Friedrich-Alexander-Universität
Naturwissenschaftliche Fakultät

An introduction to the turnpike phenomenon with examples and links to recent results

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Department Mathematik

August 23, 2024 - X Partial differential equations, optimal design and numerics, Benasque

1. The turnpike property - what is it?

1.1 The turnpike phenomenon

The Turnpike Phenomenon: What is it?

Dynamic optimal control

- Let a **time interval** $[0,\ T]$ be given.
- Consider a **control-to-state map**

$$u(t) \mapsto \Phi(y_0, u)(t) = y(t)$$

  ($t \in [0, T]$) with a given initial state $y(0) = \Phi(y_0, u)(0) = y_0$.
- Let a **desired state** y_d be given. Consider an objective function of **integral-type**

$$J_T(u) = \int_0^T \|y(t) - y_d\|^2 + \|u(t)\|^2 \, dt.$$

- The **dynamic optimal control problem** has the form $\min_{u \in U} J_T(u)$.

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Static optimal control

- Consider **steady states**

$$u^{(\sigma)} \mapsto \Phi(y^{(\sigma)}, u^{(\sigma)})(t) = y^{(\sigma)}$$

  with $y(0) = \Phi(y^{(\sigma)}, u^{(\sigma)})(0) = y^{(\sigma)}$ (independent of t).
- The **static optimal control problem** has the form $\min_{u^{(\sigma)} \in U} \|y^{(\sigma)} - y_d\|^2 + \|u^{(\sigma)}\|^2$.

The Turnpike Phenomenon: What is it?

Turnpike results relate the solutions of the *dynamic optimal control problems* (*optimal states/controls*) that are influenced by prescribed

initial/terminal data

with solutions of optimal control problems that are *independent* of given initial/terminal data (e.g. *static problems*).

Figure: Dynamic versus static

A turnpike is a road, especially an expressway, which people have to pay to drive on.

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Figure: Dynamic versus static

In the dynamic case with a time-interval $[0, T]$, the influence of the *initial data* and *terminal data* becomes **small** around $\frac{T}{2}$ for **large time horizons** T!

Wikipedia: The **New Jersey Turnpike** Creative-Commons-Lizenz

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The Turnpike Phenomenon

A historical perspective

Very early references:

- JOHN VON NEUMANN *A Model of General Economic Equilibrium*
  Über ein ökonomisches Gleichungssystem und eine Verallgemeinerung des Brouwerschen Fixpunktsatzes
- FRANK RAMSEY (1928) *A Mathematical Theory of Saving*.

Later

- PAUL A. SAMUELSON (1976) *The periodic turnpike theorem*
- Currently e.g. ALEXANDER ZASLAVSKI, LARS GRÜNE, KARL WORTHMANN, KATHRIN FLASSKAMP, TIM FAULWASSER, ROBERTO GUGLIELMI, ...

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Different types of turnpike notions:

Let $d(t)$ denote the *distance between the dynamic solutions and the solutions of the static problem at time* t.

- **Measure turnpike**: For all $\varepsilon > 0$,
  the measure of the set $\{t \in [0, T]$ where the distance $d(t) \geq \varepsilon\}$ is uniformly bounded w.r.t. T.

