Exponential average turnpike property with average observation

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Overview

Random differential equation

Model

Let us consider the probability space $(\Omega, \mathcal{F}, \mu)$ with $\omega \in \Omega$, $A(\omega) \in \mathcal{L}(\mathbb{R}^n)$ and $B(\omega) \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$. Consider the RDE with random coefficients

$$\begin{cases} x_t(t,\omega) + A(\omega)x(t,\omega) = B(\omega)u(t), \\ x(t_0) = x_0 \in \mathbb{R}^n, \end{cases}$$

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with $u(t) \in \mathbb{R}^n$ independent of ω.

Main Questions:

1. How do we define an optimal control problem in this context?

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Main Questions:

1. How do we define an optimal control problem in this context?
2. Is it possible to prove the turnpike property when x^T and $\bar{x}$ are random trajectories?
3. What is the meaning of the turnpike property in this context?

Problem formulation

In the following

$$L^2(\Omega;\mathbb{R}^n) := \left\{ x : \Omega \to \mathbb{R}^n \ : \ \mathbb{E}[\|x(\cdot)\|^2_{\mathbb{R}^n}] = \int_\Omega \|x(\omega)\|^2_{\mathbb{R}^n} d\mu(\omega) < \infty \right\}.$$

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We consider the following evolutive optimal control problem with averaged observations

$$\min_{u\in L^2(0,T;\mathbb{R}^m)} \left\{ J^T(u) = \frac{1}{2}\int_0^T \left(\|u(t)\|^2_{\mathbb{R}^m} + \|\mathbb{E}[C(\cdot)x(t,\cdot)] - z\|^2_{\mathbb{R}^n} \right) dt \right.$$
$$\left. + \langle x(T,\cdot), \varphi_T(\cdot)\rangle_{L^2(\Omega;\mathbb{R}^n)} \right\},$$

with $C(\omega) \in \mathcal{L}(\mathbb{R}^n)$ and $x = x(t,\omega) \in \mathbb{R}^n$ solving

$$\begin{cases} x_t(t,\omega) + A(\omega)x(t,\omega) = B(\omega)u(t), \\ x(t_0) = x_0. \end{cases}$$

Problem formulation

Also, consider the following minimization stationary problem

$$\min_{u\in\mathbb{R}^m}\left\{J^s(u)=\frac{1}{2}\left(\|u\|^2_{\mathbb{R}^m}+\|\mathbb{E}[C(\cdot)x(\cdot)]-z\|^2_{\mathbb{R}^n}\right)\right\},$$

with $x(\omega)$ the solution of $A(\omega)x(\omega)=B(\omega)u$.

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$$\|x^T(t) - \overline{x}\|_{L^2(\Omega;\mathbb{R}^n)} + \|u^T(t) - \overline{u}\|_{\mathbb{R}^m} \leq \mathscr{C}(e^{-\delta(T-t)} + e^{-\delta t}),$$

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for all $t \in (0, T)$. In particular, the previous inequality implies

$$\|\mathbb{E}(x^T(t)) - \mathbb{E}(\overline{x})\|_{\mathbb{R}^n} + \|u^T(t) - \overline{u}\|_{\mathbb{R}^m} \leq \mathscr{C}(e^{-\delta(T-t)} + e^{-\delta t}),$$

for all $t \in (0, T)$.

Optimality conditions for the evolutionary system

Assume that $A, C \in C(\Omega; \mathcal{L}(\mathbb{R}^n))$ and $B \in C(\Omega; \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n))$, and are uniformly bounded.

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Proposition:

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Proposition:

There exist a unique optima control $u^T \in L^2(0, T; \mathbb{R}^m)$ for the evolutive problem, and unique optimal state x^T associated to u^T. Furthermore,

$$u^T(t) = -\mathbb{E}[B^* \varphi^T(t, \cdot)],$$

where φ^T solves

$$\begin{cases} -\varphi_t(t,\omega) + A^*(\omega)\varphi(t,\omega) = C^*(\omega)\left(\mathbb{E}[C(\cdot)x^T(t,\cdot)] - z\right), & t > 0, \\ \varphi(T,\omega) = \varphi_T(\omega), \end{cases}$$

Main results

Necessary hypotheses

Motivated by the notions of exponentially stabilizable and detectable, we assume two hypotheses

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Assumption 1: There exists a feedback operator $K_C \in C^0(\Omega, \mathcal{L}(\mathbb{R}^n))$ uniformly bounded and $\alpha_C > 0$ such that

$$(Av + \mathbb{E}[K_C C v], v)_{L^2(\Omega;\mathbb{R}^n)} \geq \alpha_C \|v\|^2_{L^2(\Omega;\mathbb{R}^n)},$$

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for every $v \in L^2(\Omega; \mathbb{R}^n)$.

