Optimal Boundary Control for the Transport Equation under Uncertainty: A Turnpike Result

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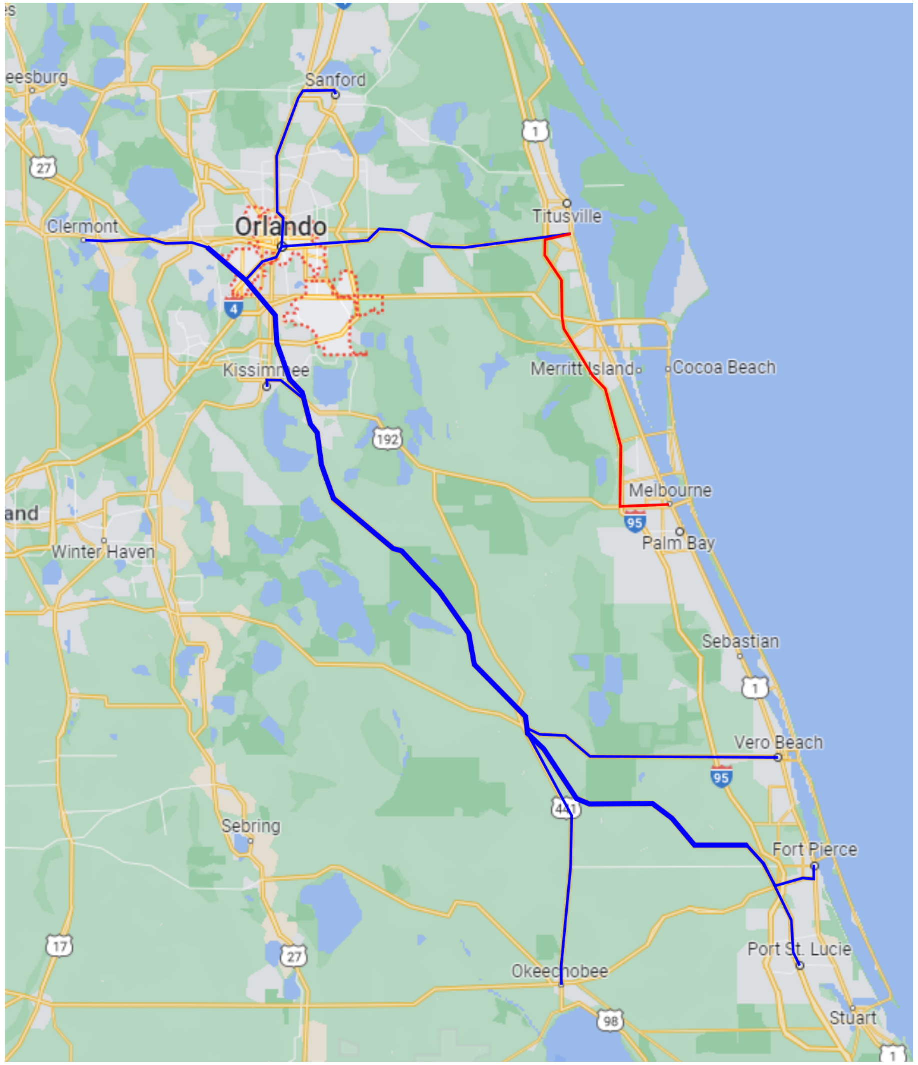
Motivation

The Turnpike Phenomenon



- There is a fastest route between any two points and if the origin and destination are close together and far from the turnpike, the best route may not touch the turnpike.
- But if origin and destination are far enough apart, it will always pay to get on to the turnpike and cover distance at the best rate of travel, even if this means adding a little mileage at either end.

