The moment method in action

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X Partial differential equations, optimal design and numerics
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Outline

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What is a moment problem ?

- H a Hilbert space • $E \subset H\backslash\{0\}$ a family in H • $\omega = (\omega_e)_{e\in E} \subset \mathbb{R}$ a family of numbers

Moment problem

We look for an element $x \in H$ such that

$$(e, x)_H = \omega_e, \ \forall e \in E. \tag{P}$$

Main question

Given the family E, for which data ω is there a solution to (P) ?

Remarks

- Uniqueness $\iff$ $E^\perp = \{0\}$ $\iff$ E is complete in H.
- Necessary existence conditions : there exists $C > 0$ such that

$$|\omega_e| \leqslant C\|e\|_H, \qquad \forall e \in E,$$

$$|\omega_e - \omega_f| \leqslant C\|e - f\|_H, \qquad \forall e, f \in E,$$

$$\dots$$

$$\left|\sum_{e\in E} \alpha_e \omega_e\right| \leqslant C \left\|\sum_{e\in E} \alpha_e \, e\right\|_H, \qquad \forall (\alpha_e)_{e\in E} \subset \mathbb{R} \text{ finitely supported .}$$

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Proposition (Finite case)

If E is finite and linearly independent, then (P) *has a solution for every ω.*

Sketch of proof

- There exists a **biorthogonal family** $(q_e)_{e \in E} \subset H$

$$(e, q_f)_H = \delta_{e,f}, \quad \forall e, f \in E.$$

- A solution of (P) is given by $x = \sum_{f \in E} \omega_f q_f$.
- Note that the minimal such family satisfies $\|q_e\|_H = \dfrac{1}{d(e, \mathrm{Span}(E \backslash \{e\}))}$.

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Theorem (Infinite case - usual approach)

If E is infinite, there exists a biorthogonal family $(q_e)_{e \in E}$ ***if and only if***

$$d(e, \mathrm{Span}(E \backslash \{e\})) > 0, \quad \forall e \in E.$$

In that case, we have $\|q_e\|_H = \frac{1}{d(e, \mathrm{Span}(E \backslash \{e\}))}$.

In particular,

$$\sum_{e \in E} |\omega_e| \|q_e\|_H < +\infty \qquad \Longrightarrow \qquad x = \sum_{e \in E} \omega_e q_e \text{ solves (P)} .$$

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Moment problem

We look for an element $x \in H$ such that

$$(e, x)_H = \omega_e, \ \forall e \in E. \tag{P}$$

Main question

Given the family E, for which data ω is there a solution to (P) ?

Theorem (Infinite case - block moment approach)

Assume $d(e, \mathrm{Span}(E \backslash \{e\})) > 0$, $\forall e \in E$. Let $(q_e)_{e \in E}$ be the minimal biorthogonal family.

Let $E = \bigsqcup_{G \in \mathcal{G}} G$ be a partition of E into finite subsets. We have

$$\sum_{G \in \mathcal{G}} \left\| \sum_{e \in G} \omega_e q_e \right\| < +\infty \quad \Longrightarrow \quad x = \sum_{G \in \mathcal{G}} \left(\sum_{e \in G} \omega_e q_e \right) \text{ solves (P)}.$$

Remark : $q_G = \sum_{e \in G} \omega_e q_e$ solves the partial moment problem $(e, q_G)_H = \begin{cases} \omega_e, & \text{if } e \in G, \\ 0, & \text{otherwise} \end{cases}$

Outline

Abstract linear time invariant parabolic control problems

- Two Hilbert spaces : the state space $(X, \langle .,. \rangle)$ and the control space $(U, (.,.)_U)$.
- $\mathcal{A} : D(\mathcal{A}) \subset X \mapsto X$ some unbounded operator and such that $-\mathcal{A}$ generates a continuous semigroup.
- $\mathcal{B} : U \mapsto X$ the (bounded) control operator, $\mathcal{B}^\star$ its adjoint.

