Quantification of unique continuation for the Laplace operator with rough potentials

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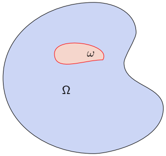
Unique continuation property

Consider a second order elliptic partial differential operator



$$Pu = \mathrm{div}(A(x).\nabla u) + W(x).\nabla u + V(x)u \quad \text{in } \Omega \subset \mathbb{R}^d.$$

$$(Pu = 0 \text{ in } \Omega \text{ and } u = 0 \text{ in } \omega) \xrightarrow{\text{When?}} u = 0 \ \text{ in } \ \Omega \text{ (UCP)}.$$

	A	W	V	Tools
Holmgren 1901	analytic	analytic	analytic	Analyticity of u
Carleman39,Hörmander63	$W^{1,\infty}$	L^∞	L^∞	L^2-Carleman estimates
Jerison-Kenig 1985	Id	0	$L^{\frac{d}{2}}$	$L^p - L^{p'}$-Carleman $\||x|^{-\tau} u\|_{L^{\frac{2d}{d-2}}} \lesssim \||x|^{-\tau} \triangle u\|_{L^{\frac{2d}{d+2}}}$
Barcelo-Kenig-Ruiz-Sogge87	Id	$L^{\frac{3d-2}{2}}$	$L^{\frac{d}{2}}$	$L^p - L^{p'}$-Carleman Only
Wolff 1992 Koch-Tataru 2001	Id $W^{1,\infty}$	L^d L^d	$L^{\frac{d}{2}}$ $L^{\frac{d}{2}}$	$L^p - L^{p'}$-Carleman + Wolff's argument
$V \in L^{d/2}$ and $W \in (L^d)^d$ are the optimal spaces to obtain unique continuation Koch-Tataru 2002.				

Table: Unique continuation results for the operator P.

Unique continuation property

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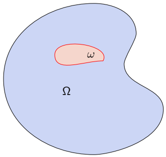

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Global quantitative unique continuation

If Pu *is "small" in* Ω, and u *is "small" in* ω, then u *is "small" in* Ω.

Analytically, $\|u\|_\Omega \leqslant C(\|Pu\|_\Omega, \|u\|_\omega)$.

Global quantitative unique continuation

Theorem (Dehman-Ervedoza-Thabouti '23)

Let $d \geqslant 3$, $\Omega \subset \mathbb{R}^d$ C^3*-bounded domain,* $\overline{\omega} \subset \Omega$ *a non-empty open subset.* $\exists C = C(\Omega, \omega) > 0$ *such that for any* $V \in L^{q_0}(\Omega)$, $W \in L^{q_1}(\Omega; \mathbb{C}^d)$, *the corresponding solution* $u \in H_0^1(\Omega)$

$$\Delta u = Vu + W \cdot \nabla u \quad \text{in } \Omega,$$

satisfies, for $q_0 > d/2$, $q_1 > (3d-2)/2$

$$\|u\|_{L^2(\Omega)} \leqslant C e^{C\left(\|V\|_{L^{q_0}}^{\gamma(q_0)} + \|W_1\|_{L^{q_1}}^{\delta(q_1)}\right)} \|u\|_{L^{\frac{2d}{d-2}}(\omega)},$$

with

$$\gamma(q) = \begin{cases} \dfrac{1}{\frac{3}{2}\left(1 - \frac{d}{2q}\right) + \frac{1}{2q}} & \text{if } q \geqslant d, \\[2ex] \dfrac{1}{\left(\frac{3}{4} + \frac{1}{2d}\right)\left(2 - \frac{d}{q}\right)} & \text{if } q \in \left(\frac{d}{2}, d\right] \end{cases} \; ; \quad \delta(q) = \frac{2}{1 - \frac{3d-2}{2q}}.$$

Global quantitative unique continuation

Theorem (Dehman-Ervedoza-Thabouti '23)

Let $d \geqslant 3$, $\Omega \subset \mathbb{R}^d$ C^3-bounded domain, $\overline{\omega} \subset \Omega$ a non-empty open subset. $\exists C = C(\Omega, \omega) > 0$ such that for any $V \in L^{q_0}(\Omega)$, $W \in L^{q_1}(\Omega; \mathbb{C}^d)$, the corresponding solution $u \in H_0^1(\Omega)$

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Aim : Handle potentials W in L^{q_1}, where $d < q_1 \leqslant \frac{3d-2}{2}$?

