Finally for $i \in \{1,2\}$, let $r_i(t,x) = e^{-s(t)\eta_i(x)}$.

Discrete Carleman Estimate

Theorem (Discrete Carleman type inequality)

Let $\varphi^M \in \mathcal{H}$, and let $\lambda > 0$ be sufficiently large. Then for sufficiently large τ, and for Δt, $\delta > 0$ such that $\frac{\tau^3 \Delta t}{\delta^4}$ is sufficiently small, the solution of discrete adjoint system (4) *satisfies*

$$\tau^3 \sum_{i=1}^{2} \sum_{n=0}^{M-1} \int_0^L (\theta^n)^3 (r_i^n)^2 \, |\varphi_i^n|^2 + \tau \sum_{i=1}^{2} \sum_{n=0}^{M-1} \int_0^L (\theta^n)(r_i^n)^2 \, |\partial_x(\varphi_i^n)|^2$$

$$+ \tau^2 \sum_{n=0}^{M-1} (\theta^n)(r_1^n(1))^2 \big|\varphi_1^n(1)\big|^2$$

$$\leq C\tau \sum_{n=0}^{M-1} (\theta^n) \, |(r_2^n(0))|^2 \, |\partial_x(\varphi_2^n)(0)|^2$$

$$+ C \, (\Delta t)^{-1} \sum_{i=1}^{2} \left(\int_0^1 |(r_i \, \varphi_i)^0|^2 + \int_0^1 |(r_i \, \varphi_i)^M|^2 + \int_0^1 |(r_i \, \partial_x \varphi_i)^M|^2 \right),$$

where the constant $C > 0$ depends on $\gamma_1, \gamma_2, \alpha, T$ and λ.

Relaxed Observability Inequality

Theorem (Relaxed observability inequality)

For sufficiently small discrete parameter δ and Δt, $\exists$ constants $K_0, K_1, K_2 > 0$ such that any solution to (4) with $\varphi^M \in \mathcal{H}$ satisfies

$$\|\varphi^0\|_{\mathcal{H}}^2 \leq C_{obs}\left(\sum_{n=0}^{M-1} |\partial_x \varphi_2^n(0)|^2 + e^{-\frac{K_2}{(\Delta t)^{1/4}}} \left\|\varphi^M\right\|_{\mathcal{H}}^2\right), \tag{6}$$

where $C_{obs} = e^{K_1(1+1/T)}$.

Controllability result

Theorem ($\phi(\Delta t)$-controllability in $\mathcal{H}'$)

Let the discretization parameter Δt be sufficiently small. Then, for any initial data $u_0 \in \mathcal{H}'$ and any function ϕ satisfying

$$\liminf_{\Delta t \to 0} \frac{\phi(\Delta t)}{e^{-C_2/(\Delta t)^{1/4}}} > 0,$$

there exists a sequence of controls $\{v^{n+1}\}_{n=0}^{M-1}$ satisfies

$$\sum_{n=0}^{M-1} |v^{n+1}|^2 \leq C \, \|u_0\|^2_{\mathcal{H}'} \, ,$$

such that the associated solution $\{u^{n+1}\}_{n=0}^{M-1}$ of (2) satisfying

$$\|u^{n+1}\|^2_{\mathcal{H}'} + \sum_{n=0}^{M-1} \|u^{n+1}\|^2_E \leq C\|u_0\|^2_{\mathcal{H}'} \quad \text{for } n \in \{0, 1, \dots, M-1\},$$

and have the following estimate at $t_M = T$

$$\|u^M\|_{\mathcal{H}'} \leq C\sqrt{\phi(\Delta t)} \, \|u_0\|_{\mathcal{H}'} \, ,$$

where $C > 0$ is a constant, depending on ϕ and T.

Convergence of Discrete control

Define approximations $V_M \in L^2(0, T)$, and $U_M = \Big((U_M)_1, (U_M)_2 \Big) \in L^2(0, T; E)$ as

$$V_M(t) = \sum_{n=0}^{M-1} 1_{(t_n, t_{n+1}]}(t)\, v^{n+1}, \quad t \in (0, T),$$

$$(U_M)_i(t, x) = 1_{[t_0, t_1]}(t) \frac{u_i^1(x)}{2} + \sum_{n=1}^{M-1} 1_{(t_n, t_{n+1}]}(t) \left(\frac{u_i^{n+1} + u_i^n}{2} \right)(x), \quad (t, x) \in Q, i \in \{1, 2\}.$$

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Theorem (Convergence Result)

There exist functions $v \in L^2(0,T)$ and $u \in L^2(0,T;E)$ such that

$$V_M \rightharpoonup v \text{ in } L^2(0,T), \text{ and } U_M \rightharpoonup u \text{ in } L^2(0,T;E).$$

Furthermore, the pair of functions (u,v) solves the continuos system (1) such that the state u satisfies

$$u(T,\cdot) = (0,0) \text{ on } (0,1) \text{ a.e.}$$

Idea of the proof-I

- Let $g = (g_1, g_2) \in L^2(0, T; E)$. For $n \in \{0, M-1\}$, define the functions

$$g_i^{n+1}(x) = \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} g_i(t,x)\,dt, \quad \text{for } i \in \{1,2\}.$$