Our controlled parabolic problem is

$$(S) \quad \begin{cases} \partial_t y + \mathcal{A}y = \mathcal{B}u & \text{in }]0, T[, \\ y(0) = y_0, \end{cases}$$

- $y_0 \in X$ is the initial data and $u \in L^2(]0, T[, U)$ is the control we are looking for.

Theorem (Well-posedness of (S) in a dual sense)

For any $y_0 \in X$ and $u \in L^2(0, T; U)$, there exists a unique $y = y_{u,y_0} \in \mathcal{C}^0([0, T], X)$ such that

$$\langle y(\tau), \phi \rangle - \left\langle y_0, e^{-\tau\mathcal{A}^\star}\phi \right\rangle = \int_0^\tau \left(u(t), \mathcal{B}^\star e^{-(\tau - t)\mathcal{A}^\star}\phi \right)_U dt, \quad \forall \tau \in [0, T], \forall \phi \in X.$$

Null-controllability

Let $T > 0$ be given. We say that (S) is null-controllable at time T, if

$$\forall y_0 \in X, \ \exists u \in L^2(0, T; U), \ \text{ such that } \ y_{u,y_0}(T) = 0.$$

Null-controllability vs. moment problem

A function $u \in L^2(0,T;U)$ is a null-control for our system and the initial data y_0 if and only if

$$\int_0^T \left(u(T-t), \mathcal{B}^\star e^{-t\mathcal{A}^\star}\phi \right)_U dt = -\left\langle y_0, e^{-T\mathcal{A}^\star}\phi \right\rangle, \quad \forall \phi \in X. \tag{GMP}$$

This is a moment problem in $L^2(0,T;U)$!

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This is a moment problem in $L^2(0,T;U)$!

Remarks

- It is enough to test against the elements of any **complete** family Φ of elements in X.
- Solving (GMP) is *a priori* as difficult as solving the initial control problem or not ... see M. Morancey's talk.
- (GMP) can be reduced to a more tractable moment problem if we manage to find Φ such that the "test functions"

$$\left(t \mapsto \mathcal{B}^\star e^{-t\mathcal{A}^\star} \phi \right)_{\phi \in \Phi}$$

  have simple enough expressions.

Null-controllability vs. moment problem

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This is a moment problem in $L^2(0, T; U)$!

First level of simplification : spectral structure of $\mathcal{A}^\star$

Assume that $\mathcal{A}^\star$ possesses a family of eigenfunctions $\Phi = \{\phi_\lambda, \lambda \in \Lambda\}$ which is complete in X.
Then (GMP) is equivalent to find $v = u(T - .)$

$$\int_0^T e^{-\lambda t}\left(v(t), \mathcal{B}^\star \phi_\lambda\right)_U dt = -e^{-\lambda T}\langle y_0, \phi_\lambda\rangle, \quad \forall \lambda \in \Lambda.$$

Null-controllability vs. moment problem

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Then (GMP) is equivalent to find $v = u(T - .)$

$$\int_0^T e^{-\lambda t}(v(t), \mathcal{B}^\star\phi_\lambda)_U \, dt = -e^{-\lambda T}\langle y_0, \phi_\lambda\rangle, \quad \forall \lambda \in \Lambda.$$

Second level of simplification : scalar control

Assume we are dealing with a scalar control problem : $U = \mathbb{R}$, $\mathcal{B}^\star : U \mapsto \mathbb{R}$.
Then (GMP) is equivalent to find $v = u(T - .) \in L^2(0, T; \mathbb{R})$

$$\int_0^T e^{-\lambda t} v(t) \, dt = -e^{-\lambda T}\frac{\langle y_0, \phi_\lambda\rangle}{\mathcal{B}^\star\phi_\lambda}, \quad \forall \lambda \in \Lambda.$$

Outline

Partial (scalar, exponential) moment problems

Data

- A family $\Lambda \subset \mathbb{C}^+$
- A finite subset $G \subset \Lambda$
- A non trivial $\omega_G = (\omega_\lambda)_{\lambda \in G} \subset \mathbb{C}$
- A time $T \in (0, +\infty]$