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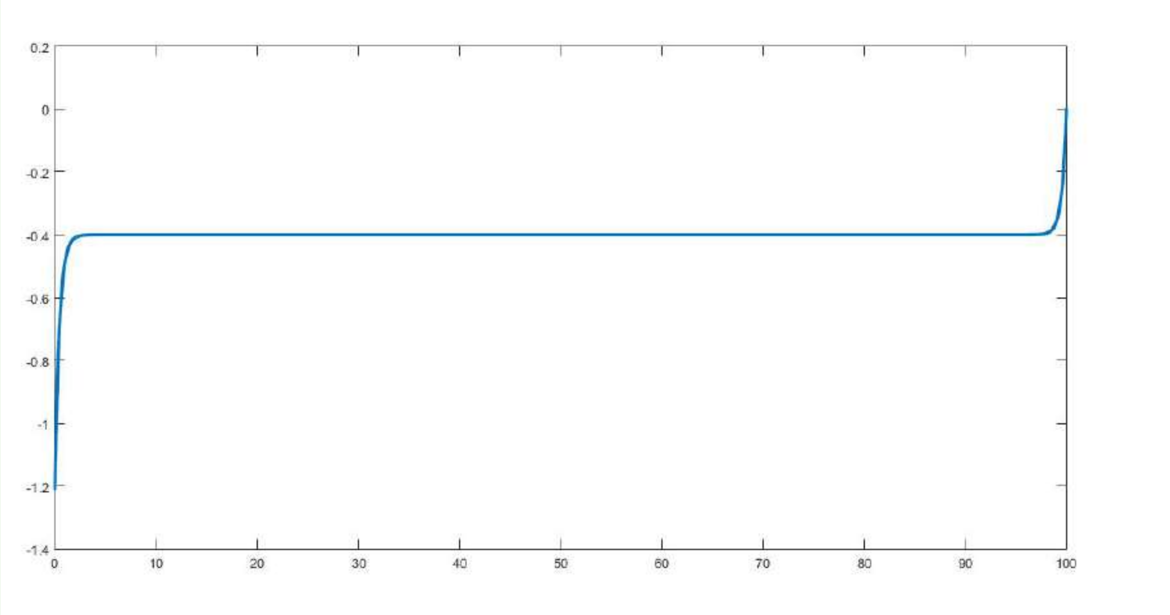
- **Exponential turnpike** There exist $C_1 > 0$, $\omega > 0$ such that for arbitrarily large T we have

$$|d(t)| \leq C_1 \left[\exp(-\omega\, t) + \exp(-\omega(T - t)\right] \text{ for all } t \in [0, T].$$

The Turnpike Phenomenon

A typical situation

If a problem has the turnpike property, the typical structure of the optimal control and the optimal state is of this type:



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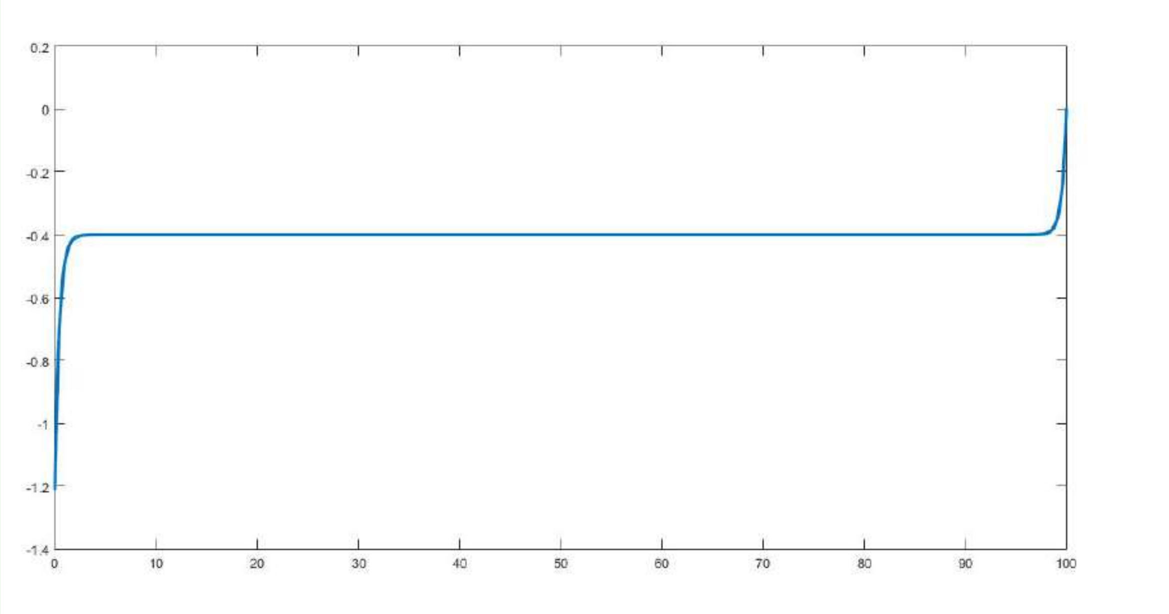

