Controllability of a multi-dimensional system of coupled parabolic equations

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X Partial differential equations, optimal design and numerics, Benasque

August 2024

Parabolic systems, null-controllability

Parabolic system (heat-like equation) :

$$\partial_t y + Ay = Bv,$$

where A is a (vectorial) elliptic operator, B is a control operator.

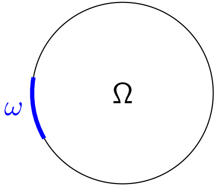
Null-controllability

Given an initial data y_0, and a time T, we want to find a control v, such that the solution of the system satisfies $y(T) = 0$.

The problem

Null-controllability of:

$$\begin{cases} \partial_t y_1 - \Delta y_1 = 0 \text{ in } [0,T] \times \Omega \\ \partial_t y_2 - \Delta y_2 + y_1 = 0 \text{ in } [0,T] \times \Omega \\ y_1 = 1_\omega v \text{ in } [0,T] \times \partial\Omega \\ y_2 = 0 \text{ in } [0,T] \times \partial\Omega \\ y(0,\cdot) = y_0 \end{cases}$$

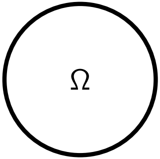


Main issues:

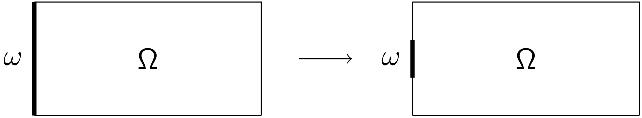
1. Less control than components, coupling
2. Boundary controllability
3. Multi-dimensional

Results from the literature

- Polar coordinates, properties of eigenvalues, moment method, [H.O.Fattorini, D.L.Russell, 1975].



- Control on a subset of the boundary, [A.Benabdallah, F.Boyer, M.González-Burgos, G.Olive, 2014].

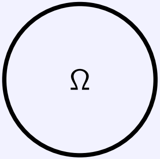


The result

Theorem [H.O.Fattorini, D.L.Russell, 1975]

For any $T > 0$, $y_0 \in H^{-1}(\Omega)$ the problem

$$\begin{cases} \partial_t y - \Delta y = 0 \text{ in } (0,T) \times \Omega, \\ y = v \text{ in } (0,T) \times \partial\Omega, \\ y(0) = y_0, \end{cases}$$



is null-controllable.

Observability and controllability

Theorem [S.Dolecki, D.L.Russell, 1977]

Let $T > 0$, the following properties are equivalent:

1. $\forall y_0 \in H^{-1}(\Omega)$, $\exists v \in L^2(0, T; U)$ such that the solution y of
$$\begin{cases} \partial_t y + Ay = Bv, \\ y(0) = y_0, \end{cases} \tag{1}$$
satisfies $y(T) = 0$ and $\|v\|_{L^2(0,T;U)} \leq C \|y_0\|_{H^{-1}(\Omega)}$.
2. $\forall q_T \in H_0^1(\Omega)$, the solution q of
$$\begin{cases} -\partial_t q + A^* q = 0, \\ q(T, \cdot) = q_T, \end{cases} \tag{2}$$
satisfies $\|q(0)\|_{H_0^1(\Omega)} \leq C \|B^* q\|_{L^2(0,T;U)}$.

2D problem → 1D problems

If $q_T = \sum_{\substack{m=0 \\ k\in\{0,1\}}}^{\infty} q_T^{mk}(r) Y_{mk}(\theta)$ and q satisfies

$$\begin{cases} -\partial_t q - \Delta q = 0 \text{ in } (0,T)\times\Omega, \\ q = 0 \text{ in } (0,T)\times\Omega, \\ q(T) = q_T, \end{cases}$$

then $q^{mk} = \langle q, Y_{mk}\rangle_{L^2(0,2\pi)}$ satisfies

$$\begin{cases} -\partial_t q^{mk} + L_m^* q^{mk} = 0 \text{ in } (0,T)\times(0,1), \\ q(\cdot,1) = 0 \text{ in } (0,T), \\ q^{mk}(T) = q_T^{mk}, \end{cases}$$

with $L_m = -\frac{1}{r}\partial_r(r\partial_r) + \frac{m^2}{r^2} id.$

Controllability 1D → Observability 1D

By the moment method, we can show that the system

$$\begin{cases} \partial_t y + L_m y = 0 \text{ in } (0,T) \times (0,1), \\ y_1(\cdot,1) = v \text{ in } (0,T), \\ y(0) = y_0 \in (\mathcal{V}_m)', \end{cases}$$

is null-controllable with a control cost $\|v\|_{L^2(0,T)} \leq Ce^{\frac{C}{T}} \|y_0\|_{(\mathcal{V}_m)'}$.