Improved quantitative unique continuation result

Theorem (Caro-Ervedoza-Thabouti '24)

Let $d \geqslant 3$, $\Omega \subset \mathbb{R}^d$ be a C^3-bounded domain, and ω and $\mathcal{O}$ be non-empty open subsets of Ω with $\omega \subset \overline{\omega} \subset \mathcal{O} \subset \overline{\mathcal{O}} \subset \Omega$. Then $\exists C = C(\omega, \mathcal{O}, \Omega) > 0$, $\alpha \in (0,1)$ depending only on ω, $\mathcal{O}$ and Ω that for any solution $u \in H^1(\Omega)$ of

$$\Delta u = W \cdot \nabla u \quad \text{in } \mathcal{D}'(\Omega),$$

$$W \in L^q(\Omega; \mathbb{C}^d), \quad \text{with } q \in (d, \infty].$$

we have

$$\|u\|_{H^1(\mathcal{O})} \leqslant C e^{C\|W\|_{L^q}^{\delta(q)}} \|u\|_{H^1(\omega)}^{\alpha} \|u\|_{H^1(\Omega)}^{1-\alpha},$$

with

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Tools :

- Wolff's Lemma.
- A refined version of L^p Carleman estimates.

Wolff's Lemma

Lemma (T.Wolff 1992)

Suppose μ is a positive measure in $\mathbb{R}^d$ which has faster than exponential decay in the following sense

$$\lim_{T\to\infty} T^{-1}\log(\mu\{x \in \mathbb{R}^d, |x| \geqslant T\}) = -\infty.$$

For $k \in \mathbb{R}^d$, define a measure μ_k by $d\mu_k(x) = e^{k\cdot x} d\mu(x)$. Suppose $\mathcal{C} \subset \mathbb{R}^d$ is a compact convex set. Then there is a sequence $\{k_j\} \subset \mathcal{C}$ and disjoint convex sets $\{E_{k_j}\} \subset \mathbb{R}^d$ so that the measures $d\mu_{k_j}$ are concentrated in E_{k_j},

$$\mu_{k_j}(\mathbb{R}^d \setminus E_{k_j}) \leqslant \frac{1}{2}\|\mu_{k_j}\|, \qquad \text{(concentration property)}$$

and such that

$$\sum_j |E_{k_j}|^{-1} \geqslant C_W^{-1}|\mathcal{C}|, \qquad \text{(summation property)}$$

where C_W is a positive constant depending only on d.

L^p Carleman-Wolff type estimates

Weight function: Let $\varphi \in C^3(\overline{\Omega})$, $\Omega \subset \mathbb{R}^d$ bounded domain and $\varnothing \neq \overline{\omega} \subset \Omega$, be such that $\exists \alpha, \beta > 0$

- $\inf_{x \in \overline{\Omega} \setminus \omega} |\nabla \varphi(x)| > \alpha$
- $\forall x \in \overline{\Omega} \setminus \omega$, $\forall \xi \in \mathbb{R}^d$ with $|\nabla \varphi(x)| = |\xi|$ and $\nabla \varphi(x) \cdot \xi = 0$,
  $$(D^2\varphi)\nabla\varphi(x) \cdot \nabla\varphi(x) + (D^2\varphi)\xi \cdot \xi \geqslant \beta |\nabla \varphi(x)|^2.$$

} Sub-ellipticity conditions.

