# Local Controllability to the Trajectories of a Gray-Scott System


Mohmedmunavvar Mubarak Bapu

( Joint work with M. Biswas )

Department of Mathematics and Statistics, IIT Kanpur


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- The Gray-Scott system models a chemical reaction

$$U + 2V \to 3V, \tag{1}$$
$$V \to P, \tag{2}$$

where the reaction (1) consumes chemical $U$ and produce $V$ and in the reaction (2) $P$ is an inert product.

- The Gray-Scott system describes two equations for reacting chemicals $U$ and $V$ with concentrations $u$ and $v$ respectively:

$$\begin{cases} \partial_t u = d_u \Delta u - uv^2 + F(1-u), \\ \partial_t v = d_v \Delta v + uv^2 - (F+k)v, \end{cases} \tag{3}$$

the cubic terms $uv^2$ and $-uv^2$ corresponds to chemical reaction (1). The linear term $kv$ comes from the chemical reaction (2) at rate $k$ and positive constant $F > 0$ denotes the rate at which $U$ is fed.

For $T > 0$, consider the reaction-diffusion ODE-PDE model

$$\begin{cases} \partial_t u = -uv^2 + F(1-u) + h\mathbf{1}_{\omega(t)} & \text{in } (0,T)\times(0,1), \\ \partial_t v = d_v \partial_{xx} v + uv^2 - (F+k)v & \text{in } (0,T)\times(0,1), \\ \partial_x v = 0 & \text{on } (0,T)\times\{0,1\}, \\ (u,v)(0,\cdot) = (u_0,v_0) & \text{in } (0,1). \end{cases} \tag{4}$$

## Theorem: (Víctor H.S. and Kévin L.B., 2021)

Let $u_\pm = \frac{1}{2}(1 \pm \sqrt{1-4\gamma^2 F})$, $v_\mp = \frac{1}{2\gamma}(1 \mp \sqrt{1-4\gamma^2 F})$, then there exist $\delta > 0$, $C > 0$ such that for every $d_v \in (1,\infty)$, $(u_0, v_0) \in L^2(0,1) \times H^1(0,1)$ satisfying

$$\|(u_0 - u_\pm, v_0 - v_\mp)\|_{L^2(0,1)\times H^1(0,1)} < \delta,$$

there exist a control $h \in L^2((0,T)\times(0,1))$ such that the solution of $(u,v)$ of (4) satisfying

$$\|u\|_{H^1(0,T;L^2(0,1))} + \|v\|_{L^\infty(0,T;H^1(0,1))} + \|h\|_{L^2((0,T)\times(0,1))} \le C$$

and

$$(u,v)(T,\cdot) = (u_\pm, v_\mp).$$

We focused on the following controlled parabolic system with the state functions $u \equiv u(t,x)$ and $v \equiv v(t,x)$

$$\begin{cases} \partial_t u = \Delta u - uv^2 - F(u-1) + \mathbb{1}_\omega h & \text{in } Q_T, \\ \partial_t v = \Delta v + uv^2 - (k+F)v & \text{in } Q_T, \\ \partial_\nu u = 0, \ \partial_\nu v = 0 & \text{on } \Sigma_T, \\ u(0,x) = u_0(x), \ v(0,x) = v_0(x) & x \in \Omega, \end{cases} \tag{CS}$$

- For $\Omega \subset \mathbb{R}^N (N = 1,2,3)$ and $T > 0$, $Q_T = (0,T) \times \Omega$, $\Sigma_T = (0,T) \times \partial\Omega$.
- Also, $\omega$ a non-empty open subset of $\Omega$.
- Define the space

$$V^1(Q_T) := \{y | y \in L^2(0,T; H^1(\Omega)), \ \partial_t y \in L^2(0,T; H^1(\Omega)^*)\}.$$

Consider the system

$$\begin{cases} \bar{u}_t = \Delta\bar{u} - \bar{u}\bar{v}^2 - F(\bar{u}-1) & \text{in } Q_T, \\ \bar{v}_t = \Delta\bar{v} + \bar{u}\bar{v}^2 - (k+F)\bar{v} & \text{in } Q_T, \\ \partial_\nu \bar{u} = 0,\ \partial_\nu \bar{v} = 0 & \text{on } \Sigma_T, \\ \bar{u}(0,x) = \bar{u}_0(x),\ \bar{v}(0,x) = \bar{v}_0(x) & x \in \Omega. \end{cases} \tag{trj}$$