[Dorfman, Samuelson, Solow, 1958]: Linear Programming and Economic Analysis. *New York: McGraw-Hill*

Motivation

The Turnpike Phenomenon

Time dependent optimal control

$$\begin{aligned} \min_{u^\delta \in L^\infty(0,T;\mathbb{R}^m)} \quad & \int_0^T f_0(x(t), u^\delta(t))\, dt \\ \text{s.t.} \quad & x'(t) = A\,x + B\,u^\delta \\ & x(0) = x_0, \quad x(T) = x_1 \end{aligned}$$

Static state optimal control

$$\begin{aligned} \min_{u^\sigma \in \mathbb{R}^m} \quad & f_0(x, u^\sigma) \\ \text{s.t.} \quad & A\,x + B\,u^\sigma = 0 \end{aligned}$$

Motivation

The Turnpike Phenomenon

Time dependent optimal control

$$\min_{u^\delta \in L^\infty(0,T;\mathbb{R}^m)} \int_0^T f_0(x(t), u^\delta(t))\, dt$$

$$\text{s.t.} \quad x'(t) = A\,x + B\,u^\delta$$

$$x(0) = x_0, \quad x(T) = x_1$$

Static state optimal control

$$\min_{u^\sigma \in \mathbb{R}^m} f_0(x, u^\sigma)$$

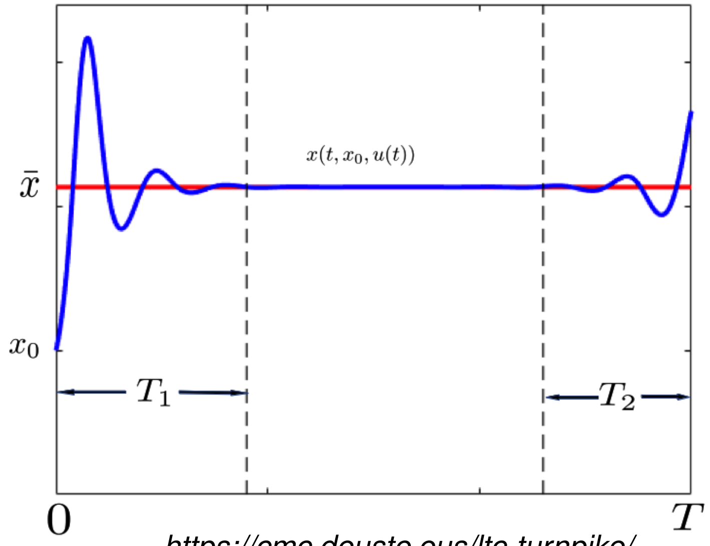
$$\text{s.t.} \quad A\,x + B\,u^\sigma = 0$$

Theorem: Turnpike result

There exist constants $C_1 > 0$ and $C_2 > 0$, s.t. for every time $T > 0$ the time dependent optimal control problem has a unique solution $(u^{\delta,*}, x^{\delta,*})$, which satisfies

$$\|u^{\delta,*}(t) - u^{\sigma,*}\| + \|x^{\delta,*}(t) - x^{\sigma,*}\| \le C_1 \Big(\exp\big(-C_2 t\big) + \exp\big(-C_2(T-t)\big)\Big),$$

where $(u^{\sigma,*}, x^{\sigma,*})$ is the (unique) solution of the static problem.



https://cmc.deusto.eus/ltc-turnpike/

A. Porretta, E. Zuazua (2013): *Long Time versus Steady State Optimal Control.* SIAM J. Control Optim. 51(6), 4242–4273

E. Trélat, E. Zuazua (2015): *The Turnpike Property in finite-dimensional nonlinear optimal control.* J. Differential Equations 258, pp. 81–114

A Turnpike Result for the Transport Eq.

Deterministic Optimal Control

For $c > 0$ consider the transport equation in one dimension

$$r_t(t,x) + c\, r_x(t,x) = m\, r(t,x),$$

with initial condition and boundary control

$$r(0,x) = r_{\text{ini}}(x) \qquad \text{and} \qquad r(t,0) = u(t).$$

For convex functions f and g consider the optimal control problems

Dynamic Optimal Control Problem

$$\begin{aligned} \min_{u \in L^2(0,T)} \quad & J_T(u) = \int_0^T f\big(u(t)\big) + g\big(r(t,L)\big)\, dt \\ \text{s.t.} \quad & r_t(t,x) + c r_x(t,x) = m\, r(t,x), \\ & r(0,x) = r_{\text{ini}}(x), \\ & r(t,0) = u(t). \end{aligned}$$

Static Optimal Control Problem

$$\begin{aligned} \min_{u \in \mathbb{R}} \quad & J(u) = f\big(u\big) + g\big(r(L)\big) \\ \text{s.t.} \quad & c r_x(x) = m\, r(x), \\ & r(0) = u. \end{aligned}$$

A Turnpike Result for the Transport Eq.