Theorem (Caro-Ervedoza-Thabouti '24)

Let $d \geqslant 3$. Then $\forall K$ compact subset of Ω, $\exists C = C(\Omega, \omega, \|\varphi\|_{C^3(\overline{\Omega})}) > 0$, $\tau_0 \geqslant 1$ such that $\forall u \in H^1(\Omega)$ satisfying $\operatorname{supp} u \subset K$ and

$$\Delta u = f_2 + f_{2*'} \quad \text{in } \mathcal{D}'(\Omega),$$

with $(f_2, f_{2'_*})$ satisfying

$$f_2 \in L^2, \quad \text{and } f_{2'_*} \in L^{\frac{2d}{d+2}}, \qquad \text{we have }, \forall \tau \geqslant \tau_0,$$

$$\tau^{\frac{3}{2}} \|e^{\tau\varphi} u\|_{L^2(\Omega)} + \tau^{\frac{1}{2}} \|e^{\tau\varphi} \nabla u\|_{L^2(\Omega)} \leqslant C \left(\|e^{\tau\varphi} f_2\|_{L^2(\Omega)} + \tau^{\frac{3}{4} - \frac{1}{2d}} \|e^{\tau\varphi} f_{2'_*}\|_{L^{\frac{2d}{d+2}}(\Omega)} + \tau^{\frac{3}{2}} \|e^{\tau\varphi} u\|_{H^1(\omega)} \right),$$

and, for all measurable sets E of Ω,

$$\tau^{\frac{3}{4}+\frac{1}{2d}} \|e^{\tau\varphi} u\|_{L^{\frac{2d}{d-2}}(\Omega)} + \tau^{\frac{3}{4}+\frac{1}{2d}} \min\left\{ \frac{1}{\tau |E|^{\frac{1}{d}}}, 1 \right\} \|e^{\tau\varphi} \nabla u\|_{L^2(E)} \leqslant C \Big(\|e^{\tau\varphi} f_2\|_{L^2(\Omega)}$$

$$+ \tau^{\frac{3}{4}+\frac{1}{2d}} \|f_{2'_*}\|_{L^{\frac{2d}{d+2}}(\Omega)} + \tau^{\frac{3}{2}} \|e^{\tau\varphi} u\|_{H^1(\omega)} \Big).$$

A specific geometric setting

Lemma

Let $R > 0$ *and* $d \geqslant 3$. *Under the assumptions of Theorem 1 with the following setting:*

$$\Omega = B_0(4R),\ \mathcal{O} = B_0(R),\ \text{and } \omega = B_0(4R) \setminus B_{4Re_1}(4R),$$

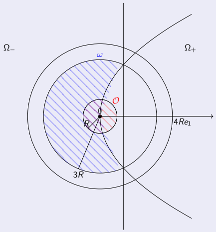

Figure: This allows to propagate information from the blue to the red region.

$\rightsquigarrow \exists C = C(R,d) > 0$ *and* $\alpha \in (0,1)$ *depending only on* R *and* d *that for any solution*

$$\Delta u = W \cdot \nabla u \quad \text{in } \Omega,$$

with $W \in L^q$, $q \in (d, +\infty]$, *satisfies*

$$\|u\|_{H^1(\mathcal{O})} \leqslant C e^{C\|W\|_{L^q}^{\delta(q)}} \|u\|^{\alpha}_{H^1(\omega)} \|u\|^{1-\alpha}_{H^1(\Omega)},$$

with $\delta(q) = 2/(1 - d/q)$.