Notation

$$e[\lambda] := \left(t \in (0, +\infty) \mapsto e^{-\lambda t}\right) \in L^2(0, +\infty).$$

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Partial moment problem

$$\text{Find } q = q_{G,\omega,T} \in L^2(0,T) \text{ s.t. } \begin{cases} (e[\lambda], q)_{L^2(0,T)} = \omega_\lambda, & \text{for any } \lambda \in G, \\ (e[\lambda], q)_{L^2(0,T)} = 0, & \text{for any } \lambda \in \Lambda \backslash G. \end{cases} \quad \text{(PM)}$$

Partial (scalar, exponential) moment problems

Main result

Data

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- A non trivial $\omega_G = (\omega_\lambda)_{\lambda \in G} \subset \mathbb{C}$
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Assumptions

- Parabolic sector : $\Lambda \subset S_\eta$ for some $\eta \in (0, \pi/2)$.

Partial moment problem

Find $q = q_{G,\omega,T} \in L^2(0,T)$ s.t.

$$\begin{cases} (e[\lambda], q)_{L^2(0,T)} = \omega_\lambda, & \text{for any } \lambda \in G, \\ (e[\lambda], q)_{L^2(0,T)} = 0, & \text{for any } \lambda \in \Lambda \backslash G. \end{cases} \tag{PM}$$

Theorem (Necessary condition : the price to pay for orthogonality)

A solution to (PM) *exists* **if and only if** $\displaystyle\sum_{\lambda \in \Lambda} \frac{1}{|\lambda|} < +\infty.$

Sketch of proof

(Müntz, 1914) **(Schwartz, 1943)**

$$d_{L^2(0,\infty)}\Big(e[\lambda], \mathrm{Span}(e[\mu], \mu \neq \lambda)\Big) = \frac{1}{\sqrt{2\,\mathrm{Re}\,\lambda}} \prod_{\substack{\mu \in \Lambda \\ \mu \neq \lambda}} \left| \frac{1 - \frac{\lambda}{\mu}}{1 + \frac{\lambda}{\bar{\mu}}} \right|.$$

Partial (scalar, exponential) moment problems

Main result

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- A finite subset $G \subset \Lambda$
- A non trivial $\omega_G = (\omega_\lambda)_{\lambda \in G} \subset \mathbb{C}$
- A time $T \in (0, +\infty]$

Assumptions

- Parabolic sector : $\Lambda \subset S_\eta$ for some $\eta \in (0, \pi/2)$.
- Asymptotics : $N_\Lambda(r) \leqslant \kappa r^\theta$ for any $r > 0$ with $\theta \in (0, 1)$.
- Group size : $\# G \leqslant n$ and $\operatorname{diam}(G) \leqslant \rho$.
- Separation : $d(\operatorname{conv}(G), \Lambda \backslash G) \geqslant \gamma$.

Partial moment problem

Find $q = q_{G,\omega,T} \in L^2(0, T)$ s.t.

$$\begin{cases} (e[\lambda], q)_{L^2(0,T)} = \omega_\lambda, & \text{for any } \lambda \in G, \\ (e[\lambda], q)_{L^2(0,T)} = 0, & \text{for any } \lambda \in \Lambda \backslash G. \end{cases} \tag{PM}$$

Theorem

There exists $C > 0$ depending only on $\eta, \kappa, \theta, n, \rho, \gamma$ such that : (B., '23+)
for any $T > 0$, there exists a solution $q_{G,\omega,T}$ to (PM) that satisfies

$$\| q_{G,\omega,T} \|_{L^2(0,T)} \leqslant C e^{C r_G^\theta + C T^{-\frac{\theta}{1-\theta}}} \max_{L \subset G} |\omega[L]|,$$

where $r_G = \min_{\lambda \in G} \operatorname{Re} \lambda$ and $\omega[L]$ denotes the divided difference associated to L and ω.