Deterministic Optimal Control

(A1) For $\varepsilon > 0$ let functions f and g satisfy

$$\big(f'(x_1) - f'(x_2)\big)\big(x_1 - x_2\big) + \big(g'(y_1) - g'(y_2)\big)\big(y_1 - y_2\big) \;\geq\; \varepsilon \, \|x_1 - x_2\|_2^2.$$

(A2) Let the derivative of g be Lipschitz continuous with Lipschitz constant L_k, i.e.,

$$\|g'(y_1) - g'(y_2)\|_2 \;\leq\; L_k \, \|y_1 - y_2\|_2.$$

A Turnpike Result for the Transport Eq.

Deterministic Optimal Control

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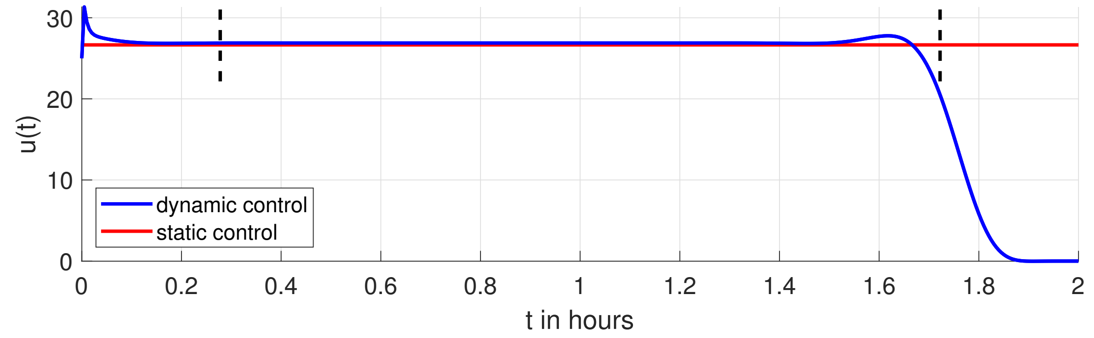
Theorem: Deterministic Turnpike

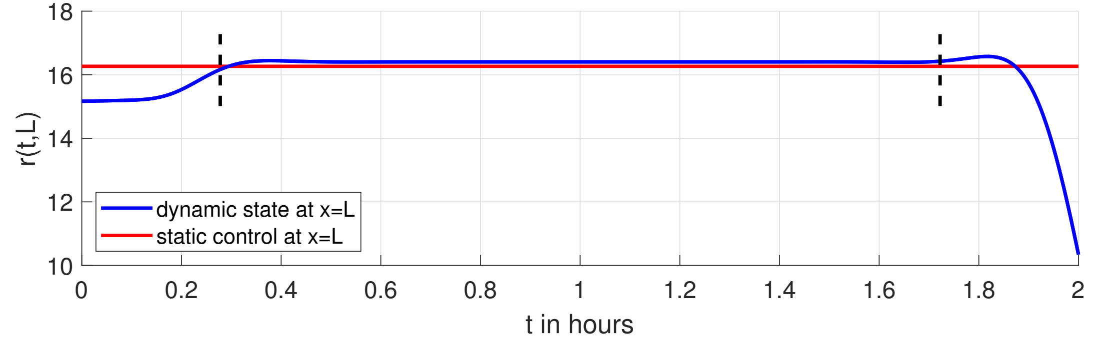
The optimal solution $u^\delta(t)$ of the dynamic optimal control problem

$$\begin{aligned} \min_{u\in L^2(0,T)} \quad & J_T(u) = \int_0^T f\big(u(t)\big) + g\big(r(t,L)\big)\,dt \\ \text{s.t.} \quad & r_t(t,x) + c r_x(t,x) = m\,r(t,x), \\ & r(0,x) = r_{\text{ini}}(x),\; r(t,0) = u(t). \end{aligned}$$

and the optimal solution u^σ of the corresponding static problem satisfy the integral turnpike property

$$\int_0^T \|u^\delta(t) - u^\sigma\|_2^2\,dt \;\leq\; \mathcal{C}.$$





A Turnpike Result for the Transport Eq.