Family of weight functions

Lemma

Within the same setting as in the previous Lemma, there exist $\epsilon > 0$, a family of weight functions $\{\varphi_k\}$, where $k \in \mathbb{R}^d$, such that

1. *The functions $\{\varphi_k\}$, for all $k \in \Sigma_\epsilon = \{k \in \mathbb{R}^d \setminus \{0\}$ with $|\frac{k}{|k|} - e_1| \leqslant \varepsilon\}$, satisfy the sub-ellipticity conditions with some uniform positive constants $\alpha > 0$ and $\beta > 0$.*
2. *The functions $\{\varphi_k\}$ satisfy the Wolff's Lemma in the following sense : If $d\mu$ is a positive compactly supported measure in Ω, we define the family $d\mu_k(x) = e^{\varphi_k(x)} d\mu$, then for $\mathcal{C} \subset \mathbb{R}^d$ there exist $\{k_j\} \subset \mathcal{C}$ and disjoint sets $E_{k_j} \subset \mathbb{R}^d$ so that measures $d\mu_{k_j}$ satisfy the* (concentration property) *and $\{E_{k_j}\}$ satisfy* (summation property).

Proof: Step 1: Application of the Carleman estimates.

For $u \in H^1(B_0(4R))$ with $\Delta u = W \cdot \nabla u, \quad \text{in } B_0(4R)$, we set $v = \eta u$, where η is a cut-off that takes 1 in $B_0(3R)$ and vanishes in a neighbourhood of $B_0(4R)$, so that we have

$$\Delta v = W \cdot \nabla v + f_\eta \quad \text{in } B_0(4R),$$

where $\text{supp}(f_\eta) \subset \mathcal{A}_0(3R, 4R)$.

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where $\text{supp}(f_\eta) \subset \mathcal{A}_0(3R, 4R)$. For $W \in L^q(B_0(4R); \mathbb{C}^d)$, $q > d$

$$W = W_d + W_\infty, \qquad \text{with } W_d \in L^d(B_0(4R); \mathbb{C}^d), \; W_\infty \in L^\infty(B_0(4R); \mathbb{C}^d).$$

Proof: Step 1: Application of the Carleman estimates.

For $u \in H^1(B_0(4R))$ with $\Delta u = W \cdot \nabla u$, in $B_0(4R)$, we set $v = \eta u$, where η is a cut-off that takes 1 in $B_0(3R)$ and vanishes in a neighbourhood of $B_0(4R)$, so that we have

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$$W = W_d + W_\infty, \qquad \text{with } W_d \in L^d(B_0(4R); \mathbb{C}^d),\ W_\infty \in L^\infty(B_0(4R); \mathbb{C}^d).$$

Applying L^p Carleman estimates with $f_{2'_*} = W_d \cdot \nabla v$, and $f_2 = W_\infty \cdot \nabla v + f_\eta$,

$$\text{if} \qquad \|W_\infty\|_{L^\infty} \lll |k|^{\frac{1}{2}}, \tag{1}$$

that for all $k \in \Sigma_\varepsilon$ with $|k| \geqslant \tau_0$,

$$\bullet |k|^{\frac{3}{2}} \|e^{|k|\varphi_k} v\|_{L^2(B_0(4R))} + |k|^{\frac{1}{2}} \|e^{|k|\varphi_k} \nabla v\|_{L^2(B_0(4R))} \leqslant C_0 \left(\|e^{|k|\varphi_k} f_\eta\|_{L^2(B_0(4R))} + |k|^{\frac{3}{4} - \frac{1}{2d}} \|e^{|k|\varphi_k} W_d \cdot \nabla v\|_{L^{\frac{2d}{d+2}}(B_0(4R))} + |k|^{\frac{3}{2}} \|e^{|k|\varphi_k} v\|_{H^1(\omega)} \right).$$

and

$$\bullet |k|^{\frac{3}{4} + \frac{1}{2d}} \|e^{|k|\varphi_k} v\|_{L^{\frac{2d}{d-2}}(B_0(4R))} + |k|^{\frac{3}{4} + \frac{1}{2d}} \min\left\{ \frac{1}{|k||E|^{\frac{1}{d}}}, 1 \right\} \|e^{|k|\varphi_k} \nabla v\|_{L^2(E)} \leqslant C_0 \Big(\|e^{|k|\varphi_k} f_\eta\|_{L^2(B_0(4R))} + |k|^{\frac{3}{4} + \frac{1}{2d}} \|e^{|k|\varphi_k} W_d \cdot \nabla v\|_{L^{\frac{2d}{d+2}}(B_0(4R))} + \|W_\infty \cdot \nabla v\|_{L^2(B_0(4R))} + |k|^{\frac{3}{2}} \|e^{|k|\varphi_k} v\|_{H^1(\omega)} \Big).$$

Step 2: Wolff's argument.

