Turnpike behaviour for systems that are partially uncontrollable

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X Partial differential equations, optimal design and numerics
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Turnpike properties for partially uncontrollable systems, Automatica, (2023)

The problem framework

We analyse the control system

$$x'(t) + Ax(t) = Bu(t)$$
$$x(0) = x_0,$$

accomopanied by the following optimization problem

$$\min J_T(u) = \min_u \frac{1}{2} \int_0^T \left(|u(t) - u_d|_U^2 + |Cx(t) - z_d|_Z^2 \right) dt + p_d \cdot x(T), \qquad (\mathcal{P}_T)$$

- A a (possibly) unbounded operator on X
- $u \in L^2_{\mathrm{loc}}([0, \infty); U)$ control
- $C \in L(X, Z)$ is observation operator
- X, U, Z Hilbert spaces
- $B \in \mathcal{L}(U, X)$ control operator
- u_d and z_d are **time independent** desirable control and observation
- $p_d \in X$ a linear regularization of the final state

Turnpike

Under suitable conditions:

- observability and controllability (or detectability and stabilizability),
- dissipativity

Turnpike - another approach

The optimality system for the problem $(\mathcal{P}_T)$ reads

$$\begin{aligned} x_T'(t) + Ax_T(t) &= -B(B^* p_T(t) - u_d) \\ x_T(0) &= x_0 \end{aligned} \qquad \begin{aligned} -p_T'(t) + A^* p_T(t) &= C^*(Cx_T(t) - z_d) \\ p_T(T) &= p_d, \end{aligned}$$

and the optimal control is given by

$$u_T = -B^* p_T + u_d.$$

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and the optimal control is given by

$$u_T = -B^* p_T + u_d.$$

The corresponding stationary problem

$$\min_{u \in U} J_s(u) = \min \left\{ \frac{1}{2} \left(|u - u_d|_U^2 + |Cx - z_d|_Z^2 \right) |Ax = Bu \right\}$$

Its optimality system reads

$$A\bar{x} = -B(B^* \bar{p} - u_d) \qquad A^* \bar{p} = C^*(C\bar{x} - z_d),$$

and the unique solution is

$$\bar{u} = -B^* \bar{p} + u_d.$$

We want to estimate

$$u_T - \bar{u}.$$

Subtracting two optimality systems

$$\begin{aligned} y_T'(t) + Ay_T(t) &= -BB^* q_T(t) & \qquad -q_T'(t) + A^* q_T(t) &= C^* C y_T(t) \\ y_T(0) &= x_0 - \bar{x} & q_T(T) &= p_d - \bar{p}, \end{aligned} \tag{1}$$

where $y_T = x_T - \bar{x}$ and $q_T = p_T - \bar{p}$.
The obtained system is independent of the target data z_d and u_d.

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The obtained system is independent of the target data z_d and u_d.

We decompose the solution to (1) into two parts

$$\begin{aligned} y_T &= y_{T,1} + y_{T,2} \\ q_T &= q_{T,1} + q_{T,2}, \end{aligned}$$

First components $(y_{T,1}, q_{T,1})$ satisfy

$$\begin{aligned} y_{T,1}'(t) + Ay_{T,1}(t) &= -BB^* q_{T,1}(t) \qquad & -q_{T,1}'(t) + A^* q_{T,1}(t) &= C^* C y_{T,1}(t) \\ y_{T,1}(0) &= x_0 - \bar{x} & q_{T,1}(T) &= 0. \end{aligned}$$

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$$\begin{aligned} y_T &= y_{T,1} + y_{T,2} \\ q_T &= q_{T,1} + q_{T,2}, \end{aligned}$$

Second components $(y_{T,2}, q_{T,2})$ satisfy

$$\begin{aligned} y_{T,2}'(t) + Ay_{T,2}(t) &= -BB^* q_{T,2}(t) & \qquad -q_{T,2}'(t) + A^* q_{T,2}(t) &= C^* C y_{T,2}(t) \\ y_{T,2}(0) &= 0 & q_{T,2}(T) &= p_d - \bar{p}. \end{aligned}$$

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(2) is the optimality system for the problem

$$\min_{v \in L^2(0,T;U)} J_{T,1}(v) = \min_{v \in L^2(0,T;U)} \frac{1}{2} \int_0^T \left(|v(t)|_U^2 + |Cy(t)|_Z^2 \right) dt,$$

where y stands for the solution to

$$\begin{aligned} y'(t) + Ay(t) &= Bv(t) \\ y(0) &= x_0 - \bar{x}. \end{aligned}$$