Assumption 2: There exists a feedback operator $K_B \in C^0(\Omega, \mathcal{L}(\mathbb{R}^m; \mathbb{R}^n))$ uniformly bounded and $\alpha_B > 0$ such that

$$(A^* v + \mathbb{E}[K_B B^* v], v)_{L^2(\Omega;\mathbb{R}^n)} \geq \alpha_B \|v\|^2_{L^2(\Omega;\mathbb{R}^n)},$$

for every $v \in L^2(\Omega; \mathbb{R}^n)$.

Energy estimation

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$$\|x(t,\cdot)\|^2_{L^2(\Omega;\mathbb{R}^n)} \leq K_1 \int_0^t \left(\|u(s)\|^2_{\mathbb{R}^m} + \|\mathbb{E}[C(\cdot)x(s,\cdot)]\|^2_{\mathbb{R}^n} \right) ds + \|x_0(\cdot)\|^2_{L^2(\Omega;\mathbb{R}^n)},$$

and

$$\|\varphi(t,\cdot)\|^2_{L^2(\Omega;\mathbb{R}^n)} \leq K_2 \int_t^T \left(\|\mathbb{E}[B^*(\cdot)\varphi(s,\cdot)]\|^2_{\mathbb{R}^m} + \|\mathbb{E}[C(\cdot)x(s,\cdot)] - z\|^2_{\mathbb{R}^n} \right) ds + \|\varphi_T(\cdot)\|^2_{L^2(\Omega;\mathbb{R}^n)}.$$

Also, we have the stationary inequalities

$$\|v\|_{L^2(\Omega:\mathbb{R}^n)} \leq K_3 (\|Av\|_{L^2(\Omega;\mathbb{R}^n)} + \|\mathbb{E}[Cv]\|_{\mathbb{R}^n}),$$

and

$$\|v\|_{L^2(\Omega:\mathbb{R}^n)} \leq K_4 (\|A^*v\|_{L^2(\Omega;\mathbb{R}^n)} + \|\mathbb{E}[B^*v]\|_{\mathbb{R}^n}),$$

Well-posedness of the stationary problem

Theorem (Uniqueness, existence and characterization of minimizer)

Under the assumptions **1** and **2**, there exists a unique optimal pair $(\overline{x}, \overline{u})$. Moreover, $\overline{u}$ can be characterized as

$$\overline{u} = -\mathbb{E}[B^*(\cdot)\overline{\varphi}(\cdot)],$$

where $\overline{\varphi}$ is the solution of

$$A^*(\omega)\overline{\varphi} = C^*(\omega)\left(\mathbb{E}[C(\cdot)\overline{x}(\cdot)] - z\right), \tag{1}$$

Main results

Theorem (Exponential Average turnpike property)

Under the assumptions **1** and **2**, there exist two constants $\mathscr{C}, \delta > 0$ independent of T such that

$$\|x^T(t,\cdot) - \overline{x}(\cdot)\|_{L^2(\Omega;\mathbb{R}^n)} + \|\varphi^T(t,\cdot) - \overline{\varphi}(\cdot)\|_{L^2(\Omega;\mathbb{R}^n)} + \|u^T(t) - \overline{u}\|_{\mathbb{R}^m} \leq \mathscr{C}(e^{-\delta(T-t)} + e^{-\delta t}), \quad (2)$$

for every $t \in [0, T]$,

Proof (Sketch)

L. Grüne and M. Schaller and A. Schiela (2019).
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$$\underbrace{\begin{pmatrix} -C^*(\omega)\mathbb{E}[C(\cdot)\cdot] & -\frac{d}{dt} + A^*(\omega) \\ 0 & E_T \\ \frac{d}{dt} + A(\omega) & B(\omega)\mathbb{E}[B^*(\cdot)\cdot] \\ E_0 & 0 \end{pmatrix}}_{\Lambda} \underbrace{\begin{pmatrix} m \\ n \end{pmatrix}}_{z} = \underbrace{\begin{pmatrix} 0 \\ n_T \\ 0 \\ m_0 \end{pmatrix}}_{y},$$

where $E_0 m := m(0)$ y $E_T n := n(T)$.

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$$\Lambda \mathcal{Z} = \mathcal{Y}.$$

We prove that Λ^{-1} is well defined and that there exists $K > 0$ independent of the time horizon such that

$$\|\Lambda^{-1}\|_{\mathcal{L}((L^2(\Omega;\mathbb{R}^n))^2,(\mathcal{X})^2)} < K,$$

where $\mathcal{X} = C([0, T]; L^2(\Omega; \mathbb{R}^n))$.