Let $n \in \mathbb{N}$ be large, set $\mathcal{C}_n = \{k \in \mathbb{R}^d : |k - ne_1| \leqslant \gamma n\}$, γ small enough so that for all $k \in \mathcal{C}_n$, $k \in \Sigma_\varepsilon$, applying Wolff's Lemma with the measure

$$d\mu_k(x) = |e^{|k|\varphi_k(x)} W_d(x) \cdot \nabla v(x)|^{\frac{2d}{d+2}} 1_{B_0(4R)}(x)\, dx.$$

$\rightsquigarrow \exists C_W > 0$; $\forall n \in \mathbb{N}$, $\exists I_n$, $(k_{i,n})_{i \in I_n} \subset \mathcal{C}_n$ and corresponding family of pairwise disjoint sets $(E_{k_{i,n}})_{i \in I_n}$ such that $\forall i \in I_n$, we have

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- $\|e^{|k_{i,n}|\varphi_{k_{i,n}}} W_d \cdot \nabla v\|_{L^{\frac{2d}{d+2}}(B_0(4R))} \leqslant 2\|e^{|k_{i,n}|\varphi_{k_{i,n}}(x)} W_d(x) \cdot \nabla v(x)\|_{L^{\frac{2d}{d+2}}(E_{k_{i,n}})}$,

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- $\sum_{i \in I_n} |E_{k_{i,n}}|^{-1} \geqslant C_W^{-1} n^d$.

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- $\|e^{|k_{i,n}|\varphi_{k_{i,n}}} W_d \cdot \nabla v\|_{L^{\frac{2d}{d+2}}(B_0(4R))} \leqslant 2\|e^{|k_{i,n}|\varphi_{k_{i,n}}(x)} W_d(x) \cdot \nabla v(x)\|_{L^{\frac{2d}{d+2}}(E_{k_{i,n}})}$,
- $\sum_{i \in I_n} |E_{k_{i,n}}|^{-1} \geqslant C_W^{-1} n^d$.

Hence, if the condition,

$$\|W_d\|^d_{L^d(B_0(4R))} < C_W^{-1}(8C_0(1+\gamma))^{-d},$$

is satisfied, then for all $n \in \mathbb{N}$, there exists $i_* \in I_n$ such that

$$8C_0\|W_d\|_{L^d(E_{k_{i_*,n}})} \leqslant \frac{1}{|k_{i_*,n}||E_{k_{i_*,n}}|^{\frac{1}{d}}}.$$

Quantification

Appling the last two estimates for n larger than $\tau_0/(1-\gamma)$, with $k = k_{i_*,n}$ denoted by k_n

$$\bullet |k_n|^{\frac{3}{2}} \|e^{|k_n|\varphi_{k_n}} v\|_{L^2(B_0(4R))} + |k_n|^{\frac{1}{2}} \|e^{|k_n|\varphi_{k_n}} \nabla v\|_{L^2(B_0(4R))} \leqslant C_1 \|e^{|k_n|\varphi_{k_n}} f_\eta\|_{L^2(B_0(4R))} + C_1 \left(4|k_n|^{\frac{3}{4}-\frac{1}{2d}} \|W_d\|_{L^d(E_{k_n})} \|e^{|k_n|\varphi_{k_n}} \nabla v\|_{L^2(E_{k_n})} + |k_n|^{\frac{3}{2}} \|e^{|k_n|\varphi_{k_n}} v\|_{H^1(\omega)} \right)$$