Partial (scalar, exponential) moment problems

Examples, extensions

- "Usual" bi-orthogonal families : **(Dolecki, '73)** **(Fattorini-Russel, '74)** **(Benabdallah - B. - Gonzalez–Burgos - Olive, '14)**
  - Case 1 : The *usual* gap condition holds

$$\inf_{\substack{\lambda,\mu\in\Lambda \\ \lambda\neq\mu}} |\lambda-\mu| \geqslant \rho. \tag{Gap}$$

    We recover the known estimates of the literature with "optimal" assumptions on Λ

$$\|q_{\lambda,T}\|_{L^2(0,T)} \leqslant Ce^{C(\mathrm{Re}\,\lambda)^\theta + CT^{-\frac{\theta}{1-\theta}}}.$$

  - Case 2 : the gap condition (Gap) does not hold **(Allonsius - B. - Morancey, '20)** **(Gonzalez–Burgos - Ouaili '21)**

$$\|q_{\lambda,T}\|_{L^2(0,T)} \leqslant Ce^{C(\mathrm{Re}\,\lambda)^\theta + CT^{-\frac{\theta}{1-\theta}}} \prod_{\substack{\mu\in\Lambda \\ 0<|\lambda-\mu|<\rho}} \frac{1}{|\mu-\lambda|}.$$

Partial (scalar, exponential) moment problems

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- Taking into account multiplicities :

We get **for free** similar estimates for solutions to

$$\begin{cases} (e[\mu], q_{\lambda,0})_{L^2(0,T)} = \delta_{\lambda,\mu}, \ \forall \mu\in\Lambda, \\ (e[\mu,\mu], q_{\lambda,0})_{L^2(0,T)} = 0, \ \forall \mu\in\Lambda, \end{cases} \quad \text{and} \quad \begin{cases} (e[\mu], q_{\lambda,1})_{L^2(0,T)} = 0, \ \forall \mu\in\Lambda, \\ (e[\mu,\mu], q_{\lambda,1})_{L^2(0,T)} = \delta_{\lambda,\mu}, \ \forall \mu\in\Lambda, \end{cases}$$

with

$$e[\lambda,\lambda] := \left(t\in(0,+\infty)\mapsto (-t)e^{-\lambda t}\right) \in L^2(0,+\infty).$$

Partial moment problems arising from parabolic controllability questions

(Benabdallah - B. - Morancey, '20) (B., '23+)

Recall the original moment problem to solve

$$\int_0^T e^{-\lambda t} v(t)\, dt = e^{-\lambda T} \frac{-\langle y_0, \phi_\lambda \rangle}{\mathcal{B}^\star \phi_\lambda}, \quad \forall \lambda \in \Lambda.$$

This amount to consider

$$\omega_\lambda = e^{-\lambda T} \psi_\lambda,$$

Lemma (≈ Leibniz rule)

In a group G we have

$$\max_{L \subset G} |\omega[L]| \leqslant C e^{-r_G T} \max_{L \subset G} |\psi[L]|.$$

Theorem

With the same assumption above there exists, for any $T > 0$, a solution to

$$\text{Find } q = q_{G,\psi,T} \in L^2(0,T) \text{ s.t. } \begin{cases} (e[\lambda], q)_{L^2(0,T)} = e^{-\lambda T} \psi_\lambda, & \text{for any } \lambda \in G, \\ (e[\lambda], q)_{L^2(0,T)} = 0, & \text{for any } \lambda \in \Lambda \backslash G, \end{cases} \tag{PM}$$

that satisfies

$$\| q_{G,\psi,T} \|_{L^2(0,T)} \leqslant C e^{C r_G^\theta + C T^{-\frac{\theta}{1-\theta}}} e^{-r_G T} \max_{L \subset G} |\psi[L]|.$$

Back to the full scalar moment problem

It's time to sum up everything ...