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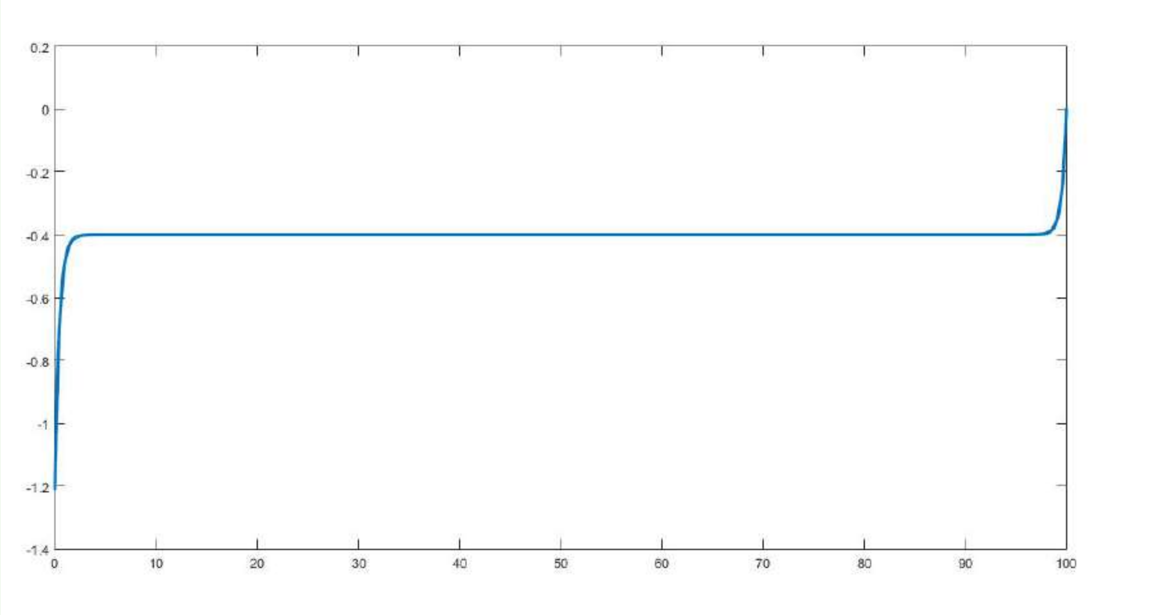

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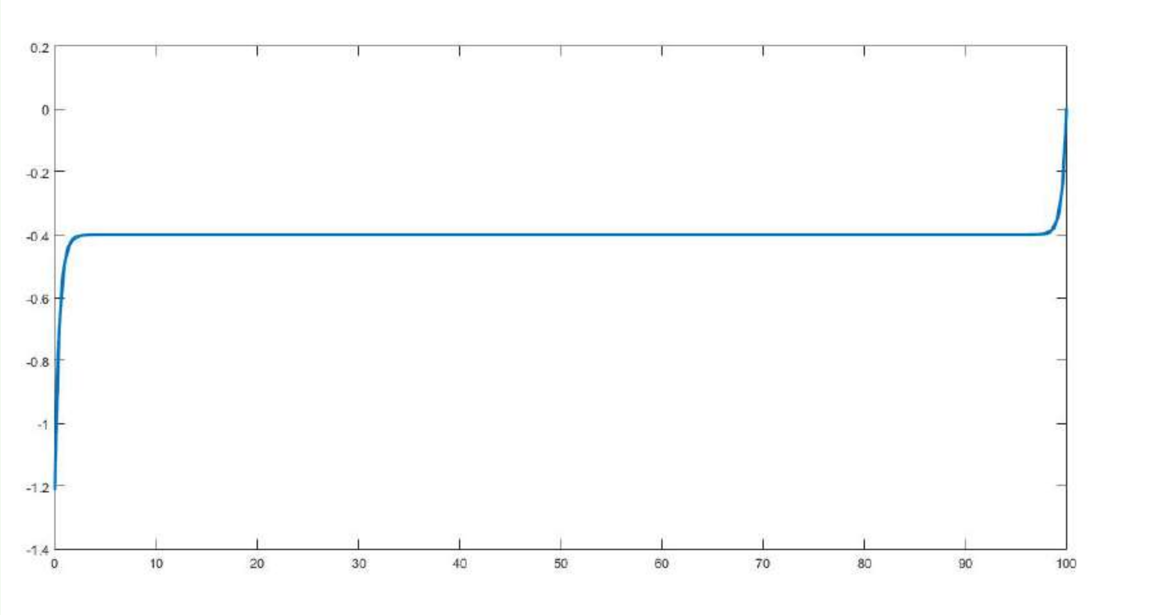