Consequently,

$$v_{T,1} = -B^* q_{T,1} = \arg\min J_{T,1}$$

and

$$\begin{aligned} \|v_{T,1}\|_{L^2(0,T;U)}^2 + \|Cy_{T,1}\|_{L^2(0,T;Z)}^2 &= 2J_{T,1}(v_{T,1}) \\ &\leq 2J_{T,1}(0) = \|CS_t(x_0 - \bar{x})\|_{L^2(0,T;X)}^2 = Q_T(x_0 - \bar{x}) \cdot (x_0 - \bar{x}), \end{aligned}$$

where

- S_T is the semigroup generated by $-A$,
- $Q_T = \int_0^T S_t^* C^* C S_t dt$ is the observability Grammian.

Second components $(y_{T,2}, q_{T,2})$ satisfy

$$\begin{aligned} y'_{T,2}(t) + Ay_{T,2}(t) &= -BB^* q_{T,2}(t) & \qquad -q'_{T,2}(t) + A^* q_{T,2}(t) &= C^* C y_{T,2}(t) \\ y_{T,2}(0) &= 0 & q_{T,2}(T) &= p_d - \bar{p}. \end{aligned} \tag{3}$$

(3) is the optimality system for the problem (the change of variable $s = T - t$)

$$\min_{z \in L^2(0,T;Z)} J_{T,2}(z) = \min_{z \in L^2(0,T;Z)} \frac{1}{2} \int_0^T \left(|z(t)|_Z^2 + |B^* q(t)|_X^2 \right) dt,$$

where q is the solution to the problem

$$\begin{aligned} q'(t) + A^* q(t) &= C^* z(t) \\ q(0) &= p_d - \bar{p}. \end{aligned}$$

Specially, $Cy_{T,2} = \arg\min J_{T,2}$.

Consequently

$$\begin{aligned} \|v_{T,2}\|^2_{L^2(0,T;U)} + \|Cy_{T,2}\|^2_{L^2(0,T;Z)} &= 2J_{T,2}(Cy_{T,2}) \\ &\leq 2J_{T,2}(0) = \|B^* S_t^* (p_d - \bar{p})\|_{L^2(0,T;X)} = \Lambda_T (p_d - \bar{p}) \cdot (p_d - \bar{p}), \end{aligned}$$

where $v_{T,2} = B^* q_{T,2}$, while Λ_T stands for the controllability Grammian

$$\Lambda_T = \int_0^T S_t B B^* S_t^* \, dt.$$

Theorem 1. (M. Gugat, M. L, '22.)

The difference of solutions to the original and the stationary optimal control problems $v_T = u_T - \bar{u}$*, together with the difference of the corresponding optimal states* $y_T = x_T - \bar{x}$*, satisfies the estimate*

$$\|u_T - \bar{u}\|^2_{L^2(0,T;U)} + \|C(x_T - \bar{x})\|^2_{L^2(0,T;Z)} \leq 2\Big(Q_T(x_0 - \bar{x}) \cdot (x_0 - \bar{x}) + \Lambda_T(p_d - \bar{p}) \cdot (p_d - \bar{p})\Big),$$

where

- Q_T *is the observability Grammian for the pair* (A, C)*,*
- Λ_T *is the controllability Grammian corresponding to the pair* (A, B)*.*

Theorem 2. (M. Gugat, M. L, '22.)

The difference of solutions to the original and the stationary optimal control problems $v_T = u_T - \bar{u}$, *together with the difference of the corresponding optimal states* $y_T = x_T - \bar{x}$, *satisfies the estimate*

$$\|u_T - \bar{u}\|^2_{L^2(0,T;U)} + \|C(x_T - \bar{x})\|^2_{L^2(0,T;Z)} \leq 2\Big(Q_T(x_0 - \bar{x})\cdot(x_0 - \bar{x}) + \Lambda_T(p_d - \bar{p})\cdot(p_d - \bar{p})\Big),$$

where

- Q_T *is the observability Grammian for the pair* (A, C),
- Λ_T *is the controllability Grammian corresponding to the pair* (A, B).

No conditions on the linear operators A, B and C!

Definition

The operator $B(C)$ is an infinite-time admissible control (observation) operator for the semigroup generated by $-A$ if the controllability (observability) Gramian $\Lambda_\infty(Q_\infty)$ is a bounded operator on X.