Proof (Sketch)

- **Step 2:** We consider a new variable change

$$\hat{m} = \frac{m}{e^{-\delta(T-t)} + e^{-\delta t}}, \quad \hat{n} = \frac{n}{e^{-\delta(T-t)} + e^{-\delta t}},$$

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and we prove that there exist $K > 0$ independent of the time horizon such that

$$\|\hat{m}\|_{\mathcal{X}} + \|\hat{n}\|_{\mathcal{X}} \leq K,$$

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$$\|\hat{m}(t)\|_{L^2(\Omega,\mathbb{R}^n)} + \|\hat{n}(t)\|_{L^2(\Omega,\mathbb{R}^n)} \leq \|\hat{m}\|_{\mathcal{X}} + \|\hat{n}\|_{\mathcal{X}} \leq K,$$

$\mathcal{X} = C([0,T]; L^2(\Omega;\mathbb{R}^n))$. We conclude the proof by returning to the original variables.

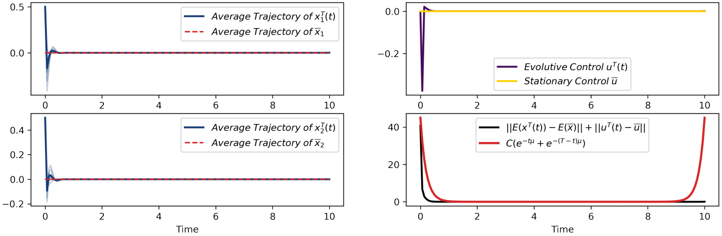
Numerical Simulations and Comments

Numerical Simulations

We consider

$$A(\omega) = \alpha(\omega)\begin{pmatrix} 2 & -5 \\ 5 & 0.1 \end{pmatrix}, \quad B(\omega) = \beta(\omega)\begin{pmatrix} 5 \\ 7 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad z = \begin{pmatrix} 4 \\ 4 \end{pmatrix}.$$

Using Gekko library in Python, we obtain the following simulations



Comments

In the paper [1] was shown that the "simultaneous" OCP

$$\min_{u\in L^2(0,T;\mathbb{R}^m)} \Big\{ J^T(u) = \frac{1}{2}\int_0^T \Big(\|u(t)\|^2_{\mathbb{R}^m} + \mathbb{E}\big[\|C(\cdot)x(t,\cdot) - z\|^2_{\mathbb{R}^n}\big]\Big)dt + \langle x(T,\cdot), \varphi_T(\cdot)\rangle_{L^2(\Omega;\mathbb{R}^n)} \Big\},$$

subject to $x = x(t,\omega) \in \mathbb{R}^n$ solving

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(and the associated stationary system) satisfies

$$\|x^T(t) - \overline{x}\|_{L^2(\Omega;\mathbb{R}^n)} + \|u^T(t) - \overline{u}\|_{\mathbb{R}^m} \leq \mathscr{C}(e^{-\delta(T-t)} + e^{-\delta t}), \quad t \in (0,T).$$

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[1] M. Hernández and R. Lecaros and S. Zamorano (2023). Averaged turnpike property for differential equations with random constant coefficients. *Mathematical Control and Related Fields*.

Numerical Comparison with the Current Work

Using the same matrices for the "average" and "simultaneous" observation OCP

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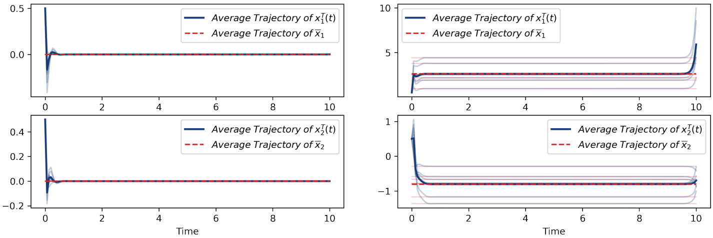

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2. Connection between exponential stability-detectability hypotheses with average control (find u such that $\mathbb{E}[x(\cdot, T)] = x_1$).

3. Riccati theory on average.

4. Hypotheses that guarantee the turnpike property for $\|\mathbb{E}[x^T(t)] - \mathbb{E}[\overline{x}]\|_{\mathbb{R}^n}$, but not for the $\|x^T(t) - \overline{x}\|_{L^2(\Omega;\mathbb{R}^n)}$.

Finally

Thanks for your attention.