For large T there are **three phases** on $[0, T]$:

- *A short transient initial phase* where the state is steered from the given initial phase close to the optimal static state.
- *A long intermediate phase* where the state remains close to the optimal static state.
- *A short terminal phase* where the state is driven rapidly towards the optimal terminal state.

The Turnpike Phenomenon

Example 1 ***Consider the optimal control problem***

$$\begin{cases} \min\limits_{u \in L^2(0,T)} \frac{1}{2} \int_0^T \|u(t)\|^2 \\ s.t.\ y'(t) + y(t) = u(t) \\ y(0) = y_0,\ y(T) = y_1 \end{cases}$$

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- We have

$$u(t) = \frac{y_1 - \exp(-T) y_0}{\frac{1 - \exp(-2T)}{2}} \exp(t - T).$$

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  There is only the arc that decays exponentially going backwards in time!
- We have the **optimal state**

$$y(t) = y_0 \frac{\sinh(T-t)}{\sinh(T)} + y_1 \frac{\sinh(t)}{\sinh(T)}.$$

  Since $\frac{\sinh(t)}{\sinh(T)} = \frac{\exp(t-T) - \exp(-t-T)}{1 - \exp(-2T)}$, $\frac{\sinh(T-t)}{\sinh(T)} = \frac{\exp(-t) - \exp(t-2T)}{1 - \exp(-2T)}$, also the **exponential turnpike property for the optimal state** y with 2 arcs follows.

The Turnpike Phenomenon

Example 2 ***Consider the optimal control problem***

$$\begin{cases} \min\limits_{u \in L^2(0,T)} \frac{1}{2} \int_0^T \|u(t)\|^2 + \|y(t)\|^2 \, dt \\ s.t. \; y'(t) + y(t) = u(t) \\ y(0) = y_0, \; y(T) = y_1 \end{cases}$$

Show that the optimal control and the optimal state have the **exponential turnpike property**!

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Show that the optimal control and the optimal state have the **exponential turnpike property**!

- The turnpike is $y = 0$, $u = 0$.
- Necessary optimality conditions imply $u'(t) = u(t) + y(t)$. Thus $u'' = 2\,u$, hence

$$u(t) = C_+ \exp(\sqrt{2}t) + C_- \exp(-\sqrt{2}t).$$

Since

$$\begin{pmatrix} \sqrt{2} - 1 & -(\sqrt{2} + 1) \\ (\sqrt{2} - 1) \exp(\sqrt{2}T) & -(\sqrt{2} + 1) \exp(-\sqrt{2}T) \end{pmatrix} \begin{pmatrix} C_+ \\ C_- \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \end{pmatrix}.$$

we obtain $C_+ = \mathcal{O}(\exp(-\sqrt{2}T))$ and $C_- = \mathcal{O}(1)$. This yields

$$u(t) = \tilde{C}_+(T) \exp(\sqrt{2}(t - T)) + \tilde{C}_-(T) \exp(-\sqrt{2}t)$$

with $\tilde{C}_\pm = \mathcal{O}(1)$.