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is null-controllable with a control cost $\|v\|_{L^2(0,T)} \leq Ce^{\frac{C}{T}} \|y_0\|_{(\mathcal{V}_m)'}$.

By the equivalence between observability and controllability we get

$$\left\| q^{mk}(0) \right\|_m^2 \leq Ce^{\frac{C}{T}} \int_0^T \left| \frac{\partial q^{mk}(t)}{\partial r} \Big|_{r=1} \right|^2 dt.$$

<u>Remark :</u> C does not depend on m or T.

Observability 1D → Observability 2D

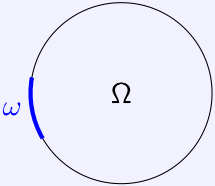
$$\|q(0)\|^2_{H^1_0(\Omega)} \leq Ce^{\frac{C}{T}} \int_0^T \left\| \frac{\partial q(t)}{\partial r}\Big|_{r=1} \right\|^2_{L^2(\partial\Omega)} dt$$

The main result

Theorem [M.T, 2024]

For any $T > 0$, $y_0 \in H^{-1}(\Omega)$ the problem

$$\begin{cases} \partial_t y_1 - \Delta y_1 = 0 \text{ in } [0,T] \times \Omega \\ \partial_t y_2 - \Delta y_2 + y_1 = 0 \text{ in } [0,T] \times \Omega \\ y_1 = 1_\omega v \text{ in } [0,T] \times \partial\Omega \\ y_2 = 0 \text{ in } [0,T] \times \partial\Omega \\ y(0,\cdot) = y_0 \end{cases}$$



is null-controllable.

The Lebeau-Robbiano strategy

[G.Lebeau, L.Robbiano, 1995]

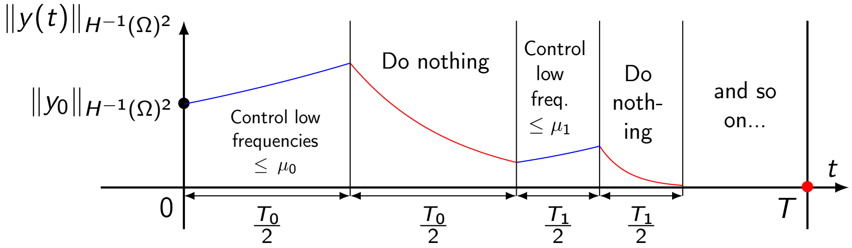


Figure. 1: The Lebeau-Robbiano method

Question : *How do we chose suitable controls ?*

Building of the control

Proposition

For any $M \in \mathbb{N}$, $\tau > 0$ and $y_0 \in H^{-1}(\Omega)^2$, there exists a control $v_M \in L^2((0,\tau) \times \omega)$ such that,

$$\|v_M\|_{L^2((0,\tau)\times\omega)} \leq C e^{\frac{C}{\tau}} e^{CM} \|y_0\|_{H^{-1}(\Omega)^2}$$
$$\|y(\tau)\|_{H^{-1}(\Omega)^2} \leq C_2 e^{\frac{C}{\tau}+CM} e^{-\frac{\tau}{2}M^2} \|y_0\|_{H^{-1}(\Omega)^2}$$

Question : *How do we obtain the dissipation $e^{-\frac{\tau}{2}M^2}$?*

Dissipation

The solution of

$$\partial y + \mathcal{A}y = 0 \text{ in } (\tau/2, \tau) \quad \text{where } \mathcal{A} = \begin{pmatrix} -\Delta & 0 \\ id & -\Delta \end{pmatrix}$$

is

$$y(\tau) = e^{-(\tau/2)\mathcal{A}} y(\tau/2)$$

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is

$$y(\tau) = e^{-(\tau/2)\mathcal{A}} y(\tau/2)$$

If we manage to have

$$y(\tau/2) \in \mathrm{Ker}(\mathcal{A}^* - \lambda)^{\perp}, \quad \forall \lambda \leq M^2,$$

we get the dissipation estimate,

$$\|y(\tau)\|_{H^{-1}(\Omega)^2} \leq C e^{-(\tau/2)M^2} \|y(\tau/2)\|_{H^{-1}(\Omega)^2}.$$

Question : *How do we get $y(\tau/2)$ in such space?*

Partial observability and controllability

Theorem [D.Allonsius, F.Boyer, 2018]

Let $T > 0$, and $E \subset H_0^1(\Omega)^2$ closed, the following are equivalent:

1. $\forall y_0 \in E'$, $\exists v \in L^2(0,T;U)$ such that the solution y of

$$\begin{cases} \partial_t y + Ay = Bv, \\ y(0) = y_0, \end{cases} \tag{3}$$

satisfies $\Pi_{E'} y(T) = 0$ and $\|v\|_{L^2(0,T;U)} \leq C \|y_0\|_{H^{-1}(\Omega)^2}$.