Let $(\bar{\mathbf{u}}, \bar{\mathbf{v}})$ be positive trajectory of system (trj) corresponding to an initial data $(\bar{\mathbf{u}}_\mathbf{0}, \bar{\mathbf{v}}_\mathbf{0})$. Then we have to find neighbourhood $\mathcal{O}$ of $(\bar{\mathbf{u}}_\mathbf{0}, \bar{\mathbf{v}}_\mathbf{0})$ in the space $L^\infty(\Omega) \times L^\infty(\Omega)$ such that for any initial data $(\mathbf{u}_\mathbf{0}, \mathbf{v}_\mathbf{0}) \in \mathcal{O}$, there exist control $\mathbf{h}$ such that solution $(\mathbf{u}, \mathbf{v})$ of system (CS) satisfies

$$u(T,x) = \bar{u}(T,x),\ v(T,x) = \bar{v}(T,x), \quad x \in \Omega \ a.e.$$

**Theorem**

*Let* $k_0 > 0$ *and* $\bar{u}_0, \bar{v}_0 \in C(\overline{\Omega})$ *satisfying* $\bar{u}_0 \geq k_0$ *and* $\bar{v}_0 \geq k_0$ *for* $x \in \overline{\Omega}$*. Let* $(\bar{u}, \bar{v})$ *be a solution of system* (trj) *corresponding to the initial condition* $(\bar{u}_0, \bar{v}_0)$*, Then, there exists* $\delta_0 > 0$ *such that for each* $(u_0, v_0)$ *satisfying*

$$\|u_0 - \bar{u}_0\|_{L^\infty(\Omega)} + \|v_0 - \bar{v}_0\|_{L^\infty(\Omega)} \leq \delta_0,$$

*with* $u_0 \geq 0, v_0 \geq 0$*, there is a control* $h \in L^\infty(Q_T)$ *such that system* (CS) *admits a unique solution* $(u, v)$ *satisfying*

$$u, v \in V^1(Q_T) \cap L^\infty(Q_T),$$

*and* $u(T, x) = \bar{u}(T, x), \;\; v(T, x) = \bar{v}(T, x)$ *for almost all* $x \in \Omega$*.*

## Approach

- First, we show the existence of a positive trajectory $(\bar{u}, \bar{v})$ for positive initial data $(\bar{u}_0, \bar{v}_0)$.
- We linearize the non-linear system (CS) around the positive trajectory $(\bar{u}, \bar{v})$ and prove the global null-controllability of the linearized system. In this step, we use the duality approach to prove the null-controllability of the linear system.
- We use the **Carleman estimate** to prove the observability inequality.
- Finally, we use **Kakutani's fixed-point theorem** to achieve controllability to the trajectory $(\bar{u}, \bar{v})$ of the non-linear system (CS) .

We get the following well-posedness result for the solution of the system:

**Theorem**

*Let $\bar{u}_0, \bar{v}_0 \in C(\overline{\Omega})$, then there exists a unique classical solution $(\bar{u}, \bar{v})$ of the system (trj) such that*

$$\bar{u}, \bar{v} \in C([0,\infty); W^{1,q}(\Omega)) \cap C^{2,1}((0,\infty) \times \overline{\Omega}).$$

*Moreover, for $T > 0$ and $\bar{u}_0 \geq 0$, $\bar{v}_0 \geq 0$ in $\overline{\Omega}$, then, there exists $M(T) > 0$*

$$0 \leq \bar{u}(t,x) \leq M(T), \quad 0 \leq \bar{v}(t,x) \leq M(T), \qquad \forall\, (t,x) \in \overline{Q_T}.$$

*Furthermore, if $\bar{u}_0(x) \geq k_0$ and $\bar{v}_0(x) \geq k_0$ in $\overline{\Omega}$ for some $k_0 > 0$, then there exist $M_1(T) > 0$, such that for all $(t,x) \in \overline{Q_T}$*

$$k_0 e^{-(F+M^2(T))T} \leq \bar{u}(t,x) \leq M_1(T), \quad k_0 e^{-(F+M^2(T))T} \leq \bar{v}(t,x) \leq M_1(T).$$

- The linearized control system corresponding to the non-linear system (CS) around $(\bar{u}, \bar{v})$:

$$\begin{cases} w_t = \Delta w - Fw - \bar{v}^2 w - 2\bar{u}\bar{v}z + \mathbb{1}_\omega h & \text{in } Q_T, \\ z_t = \Delta z - (F+k)z + \bar{v}^2 w + 2\bar{u}\bar{v}z & \text{in } Q_T, \\ \frac{\partial w}{\partial \nu} = 0 = \frac{\partial z}{\partial \nu} & \text{on } \Sigma_T, \\ w(0,x) = w_0(x), \ z(0,x) = z_0(x) & x \in \Omega. \end{cases} \tag{5}$$