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with $\tilde{C}_\pm = \mathcal{O}(1)$. Thus u has the **exponential turnpike property** (with both arcs!).
Since $y(t) = u'(t) - u(t)$, also the **exponential turnpike property for the optimal state** y follows (with rate $\sqrt{2} > 1$).

The Turnpike Phenomenon: A simple example with no exponential turnpike

Examples 3 An objective function of H^1-type with emphasis on the derivative

Let $T > 0$ be given. Consider the **dynamic** optimal control problem

$$\min \frac{1}{2} \left\| \frac{1}{T} \int_0^T y(\tau)\, d\tau \right\|^2 + \frac{1}{2} \int_0^T \|u(\tau)\|^2(\tau) d\tau$$

subject to

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- Show that the optimal control and the optimal state do **not** have the **exponential turnpike property**!
- Show that u has the **integral turnpike property**.

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The optimal state is given by a *parabola*. The optimal control is a *straight line!*

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- Show that u has the **integral turnpike property**.

The optimal state is given by a *parabola*. The optimal control is a *straight line!*
The turnpike is $y = 0$, $u = 0$. The necessary optimality conditions imply

$$\int_0^T \|u(t)\|^2\, dt = \int_0^T \mathcal{O}\left(\frac{1}{T^2}\right)\, dt = \mathcal{O}\left(\frac{1}{T}\right).$$

Hence u has the **integral turnpike property**.
We have

$$\frac{1}{T} \int_0^T y(\tau)\, d\tau = \frac{T^2}{(1 + \frac{1}{12} T^3)} \frac{y_0 + y_1}{2} = \mathcal{O}\left(\frac{1}{T}\right).$$

An L^1-example with another turnpike property

Example 4 An objective function of L^1-type

Let $T > 0$, $\gamma > 0$ and $A = A^\top$ be given. Consider the **dynamic** optimal control problem

$$\min \gamma \int_0^T |y(\tau)|\, d\tau + \frac{1}{2}\int_0^T \|u(\tau)\|^2\, d\tau$$

subject to

$$y(0) = y_0,\ y'(t) = Ay(t) + u(t),\ y(T) = 0.$$

- Does the optimal control have the **exponential turnpike property** for $-A > 0$?
- Is the optimal state identical to zero for some time?
  Yes, the optimal control drives the state to zero **in finite time** $t_0(\gamma) \in (0, T)$ if γ is sufficiently large.
  This is called the **finite-time turnpike property**.
- The optimal control is constant until it is switched off at t_0.
  For $\gamma \to \infty$ we have $t_0(\gamma) \to 0+$.

Some references on Turnpike:

Some papers about the turnpike topic

1. **Deterministic:** On the Turnpike Phenomenon for Optimal Boundary Control Problems with Hyperbolic Systems, *Martin Gugat, Falk M. Hante*, SICON 2019 https://epubs.siam.org/doi/10.1137/17M1134470

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3. With **chance constraints**:
   A Turnpike Property for Optimal Control Problems with Dynamic Probabilistic Constraints
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and

- **Network Design and Control**: Shape and Topology Optimization for the Turnpike Property for the Wave Equation
  Martin Gugat, Meizhi Qian, Jan Sokolowski The Journal of Geometric Analysis 2024
  https://link.springer.com/article/10.1007/s12220-024-01712-8

Uniform Turnpike with respect to shape

Optimal control for varying Ω

$$u \longmapsto \frac{1}{2}\int_0^T \int_\Omega (y(u) - y_d)^2 + \gamma\,(y_t(u))^2 dx\, dt + \frac{1}{2}\int_0^T \int_\Gamma (u - u_d)^2 d\Gamma(x) dt \tag{1}$$

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$$\left(\frac{\partial^2 y}{\partial^2 t}(t), \varphi\right) + a(y(t), \varphi) = (L(u)(t), \varphi) + (F(t), \varphi) \tag{2}$$

for all $\varphi \in H$ and a.e. for $t \in (0, T)$, along with the initial conditions $y(0, x) = y^0(x)$ and $y_t(0, x) = y^1(x)$.
Assume that there exists a complete orthonormal sequence of eigenfunctions.