similarly, we have

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$$\bullet |k_n|^{\frac{3}{2}} \| e^{|k_n|\varphi_{k_n}} v \|_{L^2(B_0(4R))} + |k_n|^{\frac{1}{2}} \| e^{|k_n|\varphi_{k_n}} \nabla v \|_{L^2(B_0(4R))} \leqslant C_1 \| e^{|k_n|\varphi_{k_n}} f_\eta \|_{L^2(B_0(4R))} + C_1 \left(4 |k_n|^{\frac{3}{4}-\frac{1}{2d}} \| W_d \|_{L^d(E_{k_n})} \| e^{|k_n|\varphi_{k_n}} \nabla v \|_{L^2(E_{k_n})} + |k_n|^{\frac{3}{2}} \| e^{|k_n|\varphi_{k_n}} v \|_{H^1(\omega)} \right)$$

similarly, we have

$$\bullet |k_n|^{\frac{3}{4}+\frac{1}{2d}} \| e^{|k_n|\varphi_{k_n}} v \|_{L^{\frac{2d}{d-2}}(B_0(4R))} + |k_n|^{\frac{3}{4}+\frac{1}{2d}} \min \left\{ \frac{1}{|k_n||E_{k_n}|^{\frac{1}{d}}}, 1 \right\} \| e^{|k_n|\varphi_{k_n}} \nabla v \|_{L^2(E_{k_n})} \leqslant C_1 \left(\| e^{|k_n|\varphi_{k_n}} f_\eta \|_{L^2(B_0(4R))} + \| e^{|k_n\varphi_{k_n}} W_\infty \cdot \nabla v \|_{L^2(B_0(4R))} + |k_n|^{\frac{3}{2}} \| e^{|k_n|\varphi_{k_n}} v \|_{H^1(\omega)} \right).$$

Combining two inequalities gives

$$\bullet |k_n|^{\frac{3}{2}} \| e^{|k_n|\varphi_{k_n}} v \|_{L^2} + |k_n|^{\frac{1}{2}} \| e^{|k_n|\varphi_{k_n}} \nabla v \|_{L^2} \leqslant C \left(\| e^{|k_n|\varphi_{k_n}} f_\eta \|_{L^2} + |k_n|^{\frac{3}{2}} \| e^{|k_n|\varphi_{k_n}} v \|_{H^1(\omega)} \right),$$

Step 3: Quantification. To quantify the unique continuation property, we simply need to choose appropriate values for n (recall that k_n is of the order of n), such that:

$$\| W_d \|_{L^d} \lll 1, \qquad \| W_\infty \|_{L^\infty} \lll n^{\frac{1}{2}}.$$

3-balls type estimate

Lemma

Let $R > 0$ and $d \geqslant 3$. Under the assumptions of Theorem 1 with the following setting:

$$\Omega = B_0(4R),\ \mathcal{O} = B_0(2R),\ \text{and } \omega = B_0(R);$$

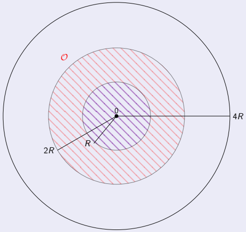

Figure: Geometry of the three balls. This allows to propagate information from the blue to the red region.

$\rightsquigarrow$ $\exists C = C(R, d) > 0$ *and* $\alpha \in (0, 1)$ *depending only on* R *and* d *that for any solution*

$$\Delta u = W \cdot \nabla u \quad \text{in } \Omega,$$

with $W \in L^q(\Omega; \mathbb{C}^d)$ *and* $q \in (d, \infty]$, *satisfies*

$$\|u\|_{H^1(\mathcal{O})} \leqslant Ce^{C\|W\|_{L^q}^{\delta(q)}} \|u\|_{H^1(\omega)}^{\alpha} \|u\|_{H^1(\Omega)}^{1-\alpha},$$

$\delta(q) = 2/(1 - d/q)$.

The general case: Proof of Theorem 1

Step 1: Compactness argument. Because of the compactness of $\overline{\mathcal{O}}$, it suffices to prove our quantitative estimate with $B_y(r)$ in place of $\mathcal{O}$ where $y \in \overline{\mathcal{O}}$ and $r > 0$ will be chosen sufficiently small.