In that case

$$\|\Lambda_T\|_{L(X)} \leq \|\Lambda_\infty\|_{L(X)}$$

$$\|Q_T\|_{L(X)} \leq \|Q_\infty\|_{L(X)}$$

for all T.

Theorem 3.

Assume that B and C are an infinite-time admissible control and observation operator for the semigroup generated by $-A$, respectively. Then the following result holds.

a) **(Integral turnpike)** *For any target data u_d and z_d,*

$$\frac{1}{T}\int_0^T u_T \xrightarrow[T\to\infty]{} \bar{u} \qquad \text{strongly in } U,$$

$$\frac{1}{T}\int_0^T Cx_T \xrightarrow[T\to\infty]{} C\bar{x} \qquad \text{strongly in } Z.$$

with the convergence rate of $O(1/\sqrt{T})$.

b) **(Measure turnpike)** *For every $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ (that depends on $x_0 - \bar{x}$ and $p_d - \bar{p}$) such that for every $T > 0$ we have*

$$\mu\left\{t \in [0,T] \middle| \, |u_T - \bar{u}|^2 + |C(x_T - \bar{x})|^2 \geq \varepsilon\right\} < C_\varepsilon.$$

c) **(Convergence of the optimal value functions)**

$$\frac{1}{T}\min J_T = \min J_s + \mathcal{O}(\frac{1}{\sqrt{T}}) \qquad \text{as } T \to \infty.$$

Parameter dependent problems

A family of the parameter-dependent control systems

$$\begin{aligned} x'(\omega,t) + A(\omega)x(\omega,t) &= B(\omega)u(t) \\ x(\omega,0) &= x_0(\omega), \end{aligned}$$

accompanied by an optimal control problem of the form

$$\begin{aligned} u_T &:= \min J_T(u) \\ &= \min_u \frac{1}{2}\int_0^T \left(|u(t)-u_d|_U^2 + \left| \int_\Omega \left(C(\omega)x(\omega,t) - z_d(\omega)\right) d\omega \right|_Z^2 \right) dt + \int_\Omega p_d(\omega)\cdot x(\omega,T)d\omega \end{aligned}$$

The corresponding stationary problem:

$$\begin{aligned} \bar{u} &: = \min_{u\in U} J_s(u) \\ &= \min \left\{ \frac{1}{2}\left(|u-u_d|_U^2 + \left| \int_\Omega \left(C(\omega)x(\omega) - z_d(\omega)\right)\, d\omega \right|_Z^2 \right) \Big| A(\omega)x = B(\omega)u \right\}. \end{aligned}$$

We want to estimate

$$u_T - \bar{u}.$$

The problem is well posed

$$u_T = -\int_\Omega B(\omega)^* p_T(\omega) d\omega + u_d$$

where p_T is the optimal dual state:

$$x_T'(\omega, t) + A(\omega) x_T(\omega, t) = -B(\omega) \int_\Omega (B(\nu)^* p_T(\nu, t) - u_d) d\nu$$

$$x_T(\omega, 0) = x_0(\omega)$$

$$-p_T'(\omega, t) + A(\omega)^* p_T(\omega, t) = C(\omega)^* \int_\Omega (C(\nu) x_T(\nu, t) - z_d(\nu)) d\nu$$

$$p_T(\omega, T) = p_d(\omega),$$

Similarly

$$\bar{u} = -\int_\Omega B(\omega)^* \bar{p}(\omega) d\omega + u_d,$$

where $\bar{p}$ is the optimal dual variable for the stationary problem.

We repeat the steps from the deterministic case.

- We consider

$$y_T = x_T - \bar{x}$$

$$q_T = p_T - \bar{p}$$

- We decompose the system satisfied by (y_T, q_T) into two parts.
- Each part is detected as optimality system for a similar kind of problem.

Consuqently, we obtain the averaged control analogue of Theorem 7.

Theorem 4.

$$\|u_T - \bar{u}\|^2_{L^2(0,T;U)} + \Big\| \int_\Omega C(\omega)\,(x_T(\omega,\cdot) - \bar{x}(\omega))\,d\omega \Big\|^2_{L^2(0,T;U)}$$
$$\leq \int_\Omega \Big(Q_T(\omega)(x_0 - \bar{x})(\omega) \cdot (x_0 - \bar{x})(\omega) + \Lambda_T(\omega)(p_T - \bar{p})(\omega) \cdot (p_T - \bar{p})(\omega) \Big)\,d\omega,$$

where $\Lambda_T(\omega)\Big(Q_T(\omega)\Big)$ stands for the controllability (observability) Grammian associated to the parameter value ω.