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Theorem (with M Qian, J Sokolowski, JGEA 2024)

Assume that $\lambda_k \geq \gamma > 0$ holds for all $k \in \{1, 2, 3, \ldots\}$ and that the initial state satisfies the regularity condition

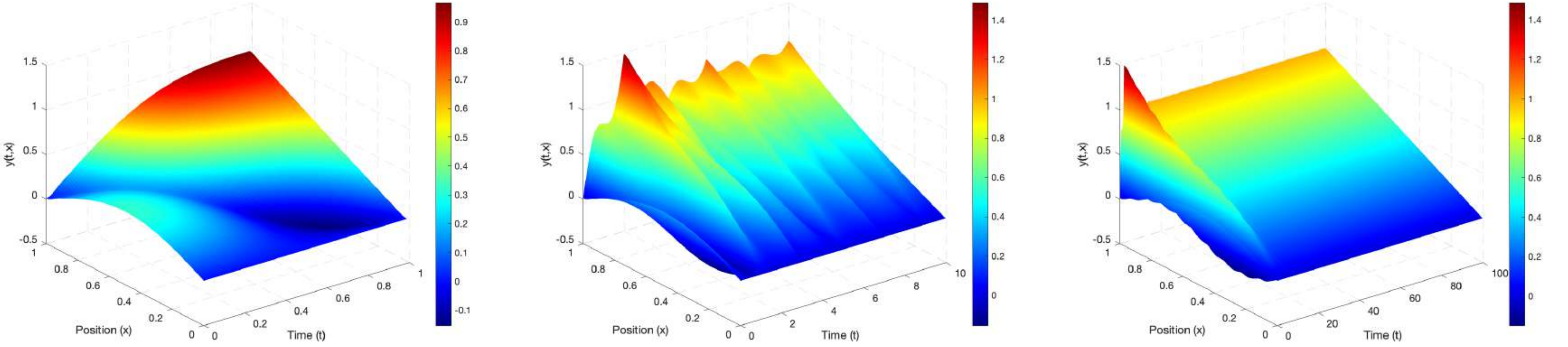
$$\sum_{k=0}^{\infty} \lambda_k\, |a_k(0)|^2 + |a_k'(0)|^2 < \infty, \tag{3}$$

that is the initial state belongs to the energy space of the elliptic problem defined by the bilinear form $a(\cdot,\, \cdot)$. If $\Omega = \Gamma$ then there exists a constant $\tilde D = \tilde D(y_0,\, y_1,\, p^\sigma)$ that is independent of T and t such that for all $t \in [0,\, T]$

$$\|\omega^T(t)\|^2_{L^2(\Omega)} + \|\nu^T(t)\|^2_{L^2(\Omega)} + \|\mu^T(t)\|^2_{L^2(\Omega)} \leq \tilde D \left[\exp(-\sqrt{\gamma}\, t) + \exp(-\sqrt{\gamma}(T - t))\right]. \tag{4}$$

The constant $\tilde D$ depends on Ω only as a function of the energy norm for the initial state.

Uniform Turnpike with respect to shape



(a) $T = 1$ (b) $T = 10$ (c) $T = 100$

Figure: Optimal state for different values of T in example 3

The relation between the turnpike phenomenon and stabilization

Here is a final remark about the relation between the turnpike property and stabilization:

- If the optimal control problem has the exponential turnpike property, for large time horizons T, the following strategy is close to optimal:
  - Use a **feedback-law** to steer the initial state *exponentially fast* towards the optimal static state (with a decay rate that is independent of T),

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- So for $T \to \infty$ for the optimal control u_{opt}^T and the control determined by the above strategy u_{approx}^T we have

$$\frac{J(u_{approx}^T)}{J(u_{opt}^T)} \to 1.$$