Deterministic Optimal Control

Sketch of the proof.

Part I: The solution of the transport equation is given by

$$r(t,x) = \begin{cases} \exp(\ mt\ )\, r_{\text{ini}}(\ x - ct\ ) & x > ct \\ \exp\left(\ m\frac{x}{c}\ \right) u\left(\ t - \frac{x}{c}\ \right) & x \leq ct \end{cases}$$

Part II: The derivative of the objective function is given by

$$J_T'(u) = f'\big(u(t)\big) + k\, g'\big(k\, u(t)\big)\, \psi(t),$$

with $k = \exp\left(m\frac{L}{c}\right)$ and $\psi(t) = \begin{cases} 1 & 0 < t < T - \frac{L}{c} \\ 0 & \text{else} \end{cases}$.

Part III: Let $u^\delta(t)$ and u^σ be the optimal dynamic and static solution. Then we have necessary optimality condition

$$f'\big(u^\delta(t)\big) - f'\big(u^\sigma\big) = k\, g'\big(k\, u^\sigma\big) - k\, g'\big(k\, u^\delta(t)\big)\, \psi(t).$$

A Turnpike Result for the Transport Eq.

Deterministic Optimal Control

Sketch of the proof.

Part IV: Starting from assumption (A1) we have

$$\varepsilon \int_0^T \|u^\delta(t) - u^\sigma\|_2^2 \, dt \le \int_0^T \Big(f'(u^\delta(t)) - f'(u^\sigma)\Big)\Big(u^\delta(t) - u^\sigma\Big) + \Big(g'(r^\delta(t,L)) - g'(r^\sigma(L))\Big)\Big(r^\delta(t,L) - r^\sigma(L)\Big) \, dt$$

$\vdots$

$\vdots$ (apply necessary optimality conditions, use integration by substitution)

$\vdots$

$$\varepsilon \int_0^T \|u^\delta(t) - u^\sigma\|_2^2 \, dt \le \int_{T-\frac{L}{c}}^T k \, g'(k \, u^\sigma)\Big(u^\delta(t) - u^\sigma\Big) \, dt$$

$$+ \int_0^{\frac{L}{c}} \Big(g'\big(\exp(mt) r_{\text{ini}}(L - ct)\big) - g'(ku^\sigma)\Big)\Big(\exp(mt) r_{\text{ini}}(L - ct) - ku^\sigma\Big) \, dt$$

$\vdots$

$\vdots$ (apply Cauchy-Schwarz inequality and (A2))

$\vdots$

$$\varepsilon \, \|u^\delta(t) - u^\sigma\|_{L^2(0,T)}^2 \le z_1 \, \|u^\delta(t) - u^\sigma\|_{L^2(0,T)} + \frac{L_k}{c} z_2^2 \, \|r_{\text{ini}}(x)\|_{L^2(0,L)}^2 + z_3 \qquad \square$$

A Turnpike Result for the Transport Eq.

Optimal Control under Uncertainty

Assume that the initial data is now perturbed by a *Wiener process*, i.e., we have

$$r_{\text{ini}}^{\omega}(x) = r_{\text{ini}}(x) + W_x \qquad \text{with} \qquad W_x = \sqrt{2\,L}\sum_{k=1}^{\infty} \xi_k \, \frac{\sin\left(\left(k-\frac{1}{2}\right)\pi \frac{x}{L}\right)}{\left(k-\frac{1}{2}\right)\pi}.$$

A Turnpike Result for the Transport Eq.