- The adjoint system corresponding to the linear system (5) is

$$\begin{cases} -\partial_t \phi = \Delta \phi - F\phi + \bar{v}^2 \phi + \bar{v}^2 \psi & \text{in} \quad Q_T, \\ -\partial_t \psi = \Delta \psi - (F+k)\psi + 2\bar{u}\bar{v}\psi - 2\bar{u}\bar{v}\phi & \text{in} \quad Q_T, \\ \partial_\nu \phi = 0, \ \partial_\nu \psi = 0 & \text{on} \quad \Sigma_T, \\ \phi(T,x) = \phi_T(x), \ \psi(T,x) = \psi_T(x) & x \in \Omega. \end{cases} \tag{6}$$

- To prove null controllability of the linearized system (5) is equivalent to prove following type observability inequality:

$$\|\phi(0,\cdot)\|^2_{L^2(\Omega)} + \|\psi(0,\cdot)\|^2_{L^2(\Omega)} \leq C \iint_{Q_T^\omega} |\phi|^2 \, dxdt.$$

- The idea to prove the above observability inequality for the system (6) is to derive a Carleman inequality for the system (6).
- Consider $\omega' \subset\subset \omega$ a nonempty open subset and a function $\alpha \in C^2(\overline{\Omega})$ such that

$$\alpha(x) > 0 \quad \text{for all} \quad x \in \Omega,$$
$$\alpha|_{\partial\Omega} = 0 \quad \text{and} \quad |\nabla\alpha(x)| > 0 \quad \text{for all} \quad x \in \overline{\Omega \setminus \omega'}.$$

- For $\lambda > 0$, we defined functions $\varphi, \beta : Q_T \to \mathbb{R}$ such that

$$\varphi(t,x) = \frac{e^{\lambda\alpha(x)}}{t(T-t)}, \quad \beta(t,x) = \frac{e^{\lambda\alpha(x)} - e^{2\lambda\|\alpha\|_{\infty,\overline{\Omega}}}}{t(T-t)}, \quad (t,x) \in Q_T$$

and

$$\theta(\lambda) = e^{2\lambda\|\alpha\|_{\infty,\overline{\Omega}}}.$$

## Lemma

*Let* $\nu =: \inf\{\overline{v}(t,x) | (t,x) \in (0,T) \times \omega\} > 0$. *Then, there exists a positive constant* $\lambda_1 = C(\Omega, \omega, \omega')(1 + \|\bar{u}\|_\infty^2 + \|\bar{v}\|_\infty^2 + k^2 + F^2) > 1$ *such that for any* $\lambda \geq \lambda_1$, *any* $s \geq \theta(\lambda)(T + T^2)$ *and* $\phi_T,\ \psi_T \in L^2(\Omega)$, *the corresponding solution* $(\phi, \psi)$ *of the equation (6) satisfies*

$$\iint_{Q_T} \left[ \lambda^4 (s\varphi)^5 |\phi|^2 + \lambda^4 (s\varphi)^3 |\psi|^2 \right] e^{2s\beta}\, dxdt \leq C_0 \iint_{Q_T^\omega} \lambda^9 (s\varphi)^7 e^{2s\beta} |\phi|^2\, dxdt.$$

*where* $C_0 = C_0(\Omega, \omega, \omega', \nu) > 0$.

## Proposition (Observability inequality)

*There exist* $\lambda,\ s > 0$ *such that, for all* $\phi_T, \psi_T \in L^2(\Omega)$, *the solution* $(\phi, \psi)$ *of (6) satisfies*

$$\|\phi(0,\cdot)\|_{L^2(\Omega)}^2 + \|\psi(0,\cdot)\|_{L^2(\Omega)}^2 \leq C \iint_{Q_T^\omega} e^{\frac{3}{2}s\beta} |\phi|^2\, dxdt, \qquad (7)$$

*for some* $C > 0$.

## Theorem

*For any* $(w_0, z_0) \in L^2(\Omega) \times L^2(\Omega)$*, there exist a control* $h \in L^\infty(Q_T)$ *such that the solution* $(w, z)$ *of system* (5) *corresponding to h satisfies* $w(T, x) = z(T, x) = 0$ *for almost all* $x \in \Omega$*. Moreover the control h satisfies*

$$\|h\|_{L^\infty(Q_T^\omega)} \leq C\Big( \|w_0\|_{L^2(\Omega)} + \|z_0\|_{L^2(\Omega)} \Big)$$

*for some* $C > 0$.