2. $\forall q_T \in E$, the solution q of

$$\begin{cases} -\partial_t q + A^* q = 0, \\ q(T, \cdot) = q_T, \end{cases} \tag{4}$$

satisfies $\|\Pi_E q(0)\|_{H_0^1(\Omega)^2} \leq C \|B^* q\|_{L^2(0,T;U)}$.

Question : *Which space E do we chose and how to prove observability?*

2D problem → 1D problems

For $M \geq 0$

$$E_M := \left\{ \sum_{\substack{m=0 \\ k\in\{0,1\}}}^{M} \langle u, Y_{mk} \rangle_{L^2(0,2\pi)} Y_{mk} \quad \middle| \quad u \in H_0^1(\Omega)^2 \right\} \subset H_0^1(\Omega)^2,$$

2D problem → 1D problems

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If $q_T = \sum q_T^{mk} Y_{mk} \in E_M$ and q satisfies

$$\begin{cases} -\partial_t q + \mathcal{A}^* q = 0, \\ q(T) = q_T, \end{cases} \quad \text{with } \mathcal{A}^* = \begin{pmatrix} -\Delta & id \\ 0 & -\Delta \end{pmatrix},$$

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If $q_T = \sum q_T^{mk} Y_{mk} \in E_M$ and q satisfies

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then $q^{mk} = \langle q, Y_{mk} \rangle_{L^2(0,2\pi)}$ satisfies

$$\begin{cases} -\partial_t q^{mk} + \mathcal{L}_m^* q^{mk} = 0, \\ q^{mk}(T) = q_T^{mk}, \end{cases} \quad \text{with} \quad \begin{matrix} \mathcal{L}_m^* = \begin{pmatrix} L_m & id \\ 0 & L_m \end{pmatrix} \\ L_m = -\frac{1}{r}\partial_r(r\partial_r) + \frac{m^2}{r^2} id. \end{matrix}$$

Controllability 1D → Observability 1D

By the moment method we can prove that the following system

$$\begin{cases} \partial_t y + \mathcal{L}_m y = 0 \text{ in } (0, T) \times (0, 1), \\ y_1(\cdot, 1) = v \text{ in } (0, T), \\ y_2(\cdot, 1) = 0 \text{ in } (0, T), \\ y(0) = y_0 \in (\mathcal{V}_m^2)', \end{cases}$$

is null-controllable with a control cost $\|v\|_{L^2(0,T)} \leq C e^{\frac{C}{T}} \|y_0\|_{(\mathcal{V}_m^2)'}$.

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And by equivalence between observability and controllability we get

$$\left\| q^{mk}(0) \right\|_m^2 \leq Ce^{\frac{C}{T}} \int_0^T \left| \frac{\partial q_1^{mk}(t)}{\partial r} \Big|_{r=1} \right|^2 dt.$$

Observability 1D → Observability 2D

Theorem (Spectral inequality) [D.Jerison, G.Lebeau, 1999]

Let $\mu > 0$, and ϕ a sum of eigenfunctions of the Laplace-Beltrami operator on $\partial\Omega$ of the form

$$\phi = \sum_{\substack{\lambda \in \Lambda \\ \lambda \leq \mu}} a_\lambda \phi_\lambda .$$

For any non-empty set $\omega \subset \partial\Omega$ there exists $C > 0$ depending only on ω such that

$$\|\phi\|_{L^2(\partial\Omega)} \leq C e^{C\sqrt{\mu}} \|\phi\|_{L^2(\omega)} .$$

$$\|q(0)\|^2_{H^1_0(\Omega)^2} \leq C e^{\frac{C}{T}} e^{CM} \int_0^T \left\| 1_\omega \frac{\partial q_1(t)}{\partial r}\Big|_{r=1} \right\|^2_{L^2(\partial\Omega)} dt.$$

We rewind the movie

Partial observability 2D $\Longrightarrow$ control which kills the low frequencies on $(0, \tau/2)$

No low frequencies $\Longrightarrow$ dissipation on $(\tau/2, \tau)$

We iterate cleverly to reach 0 at time T !

Conclusion

Key points and main issues:

- Moment method (uniform w.r.t m)
- Eigenvalues of L_m, Bessel functions
- (Partial) observability $\Leftrightarrow$ (Partial) controllability
- Spectral inequality
- The Lebeau-Robbiano method

Perspectives

- Apply this strategy for other multi-dimensional problems
- New methods ?

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Thanks for your attention !