Consider the adjoint system

$$\begin{cases} -\frac{\varphi_i^{n+1}-\varphi_i^n}{\Delta t} - \partial_x\left(\gamma_i\,\partial_x\varphi_i^n\right) = g_i^{n+1}, \ i \in \{1,2\},\\ \varphi_1^n(0) = \varphi_2^n(0) = 0, \ \varphi_1^n(1) = \varphi_2^n(1),\\ \gamma_1(1)\,\partial_x\varphi_1^n(1) + \gamma_2(1)\,\partial_x\varphi_2^n(1) + \alpha\,\varphi_1^n(1) = 0,\\ \varphi^M = 0, \end{cases}$$

For $i \in \{1,2\}$, we define the functions

$$(\varphi_M)_i(t,x) := \sum_{n=0}^{M-1} 1_{[t_n,t_{n+1}]}(t)\left(\frac{(t-t_n)}{\Delta t}\varphi_i^{n+1}(x) + \frac{(t_{n+1}-t)}{\Delta t}\varphi_i^n(x)\right), \quad (t,x) \in Q.$$

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- Using *weak formulation of the discrete control system* and the *definitions of V_M, U_M and φ_M*, we have:

$$\int_0^T \langle U_M(t), g(t)\rangle_E\,dt - \langle u_0, \varphi_M(0,\cdot)\rangle_{\mathcal{H}',\mathcal{H}} - \gamma_2(0)\int_0^T V_M(t)\,\partial_x(\varphi_M)_2(t,0)\,dt$$
$$= -\frac{1}{2}\langle u_0, \varphi_M(\Delta t,\cdot) - \varphi_M(0,\cdot)\rangle_{\mathcal{H}',\mathcal{H}}, \quad \forall\, g \in L^2(0,T;E).$$

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- Taking limit as $M \to \infty$, we get

$$\int_0^T \langle u(t), g(t)\rangle_E\,dt - \langle u_0, \varphi(0,\cdot)\rangle_{\mathcal{H}',\mathcal{H}} - \gamma_2(0)\int_0^T v(t)\,\partial_x\varphi_2(t,0)\,dt = 0, \ \forall g \in L^2(0,T;E),$$

  where φ solves the adjoint system with source term g and $\varphi_T = 0$.

Idea of the proof-II

A. López, E. Zuazua. *Uniform null-controllability for the one-dimensional heat equation with rapidly oscillating periodic density*. Annales de l'Institut Henri Poincaré C, Analyse non linéaire, Volume 19, Issue 5, 2002, Pages 543-580, ISSN 0294-1449.

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Using φ^n, we define φ_M as before.

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  Using φ^n, we define φ_M as before.

- Then for every $\varphi_T = \varphi^M \in \mathcal{H}$, the approximate control V_M satisfies

$$\int_0^T V_M(t)\,\partial_x\left(\varphi_M\right)_2(t,0)\,dt = -\phi(\Delta t)\left\langle \partial_x\widehat{\varphi}^M, \frac{\partial_x\varphi^M + \partial_x\varphi^{M-1}}{2}\right\rangle_E - \left\langle u_0, \frac{\varphi^0+\varphi^{-1}}{2}\right\rangle_{\mathcal{H}',\mathcal{H}},$$

  where $\widehat{\varphi}^M \in \mathcal{H}$.

- Passing to the limit gives

$$\int_0^T v(t)\,\partial_x\varphi_2(t,0)\,dt + \langle u_0, \varphi(0,x)\rangle_{\mathcal{H}',\mathcal{H}} = 0, \quad \forall\,\varphi_T \in \mathcal{H}.$$

Idea of the proof-II

A. López, E. Zuazua. *Uniform null-controllability for the one-dimensional heat equation with rapidly oscillating periodic density*. Annales de l'Institut Henri Poincaré C, Analyse non linéaire, Volume 19, Issue 5, 2002, Pages 543-580, ISSN 0294-1449.

- For $\varphi_T \in \mathcal{H}$, consider the following time-discrete adjoint system

$$\begin{cases} -\frac{\varphi_i^{n+1}-\varphi_i^n}{\Delta t} - \partial_x(\gamma_i\,\partial_x\varphi_i^n) = 0, \;\; i \in \{1,2\}, \\ \varphi_1^n(0) = \varphi_2^n(0) = 0, \\ \varphi_1^n(1) = \varphi_2^n(1), \\ \gamma_1(1)\,\partial_x\varphi_1^n(1) + \gamma_2(1)\,\partial_x\varphi_2^n(1) + \alpha\,\varphi_1^n(1) = 0, \\ \varphi^M = \varphi_T, \end{cases} \qquad n \in \{0,1,\dots,M-1\},$$

  Using φ^n, we define φ_M as before.
- Then for every $\varphi_T = \varphi^M \in \mathcal{H}$, the approximate control V_M satisfies

$$\int_0^T V_M(t)\,\partial_x(\varphi_M)_2(t,0)\,dt = -\phi(\Delta t)\left\langle \partial_x\widehat{\varphi}^M, \frac{\partial_x\varphi^M + \partial_x\varphi^{M-1}}{2}\right\rangle_E - \left\langle u_0, \frac{\varphi^0+\varphi^{-1}}{2}\right\rangle_{\mathcal{H}',\mathcal{H}},$$

  where $\widehat{\varphi}^M \in \mathcal{H}$.
- Passing to the limit gives

$$\int_0^T v(t)\,\partial_x\varphi_2(t,0)\,dt + \langle u_0, \varphi(0,x)\rangle_{\mathcal{H}',\mathcal{H}} = \langle u(T), \varphi_T\rangle_{\mathcal{H}',\mathcal{H}} = 0, \quad \forall\, \varphi_T \in \mathcal{H}.$$

  This proves $u(T,\cdot) = 0$ on $(0,1)$ a.e.

Thank you for your attention.