- At first, we get the following $L^2$-estimate

$$\|h\|^2_{L^2(Q_T^\omega)} \leq C_1 \Big( \|w_0\|^2_{L^2(\Omega)} + \|z_0\|^2_{L^2(\Omega)} \Big).$$

  After this, using a bootstrap argument with Sobolev embedding results to get $L^\infty$-estimate.

**Theorem (Kakutani's fixed point theorem )**

*Let $K$ be a compact convex subset of Banach space $Y$ and let $G : K \to X$ be an upper semi-continuous mapping(set-valued) with convex values $G(x)$ such that $G(x) \subset K$ for each $x \in K$. Then there is at least one $x \in K$ such that $x \in G(x)$.*

- Let $(\bar{u}, \bar{v})$ be a trajectory of the system (trj) corresponding to $(\bar{u}_0, \bar{v}_0)$, satisfying $\bar{u}_0 \geq k_0$ and $\bar{v}_0 \geq k_0$. We set $w = u - \bar{u}, z = v - \bar{v}$ with $w_0 = u_0 - \bar{u}_0$ and $z_0 = v_0 - \bar{v}_0$.
- Then easy to see that $(w, z)$ satisfies the control system

$$\begin{cases} \partial_t w = \Delta w - lw - w(z + \bar{v})^2 - (\bar{u}z + 2\bar{u}\,\bar{v})z + \mathbb{1}_\omega h & \text{in } Q_T, \\ \partial_t z = \Delta z - (l + k)z + w(z + \bar{v})^2 + (\bar{u}z + 2\bar{u}\,\bar{v})z & \text{in } Q_T, \\ \partial_\nu w = 0, \partial_\nu z = 0 & \text{on } \Sigma_T, \\ w(0, x) = w_0(x), z(0, x) = z_0(x) & x \in \Omega, \end{cases} \tag{8}$$

- So, local exact controllability to the trajectories of system (CS) is equivalent to the local null controllability of the system (8).

- Let $R > 0$, and consider set

$$\mathcal{K} = \left\{ \eta \in L^\infty(Q_T) \ : \ \|\eta\|_{\infty, Q_T} \leq R \right\}. \tag{9}$$

Then, for each $\eta \in \mathcal{K}$, we consider solution of the following linear control system:

$$\begin{cases} \partial_t w = \Delta w - lw - a(\eta)w - b(\eta)z + \mathbb{1}_\omega h & \text{in } Q_T, \\ \partial_t z = \Delta z - (l+k)z + a(\eta)w + b(\eta)z & \text{in } Q_T, \\ \partial_\nu w = 0, \partial_\nu z = 0 & \text{on } \Sigma_T, \\ w(0,x) = w_0(x), z(0,x) = z_0(x) & x \in \Omega, \end{cases} \tag{10}$$

where $a(\eta) = (\eta + \overline{v})^2$, $b(\eta) = (\overline{u}\eta + 2\overline{u}\,\overline{v})$.

- For $\eta \in \mathcal{K}$, let us define a set-valued map $\Lambda : \mathcal{K} \to 2^{L^2(Q_T)}$ by

$$\Lambda(\eta) = \left\{ z \in L^2(Q_T) \;\middle|\; \begin{array}{l} \text{There exists } h_\eta \in L^\infty(Q_T) \text{ such that} \\ \text{the solution } (w_\eta, z_\eta) \text{ of the equation (10) corresponding} \\ \text{to } \eta \text{ and } h_\eta \text{ satisfies } w_\eta(T,x) = z_\eta(T,x) = 0. \end{array} \right\} \tag{11}$$

- We proved that $\Lambda$ is an upper semi-continuous, $\Lambda(\eta)$ is convex and $\Lambda(\mathcal{K}) \subset \mathcal{K}$. So, $\Lambda$ has a fixed point which gives the required result.

[1] M Bapu and M Biswas, *Local controllability to the trajectories of a Gray-Scott system*, **submitted** .

[2] V Barbu. *Controllability and stabilization of parabolic equations*. Springer, 2018.

[3] A V Fursikov and O Yu Imanuvilov, *Controllability of evolution equations*, lecture notes series 34. Research Institute of Mathematics, Global Analysis Research Center, Seoul National University, 1996.

[4] Víctor Hernández- Santamaría and Kévin Le Balc'h. *Local controllability of the one-dimensional nonlocal Gray-Scott model with moving controls.* J. Evol. Equ., 21(4):4539–4574, 2021

[5] B Z Guo and L Zhang, *Local exact controllability to the positive trajectory for a parabolic system of chemotaxis*, Math. Control Relat. Fields, 6(1):143– 165, 2016.

Thank You!