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Theorem: TP with uncertain initial data

The optimal solution $u^{\delta}(t)$ of the dynamic optimal control problem

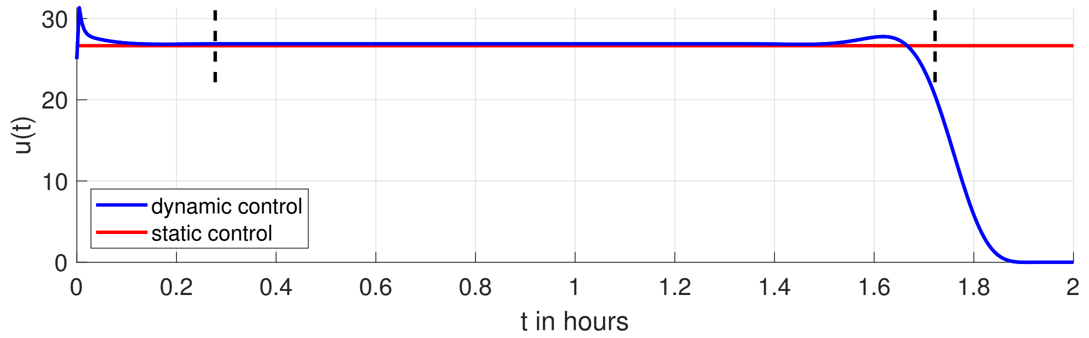
$$\min_{u\in L^2(0,T)} \quad J_T(u) = \int_0^T f\big(u(t)\big) + g\big(\mathbb{E}[r(t,L)]\big)\,dt$$

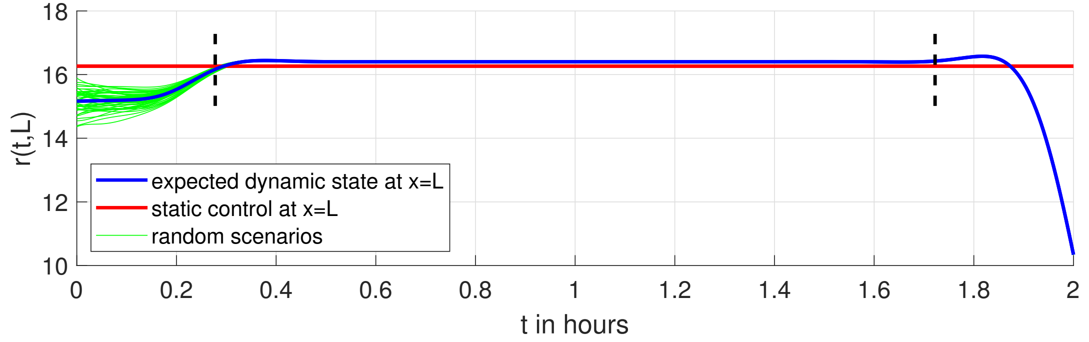
s.t. $\quad r_t(t,x) + c r_x(t,x) = m\,r(t,x),$

$\quad r(0,x) = r^{\omega}_{\text{ini}}(x), \; r(t,0) = u(t).$

and the optimal solution u^{σ} of the corresponding static problem satisfy the integral turnpike property

$$\int_0^T \|u^{\delta}(t) - u^{\sigma}\|_2^2\,dt \;\le\; \mathcal{C}.$$





A Turnpike Result for the Transport Eq.

Optimal Control under Uncertainty

We randomize the source term by a random variable ξ on an appropriate probability space: $m^\omega := \xi(\omega), \quad \omega \in \Omega$

(A3) Assume that $e_0(t)$ is uniformly bounded, where e_0 is defined as

$$e_0 : [0,T] \to \mathbb{R} \cup \{\pm\infty\} \quad t \mapsto \int_{-\infty}^{\infty} \exp(zt)\, \varrho_\xi(z)\, dz,$$

(A4) Assume that

$$\int_{-\infty}^{\infty} \exp\left(z\frac{L}{c}\right) \varrho_\xi(z)\, dz \quad < \infty$$

A Turnpike Result for the Transport Eq.

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Theorem: TP with randomized source term