Lemma

Let $n \in \mathbb{N}^*$, $\rho > 0$. *If* Λ *satisfies the following* **weak gap** *condition*

$$\#\left(\Lambda \cap D(\mu, \rho/2)\right) \leqslant n, \quad \forall \mu \in \mathbb{C},$$

then we can write

$$\Lambda = \bigsqcup_{G \in \mathcal{G}} G, \tag{1}$$

where each $G \in \mathcal{G}$ *is a finite set satisfying the assumptions we considered above*

$$\#G \leqslant n, \quad \mathrm{diam}(G) \leqslant \rho, \quad d(\mathrm{Conv}(G), \Lambda \backslash G) \geqslant \gamma.$$

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Theorem (Small time null-controllability)

Let Λ *satisfying the assumptions* **sector/asymptotics/weak gap**, *and* $\mathcal{G}$ *as in* (1).
Assume that for some $M > 0$ *we have*

$$\max_{L \subset G} |\psi[L]| \leqslant M, \quad \forall G \in \mathcal{G},$$

then for every $T > 0$, *the full moment problem (= the NC problem) has a solution* $v \in L^2(0,T)$ *s.t.*

$$\|v\|_{L^2(0,T)} \leqslant CMe^{CT^{-\frac{\theta}{1-\theta}}}.$$

Boundary control for 1D cascade parabolic systems

(Fernandez–Cara - González–Burgos - de Teresa, '10)

The following system is null-controllable at any time $T > 0$

$$\partial_t y + \begin{pmatrix} \mathcal{A} & 1 \\ 0 & \mathcal{A} \end{pmatrix} y = 0, \quad y(t,0) = \begin{pmatrix} 0 \\ u(t) \end{pmatrix}, \quad y(t,1) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

with $\mathcal{A} = -\partial_x(\gamma(x)\partial_x \cdot)$.

Back to the full scalar moment problem

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with $\mathcal{A} = -\partial_x(\gamma(x)\partial_x \cdot)$.

Uniform boundary control for discrete 1D cascade parabolic systems

(Allonsius-B.-Morancey, '18)

$$\begin{cases} \partial_t y_h + \begin{pmatrix} \mathcal{A}_h & 1 \\ 0 & \mathcal{A}_h \end{pmatrix} y_h = \mathcal{B}_h \begin{pmatrix} 0 \\ u_h(t) \end{pmatrix} \\ y_h(0) = y_{0,h}, \end{cases}$$

where $\mathcal{A}_h$ is the F.D. approximation of $\mathcal{A}$ and and $\mathcal{B}_h$ is the discrete boundary control operator.

Theorem (Relaxed uniform null-controllability)

There exists $C > 0$ and $h_0 > 0$ such that : For any $h < h_0$, any initial data $y_{0,h}$, there exists a $u_h \in L^2(0,T,U_h)$ such that

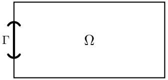
$$\|u_h\|_{L^2(0,T)} \leqslant C\|y_{0,h}\|_h,$$
$$\|y_h(T)\|_h \leqslant Ce^{-C/h^2}\|y_{0,h}\|_h.$$

Back to the full scalar moment problem

Some examples in 2D

The cascade system on a rectangle

(Benabdallah - B. - Gonzalez–Burgos - Olive, '14) (Allonsius - B., '20)



$$\partial_t y + \begin{pmatrix} -\Delta & 1 \\ 0 & -\Delta \end{pmatrix} y = 0, \quad y(t,.) = \begin{pmatrix} 0 \\ 1_\Gamma u(t,.) \end{pmatrix}. \qquad \text{(S)}$$

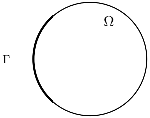
Theorem

For any non empty Γ the system (S) is null-controllable at any time $T > 0$, with the estimate

$$\|u\|_{L^2((0,T)\times\Gamma)} \leqslant Ce^{C/T}\|y_0\|.$$

The cascade system on a disk

(Trabut, '24)



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Back to the full scalar moment problem

Minimal null-control time issues

(Benabdallah - B. - Morancey, '20) (B., '23+)

Theorem

Let Λ *satisfying the assumptions* **sector/asymptotics/weak gap**, *and* $\mathcal{G}$ *as in* (1).
Assume that for some $M > 0$ *and some* $T^* > 0$, *we have*

$$\max_{L \subset G} |\psi[L]| \leqslant M e^{r_G T^*}, \quad \forall G \in \mathcal{G}, \tag{2}$$

then for every $T > T^*$, *the full moment problem (= the NC problem) has a solution* $v \in L^2(0,T)$ *s.t.*

$$\|v\|_{L^2(0,T)} \leqslant C_{T^*} M e^{C(T-T^*)^{-\frac{\theta}{1-\theta}}}.$$

Remark : Conversely if the NC at time T has a solution, then (2) holds for $T^* = T$.