The general case: Proof of Theorem 1

Step 1: Compactness argument. Because of the compactness of $\overline{\mathcal{O}}$, it suffices to prove our quantitative estimate with $B_y(r)$ in place of $\mathcal{O}$ where $y \in \overline{\mathcal{O}}$ and $r > 0$ will be chosen sufficiently small.
Step 2: Propagation of smallness within the ball $B_y(r)$. Let $x_{(0)}$ be in ω and $r_0 > 0$ such that $B_{x_{(0)}}(r_0) \Subset \omega$. As Ω is open and connected, there exists a path $\Gamma \subset \Omega$ from $x_{(0)} = \Gamma(0)$ to $y = \Gamma(1)$. Set $r_1 = \mathrm{dist}(\Gamma, \partial\Omega)$. We have $r_1 > 0$ by compactness.
Setting now $r = \inf(R, r_0, r_1/4)$, where $R = \mathrm{dist}(\mathcal{O}, \partial\Omega)/4$, we define a sequence $(x_{(j)})_j$, for $j \geqslant 0$, by $x_{(j)} = \Gamma(t_j)$ where $t_0 = 0$ and

$$t_j = \begin{cases} \inf A_j & \text{if } A_j \neq \emptyset, \\ 1 & \text{if } A_j = \emptyset, \end{cases} \qquad \text{where } A_j = \left\{ \sigma \in (t_{j-1}, 1]\,;\, \Gamma(\sigma) \notin B_{x_{(j-1)}}(r) \right\}.$$

The sequence $(x_{(j)})_j$ is finite by a compactness argument. Let $\left(x_{(0)}, \cdots, x_{(N)}\right)$ be such a sequence with $x_{(N)} = y$. Note that we have $B_{x_{(j+1)}}(r) \subset B_{x_{(j)}}(2r) \Subset \Omega$ for $j = 0, \cdots, N-1$, because of the choice we made for r above. By Lemma 7 there exist $C > 0$ and $\alpha \in (0,1)$ such that

$$\|u\|_{H^1\left(B_{x_{(j+1)}}(r)\right)} \leqslant \|u\|_{H^1\left(B_{x_{(j)}}(2r)\right)} \leqslant Ce^{C\|W\|^{\delta(q)}_{L^q(\Omega)}} \|u\|^{1-\alpha}_{H^1(\Omega)} \|u\|^{\alpha}_{H^1(B_{x_{(j)}}(r))},$$

for $j = 0, \ldots, N-1$. By consequence, we have

$$\|u\|_{H^1(B_y(r))} \leqslant Ce^{C\|W\|^{\delta(q)}_{L^q(\Omega)}} \|u\|^{1-\alpha^N}_{H^1(\Omega)} \|u\|^{\alpha^N}_{H^1\left(B_{x_{(0)}}(r)\right)}.$$

This concludes the proof of the theorem.

Comment and questions :

Main result of quantitative unique continuation holds with

$$\Delta u = Vu + W_1 \cdot \nabla u + \operatorname{div}(W_2 u) \quad \text{in } \Omega,$$

where $V \in L^{q_0}(\Omega)$, $W_j \in L^{q_j}(\Omega; \mathbb{C}^d)$, and $q_0 \in (d/2, \infty]$, $q_j \in (d, \infty]$ with $j \in \{1, 2\}$.

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$$\begin{cases} -\Delta u + a \cdot \nabla u + \nabla p = 0 \text{ in } \Omega, \\ \operatorname{div}(u) = 0, \text{ in } \Omega, \end{cases} \tag{S}$$

  [Fabre-Lebeau, 1996] with $a \in L^\infty(\Omega)$ (If u solution of (S) and $u = 0$ in $\omega \subset \Omega$, then $u = 0$ in Ω). $a \in L^d(\Omega)$?

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Any questions or comments are welcomed.