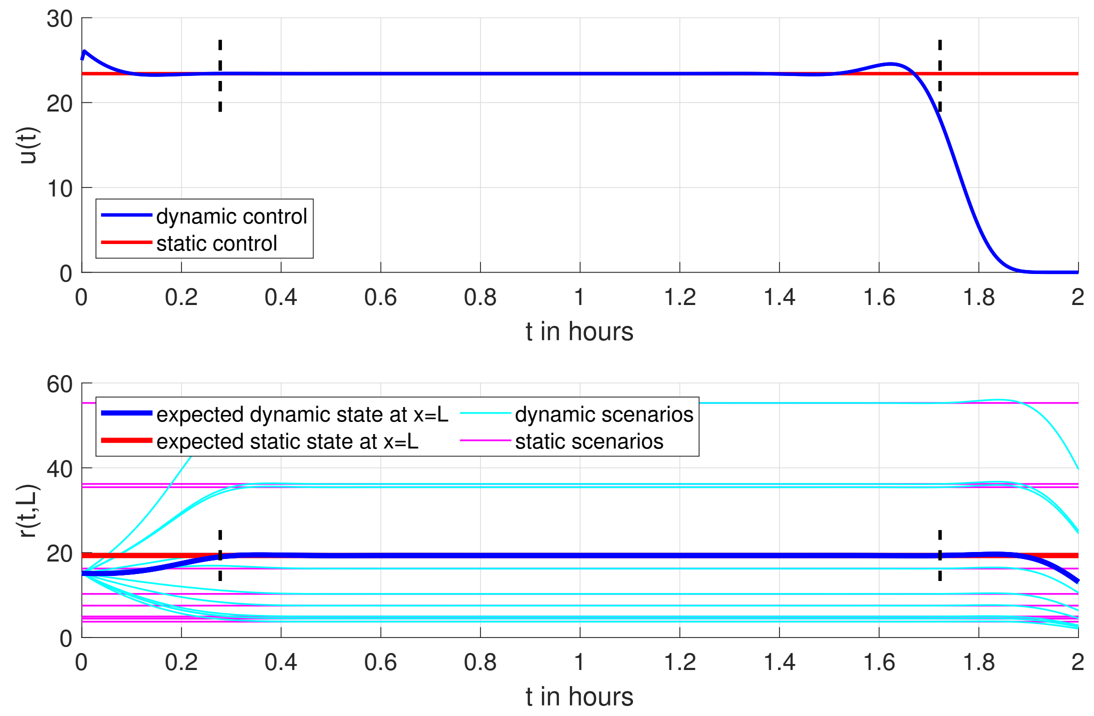
The optimal solution $u^\delta(t)$ of the dynamic optimal control problem

$$\min_{u \in L^2(0,T)} \quad J_T(u) = \int_0^T f\big(u(t)\big) + g\big(\mathbb{E}[r(t,L)]\big)\, dt$$

$$\text{s.t.} \quad r_t(t,x) + c r_x(t,x) = m^\omega\, r(t,x),$$
$$r(0,x) = r_{\text{ini}}(x), \;\; r(t,0) = u(t).$$

and the optimal solution u^σ of the corresponding static problem satisfy the integral turnpike property

$$\int_0^T \|u^\delta(t) - u^\sigma\|_2^2\, dt \;\leq\; \mathcal{C}.$$



A Turnpike Result for the Transport Eq.

Optimal Control under Uncertainty

Sketch of the proof.

Part I: Compute the solution $r^\omega(t,x)$ of the transport equation

Part II: Compute the expected value in the objective function:
For a random variable X and a function h we have $\mathbb{E}\big(h(X)\big) = \int_{-\infty}^{\infty} h(z)\, \varrho_X(z)\, dz$. Thus we have

$$\begin{aligned} J_T(u) &= \int_0^T f(u(t))\, dt + \int_0^T g\Big(\mathbb{E}\big(r^\omega(t,x)\big)\Big)\, dt \\ &= \int_0^T f(u(t))\, dt + \int_0^{\frac{L}{c}} g\Big(r_{\text{ini}}(L-ct) \int_{-\infty}^{\infty} \exp(zt)\, \varrho_\xi(z)\, dz\Big)\, dt + \int_{\frac{L}{c}}^T g\Big(u\Big(t-\frac{L}{c}\Big) \int_{-\infty}^{\infty} \exp\Big(z\frac{L}{c}\Big)\, \varrho_\xi(z)\, dz\Big)\, dt \\ &= \int_0^T f(u(t))\, dt + \int_0^{\frac{L}{c}} g\big(e_0(t)\, r_{\text{ini}}(L-ct)\big)\, dt + \int_{\frac{L}{c}}^T g\Big(e_1\, u\Big(t-\frac{L}{c}\Big)\Big)\, dt \end{aligned}$$

Part III: Compute the derivative of the objective function

Part IV: State the necessary optimality conditions

Part V: Get the Turnpike estimate by applying (A1) and (A2) $\square$

A Turnpike Result for the Transport Eq.