The minimal null control time for this problem is thus the quantity

$$T_0 = \limsup_{G \in \mathcal{G}} \frac{\ln\left(\max_{L \subset G} |\psi[L]|\right)}{r_G}.$$

Back to the full scalar moment problem

Some examples

1D boundary control - non constant coupling **(Ammar-Khodja - Benabdallah - González-Burgos - de Teresa, '16)**

$$\partial_t y + \begin{pmatrix} \mathcal{A} & a(x) \\ 0 & \mathcal{A} \end{pmatrix} y = 0, \quad y(t,0) = \begin{pmatrix} 0 \\ u(t) \end{pmatrix}, \quad y(t,1) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\rightsquigarrow$ There exists functions a such that the minimal null-control time $T_{0,a}$ is any *a priori* given number.

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⇝ There exists functions a such that the minimal null-control time $T_{0,a}$ is any *a priori* given number.

A two diffusion case - constant coupling

(Ammar-Khodja - Benabdallah - González-Burgos - de Teresa, '14)

$$\partial_t y + \begin{pmatrix} \mathcal{A} & 1 \\ 0 & -d\mathcal{A} \end{pmatrix} y = 0, \quad y(t,0) = \begin{pmatrix} 0 \\ u(t) \end{pmatrix}, \quad y(t,1) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

⇝ There exists (many) coefficients $d > 0$ such that the minimal null-control time $T_{0,d}$ is any *a priori* given number.

Back to the full scalar moment problem

Some examples

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$$\partial_t y + \begin{pmatrix} \mathcal{A} & a(x) \\ 0 & \mathcal{A} \end{pmatrix} y = 0, \quad y(t,0) = \begin{pmatrix} 0 \\ u(t) \end{pmatrix}, \quad y(t,1) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\rightsquigarrow$ There exists functions a such that the minimal null-control time $T_{0,a}$ is any *a priori* given number.

A two diffusion case - constant coupling **(Ammar-Khodja - Benabdallah - González-Burgos - de Teresa, '14)**

$$\partial_t y + \begin{pmatrix} \mathcal{A} & 1 \\ 0 & -d\mathcal{A} \end{pmatrix} y = 0, \quad y(t,0) = \begin{pmatrix} 0 \\ u(t) \end{pmatrix}, \quad y(t,1) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\rightsquigarrow$ There exists (many) coefficients $d > 0$ such that the minimal null-control time $T_{0,d}$ is any *a priori* given number.

Less involved (yet interesting) example **(B. - Benabdallah - Morancey, '20)**

$$\partial_t y + \begin{pmatrix} \mathcal{A} & 1 \\ 0 & \mathcal{A} + b(x) \end{pmatrix} y = 0, \quad y(t,0) = \begin{pmatrix} 0 \\ u(t) \end{pmatrix}, \quad y(t,1) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\rightsquigarrow$ For b small enough in L^2 the system is null-controllable at any time $T > 0$ despite the spectral condensation that occurs in the system.