Tunrpike for optimal averaged controls

Theorem 5.

Assume that for almost every $\omega \in \Omega$ we have that $B(\omega)$ and $C(\omega)$ are an infinite-time admissible control and observation operator for the semigroup generated by $-A(\omega)$, respectively.

a) (Integral turnpike) *For any target data u_d and $z_d(\omega)$, we have*

$$\frac{1}{T}\int_0^T u_T = \bar{u} + \mathcal{O}(\frac{1}{\sqrt{T}}) \qquad \text{as } T \to \infty,$$

$$\frac{1}{T}\int_0^T \int_\Omega C(\omega)x_T(\omega)d\omega = \int_\Omega C(\omega)\bar{x}(\omega)d\omega + \mathcal{O}(\frac{1}{\sqrt{T}}) \qquad \text{as } T \to \infty.$$

b) (Measure turnpike) *For every $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ (that depends on $\|x_0 - \bar{x}\|_{L^2(\Omega;X)}$ and $\|p_d - \bar{p}\|_{L^2(\Omega;X)}$) such that for every $T > 0$ we have*

$$\mu\Big\{t \in [0, T] \Big| \, |u_T - \bar{u}|^2 + \Big|\int_\Omega C(\omega)(x_T(\omega) - \bar{x}(\omega))d\omega\Big|^2 \geq \varepsilon\Big\} < C_\varepsilon.$$

c) (Convergence of the optimal value functions)

$$\frac{1}{T}\min J_T = \min J_s + \mathcal{O}(\frac{1}{\sqrt{T}}) \qquad \text{as } T \to \infty.$$

A numerical example

Consider the system for $\mathbf{x} = (x_1, x_2)$:

$$x_1' = x_1 + u$$
$$x_2' = x_2 + u.$$

with the optimization problem

$$\min \frac{1}{2} \int_0^T \left(|u(t)|^2 + |x_1(t) + x_2(t) - 1|^2 \right) \, dt.$$

$x_1 - x_2$ is independent of the control and uncontrollable.
What can we say about the observation $C\mathbf{x} = x_1 + x_2$?
The corresponding static problem is

$$\min \frac{1}{2} \left(|u|^2 + |x_1 + x_2 - 1|^2 \right)$$

subject to

$$x_1 = -u$$
$$x_2 = -u.$$

Its solution is $\bar{u} = -\frac{2}{5}$, $\bar{\mathbf{x}} = (\frac{2}{5}, \frac{2}{5})$.

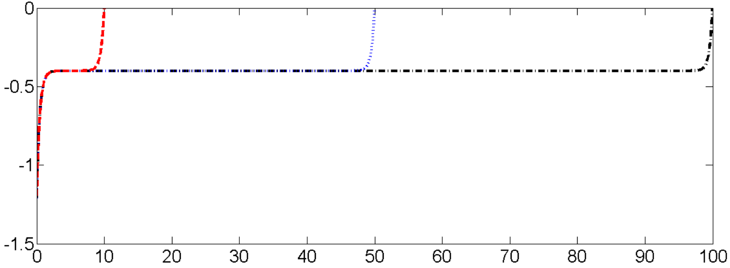

Figure 1: The graph of the optimal control $u_T(t)$.

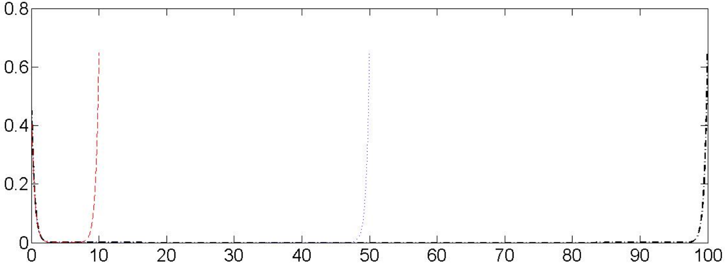

Figure 2: The graph of $x_{T,1}(t) + x_{T,2}(t) - 0.8$.

Conclusion

- Turnpike results for infinite-dimensional, LQ optimal control problems.
- The key estimate on the difference between evolutional and stationary optimal controls and observations is derived with virtually no assumptions on the operators A, B, and C.
- The turnpike properties follow directly by assuming infinite-time admissibility of control and observation operator.
- Results obtained both in deterministic and parameter dependent case.
- Exponential turnpike?

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Thanks for your attention!