Non-Autonomous and Space-Variant Tp Eq

Consider the optimal control problem governed by an non-autonomous and space-variant transport equation

$$\begin{aligned} \min_{u \in L^2(0,T)} \quad & J_T(u) = \int_0^T f\big(u(t)\big) + g\big(r(t,L)\big)\, dt \\ \text{s.t.} \quad & r_t(t,x) + c r_x(t,x) = m(x)\, r(t,x) + b(x), \\ & r(0,x) = r_{\text{ini}}(x), \\ & r(t,0) = u(t), \end{aligned}$$

and its corresponding static problem

$$\begin{aligned} \min_{u \in \mathbb{R}} \quad & J(u) = f\big(u\big) + g\big(r(L)\big) \\ \text{s.t.} \quad & c r_x(x) = m(x)\, r(x) + b(x), \\ & r(0) = u. \end{aligned}$$

$\Rightarrow$ We can get an integral Turnpike result following the proof of the previous results.

A Turnpike Result for the Transport Eq.

Space- and Time-Variant Transport Equation

Consider the optimal control problem governed by an non-autonomous, space- and time-variant transport equation

$$\begin{aligned} \min_{u \in L^2(0,T)} \quad & J_T(u) = \int_0^T f\big(u(t)\big) + g\big(r(t,L)\big)\, dt \\ \text{s.t.} \quad & r_t(t,x) + c r_x(t,x) = m(t,x)\, r(t,x) + b(t,x), \\ & r(0,x) = r_{\text{ini}}(x), \\ & r(t,0) = u(t), \end{aligned}$$

$\Rightarrow$ Numerical simulations also provide a turnpike result, but the corresponding stationary problem is not clear.

A Turnpike Result for the Transport Eq.

Linear Feedback Control

Consider the optimal control problem governed by the transport equation with feedback control

$$\begin{aligned} \min_{u \in L^2(0,T)} \quad & J_T(u) = \int_0^T f\big(u(t)\big) + g\big(r(t,L)\big)\, dt \\ \text{s.t.} \quad & r_t(t,x) + c r_x(t,x) = m\, r(t,x), \\ & r(0,x) = r_{\text{ini}}(x), \\ & r(t,0) = u(t) + r(t,L), \end{aligned}$$

and its corresponding static problem

$$\begin{aligned} \min_{u \in \mathbb{R}} \quad & J(u) = f\big(u\big) + g\big(r(L)\big) \\ \text{s.t.} \quad & c r_x(x) = m\, r(x), \\ & r(0) = u + r(L). \end{aligned}$$

$\Rightarrow$ Numerical simulations also provide a turnpike result, but the proof is not clear

A Turnpike Result for the Transport Eq.

A Semi-Linear Transport Equation

Consider an optimal boundary control problem with the transport equation nonlinear source term:

$$\begin{aligned} \min_{u \in L^2(0,T)} \quad & J_T(u) = \int_0^T f\big(u(t)\big) + g\big(r(t,L)\big)\, dt \\ \text{s.t.} \quad & r_t(t,x) + c r_x(t,x) = h\big(r(t,x)\big), \\ & r(0,x) = r_{\text{ini}}(x), \\ & r(t,0) = u(t). \end{aligned}$$

A solution of the transport equation with nonlinear source term is given by

$$r(t,x) = \begin{cases} G^{-1}\Big(\, t + G\big(\, r_{\text{ini}}(x-ct)\,\big)\Big) & x > c\,t, \\ G^{-1}\Big(\, \frac{x}{c} + G\big(\, u(t-\frac{x}{c})\big)\,\Big) & x \le c\,t, \end{cases}$$

where G is an anti-derivative of $1/h$.

$\Rightarrow$ Numerical simulations also provide a turnpike result, but the proof is not clear.

Optimal Boundary Control for the Transport Equation under Uncertainty

[Sakamoto and Schuster, 2024]: *A Turnpike Result for Optimal Boundary Control Problems with the Transport Equation under Uncertainty.* Preprint https://opus4.kobv.de/opus4-trr154/frontdoor/index/index/docId/509