Outline

Few words about some non scalar moment problem

Reminder

For a non scalar control : the moment problem to solve is more involved

$$\text{Find } v \in L^2(0,T;U) \text{ such that } \quad \int_0^T e^{-\lambda t} \left(v(t), \mathcal{B}^\star \phi_\lambda\right)_U \, dt = -e^{-\lambda T} \langle y_0, \phi_\lambda \rangle, \quad \forall \lambda \in \Lambda.$$

Abstract problem

Given a family $(b_\lambda)_{\lambda\in\Lambda} \subset U \backslash \{0\}$, a family of scalars $(\omega_\lambda)_{\lambda\in\Lambda} \subset \mathbb{C}$, can we find $v \in L^2(0,T;U)$ such that

$$(e[\lambda] b_\lambda, v)_{L^2(0,T;U)} = \omega_\lambda, \quad \forall \lambda \in \Lambda.$$

Partial version

Given $G \subset \Lambda$, find $q = q_{G,b,\omega,T}$ such that

$$\begin{cases} (e[\lambda] b_\lambda, q)_{L^2(0,T;U)} = \omega_\lambda, & \forall \lambda \in G, \\ (e[\lambda], q)_{L^2(0,T)} = 0_U, & \forall \lambda \in \Lambda \backslash G. \end{cases}$$

Partial version

Given $G \subset \Lambda$, find $q = q_{G,b,\omega,T}$ such that

$$\begin{cases} (e[\lambda]b_\lambda, q)_{L^2(0,T;U)} = \omega_\lambda, & \forall \lambda \in G, \\ (e[\lambda], q)_{L^2(0,T)} = 0_U, & \forall \lambda \in \Lambda \backslash G. \end{cases} \tag{VPM}$$

Two particular *limiting* cases

Case 1: All the $(b_\lambda)_{\lambda \in G}$ are **colinear**

(VPM) is equivalent to a scalar moment problem

$\Rightarrow$ same estimates as before depending on the divided differences $\omega[L]$ for $L \subset G$.

Case 2: All the $(b_\lambda)_{\lambda \in G}$ are **pairwise orthogonal**

The eigenvalues in G do not see each other

$$q(t) = \sum_{\mu \in G} \omega_\mu \frac{b_\mu}{\|b_\mu\|^2} \tilde{q}_\mu(t),$$

where $\tilde{q}_\mu$ is the biorthogonal in $L^2(0, T)$ to $e[\mu]$ among the family $(\Lambda \backslash G) \cup \{\mu\}$.

Partial version

Given $G \subset \Lambda$, find $q = q_{G,b,\omega,T}$ such that

$$\begin{cases} (e[\lambda]b_\lambda, q)_{L^2(0,T;U)} = \omega_\lambda, & \forall \lambda \in G, \\ (e[\lambda], q)_{L^2(0,T)} = 0_U, & \forall \lambda \in \Lambda \backslash G. \end{cases} \qquad \text{(VPM)}$$

Fifty shades of grey

(B. - Morancey, '23)

Theorem

Consider the same assumptions as before on Λ and G.

For each G, we can build:

- *an* **explicit** $n \times n$ *matrix M_G depending only on G and $(b_\lambda)_{\lambda \in G}$*
- *an* **explicit** *vector $\xi_G \in \mathbb{C}^n$ depending only on the divided differences $\omega[L]$ with $L \subset G$*

such that there exists a solution to (VPM) *that satisfies*

$$\|q_{G,b,\omega,T}\|_{L^2(0,T;U)} \leqslant C e^{C r_G^\theta + C T^{-\frac{\theta}{1-\theta}}} \left(M_G \xi_G, \xi_G\right)^{\frac{1}{2}}.$$

The red factor is optimal.

(Gonzalez–Burgos - de Teresa, '16) (Ammar-Khodja - Benabdallah - Gonzalez–Burgos - de Teresa, '16) (B. - Morancey, '24)

1D distributed control - non constant coupling

$$\partial_t y + \begin{pmatrix} \mathcal{A} & a(x) \\ 0 & \mathcal{A} \end{pmatrix} y = \begin{pmatrix} 0 \\ 1_\omega u(t,x) \end{pmatrix},$$

- If $\omega \cap \mathrm{Supp}(a) \neq \emptyset$, the system is null-controllable at any time T.
- There exists a coupling term a and two non trivial control domains ω_1 and ω_2 that do not intersect $\mathrm{Supp}(a)$such that
  - If $\omega = \omega_1$, the system is null-controllable at any time $T > 0$.
  - If $\omega = \omega_2$, the system is not even approximately controllable.

Time discrete systems

(B. - Hernandez–Santamaria, '24)

Consider a discretization of the time interval $[0, T]$ with time step τ. Set $M = T/\tau$.

$$\frac{y^{n+1} - y^n}{\tau} + \begin{pmatrix} -\partial_x^2 & 1 \\ 0 & -\partial_x^2 \end{pmatrix} y^{n+1} = 0, \quad y^{n+1}(0) = \begin{pmatrix} 0 \\ u^{n+1} \end{pmatrix}, \quad y^{n+1}(1) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad (S_\tau)$$

Moment formulation : Instead of exponentials, use this family of time discrete functions

$$p[\lambda] := \left(n \in [\![0, M]\!] \mapsto (1 + \lambda\tau)^{-n} \right) \in L^2_\tau(0, T).$$

Theorem

Assume $\Lambda \subset (0, +\infty)$, the gap condition and $N_\Lambda(r) \leqslant \kappa r^\theta$.

There exists $\varpi > 0$, τ_0 depending only on ρ, κ, θ, such that :
For any $\tau < \tau_0$ there exists a family $(q_{\lambda,T})_{\substack{\lambda \in \Lambda \\ \lambda\tau \leqslant \varpi}}$

$$(p[\mu], q_{\lambda,T})_{L^2_\tau(0,T)} = \delta_{\lambda,\mu}, \quad \forall \lambda, \mu \in \Lambda, \text{ with } \lambda\tau \leqslant \varpi, \mu\tau \leqslant \varpi,$$

$$\|q_{\lambda,T}\|_{L^2_\tau(0,T)} \leqslant C_T e^{C\lambda^\theta}.$$

Same result with multiplicities ...

Time discrete systems

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Theorem

For any initial data $y^0 \in L^2(\Omega)$, for any $\tau < \tau_0$ there exists a time-discrete control $v_\tau = (v^n)_{n \in [\![0,M]\!]}$ such that

$$\|v_\tau\|_{L^2_\tau(0,T)} \leqslant C\|y^0\|_{L^2(\Omega)},$$

$$\|y^M\| \leqslant C e^{-\frac{C}{\tau^2}} \|y^0\|_{L^2(\Omega)}$$

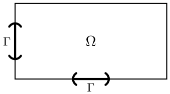
We have a similar result for fully discrete case.

Boundary controllability of a system with different diffusions

(B.-Olive, '24)

Let Ω be a rectangle and $\Gamma \subset \partial\Omega$.

$$\partial_t y + \begin{pmatrix} -\Delta & 1 \\ 0 & -d\Delta \end{pmatrix} y = 0, \quad y(t,.) = \begin{pmatrix} 0 \\ 1_\Gamma u(t,.) \end{pmatrix}$$



Theorem

If Γ intersects two **non parallel** *sides of $\partial\Omega$, then the system is null-controllable at any time $T > 0$,* **for any value of** d.

Boundary controllability of a system with different diffusions

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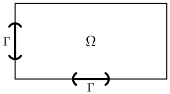

Theorem

If Γ intersects two **non parallel** *sides of $\partial\Omega$, then the system is null-controllable at any time $T > 0$,* **for any value of** *d.*

Everything boils down to a (very) weird family of moment-like problems

Here $\Omega = (0,\pi)^2$

Find two families $(u_k)_k, (v_l)_l \subset L^2(0,T)$ such that

$$\begin{cases} \displaystyle\int_0^T e^{-(k^2+l^2)t} u_k(t)\,dt + \int_0^T e^{-(k^2+l^2)t} v_l(t)\,dt = \omega_{k,l}, & \forall k,l \geqslant 1, \\ \displaystyle\int_0^T e^{-d(k^2+l^2)t} u_k(t)\,dt + \int_0^T e^{-d(k^2+l^2)t} v_l(t)\,dt = \tilde{\omega}_{k,l}, & \forall k,l \geqslant 1. \end{cases}$$

Thanks for your attention !
Any questions ?

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