Local subordination in Riesz basis analysis

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Based on the joint papers with B. Mityagin (OSU)

[1] B. Mityagin and P. Siegl [2016]. "Root system of singular perturbations of the harmonic oscillator type operators". In: *Lett. Math. Phys.* 106, pp. 147–167

[2] B. Mityagin and P. Siegl [2019]. "Local form-subordination condition and Riesz basisness of root systems". In: *J. Anal. Math.* 139, pp. 83–119

and work in progress

$$T = \underbrace{-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + |x|^\beta}_{A} + \underbrace{\mathrm{i}\,\mathrm{sgn}\, x|x|^\gamma}_{B}, \qquad \beta, \gamma \geq 0, \qquad \text{in } L^2(\mathbb{R})$$

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- two examples with $\beta = 2$

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- $\sigma(T_1) = \sigma_{\mathrm{disc}}(T_1) \subset \mathbb{R}$
- EV's are stable
- EF's form a Riesz basis
- $\sup_{t>0} \|e^{-\mathrm{i}tT_1}\| < \infty$

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- B_1 and B_2 are of “the same strength” 1/2-form-subordination:

$$|\langle B_j f, f\rangle| \lesssim \langle Af, f\rangle^p \ \|f\|^{2(1-p)}, \quad j = 1, 2, \quad p = \frac{1}{2}$$

- separable Hilbert space $\mathcal{H}$, $\mathcal{E} := \{\psi_n\} \subset \mathcal{H}$
- $\mathcal{E}$ is complete in $\mathcal{H}$: $\overline{\operatorname{span} \mathcal{E}} = \mathcal{H}$ or $\mathcal{E}^\perp = \{0\}$
- $\mathcal{E}$ is a basis in $\mathcal{H}$: every $\psi \in \mathcal{H}$ has a unique expansion

$$\psi = \sum_{n=1}^{\infty} c_n \psi_n .$$

- $\mathcal{E}$ is a Riesz basis in $\mathcal{H}$: $\mathcal{E}$ is a basis and for all $\psi \in \mathcal{H}$

$$m\|\psi\|^2 \leq \sum_{n=1}^{\infty} |\langle \psi_n, \psi \rangle|^2 \leq M\|\psi\|^2$$

or equivalently: $\exists W \in \mathscr{B}(\mathcal{H})$ with $W^{-1} \in \mathscr{B}(\mathcal{H})$ and ONB $\{e_n\}_n$ such that

$$\psi_n = W e_n .$$

Theorem [Mityagin and Siegl, 2019]

Let

$$T = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + |x|^\beta + V(x), \qquad \beta \geq 2, \qquad \text{in } L^2(\mathbb{R})$$

where

- $V = V_1 + V_2 + V_3 + V_4$ satisfies
  - $|V_1(x)| \lesssim \langle x \rangle^\gamma$ with $\gamma < \frac{\beta}{2} - 1$
  - $V_2 \in L^p(\mathbb{R})$ with $p \in [1, \infty)$
  - $V_3 \in W^{-s,2}(\mathbb{R})$ with $s \in [0, \frac{\beta-1}{2\beta})$
  - $V_4(x) = \sum_{k \in \mathbb{Z}} \nu_k \, \delta(x - x_k)$ with $\{\nu_k\} \in \ell^1(\mathbb{Z})$

Then the eigenvalues of T are eventually simple and the eigensystem contains a Riesz basis.

Strategy

- new perturbation theorems beyond classical ones (Kato, Dunford-Schwartz, Markus, Agranovich, . . .)

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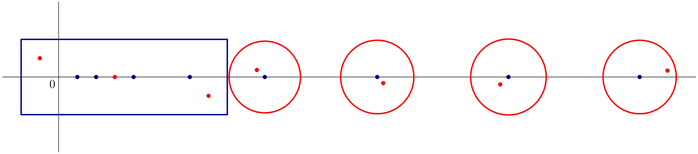
1. $\mu_{n+1} - \mu_n \to \infty$ as $n \to \infty$,
2. $\|B\| < \infty$.

Then for $T := A + B$,

- the eigenvalues $\{\lambda_n\}$ of T are eventually simple
- $\lambda_n = \mu_n + \mathcal{O}(1)$ as $n \to \infty$
- the eigensystem of T contains a Riesz basis



Theorem [Mityagin and Siegl, 2019]

Let $\mathcal{H}$ be separable, $A = A^* \geq 0$ with compact resolvent in $\mathcal{H}$ and eigenvalues $\{\mu_n\}$ of A be simple and let $\{\psi_n\}$ be the corresponding normalized eigenvectors, i.e.

$$A\psi_n = \mu_n \psi_n, \quad \|\psi_n\| = 1, \quad n \in \mathbb{N}.$$

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Assume that

1. $\exists \kappa > 0, \quad \mu_{n+1} - \mu_n \gtrsim n^{\kappa - 1}, \quad n \to \infty,$ [size of EV gaps]
2. $\exists \alpha \in \mathbb{R}$ with $2\alpha + \kappa > 1 : \quad |\langle B\psi_m, \psi_n \rangle| \lesssim \dfrac{1}{m^\alpha n^\alpha}, \quad m, n \in \mathbb{N}.$ ["local form-subordination"]

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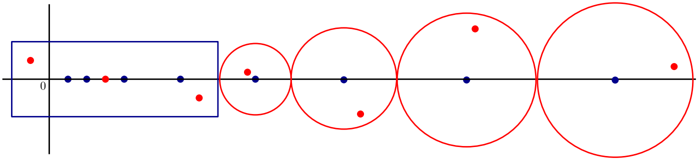
Then for $T = A + B$ [form-sum]

- the eigenvalues $\{\lambda_n\}$ of T are eventually simple,
- $\lambda_n = \mu_n + \langle B\psi_n, \psi_n\rangle + r_n(\alpha, \kappa)$ as $n \to \infty$,
- the eigensystem of T contains a Riesz basis.



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Remarks

- optimal in the sense that it cannot be weakened to $2\alpha + \kappa = 1$
- simplicity of EV's of A, $A \geq 0$ can be relaxed

  [Shkalikov, 2016; Adduci and Mityagin, 2012b]
- operator version of the local subordination earlier

  [Adduci and Mityagin, 2012a; Shkalikov, 2010]
- survey and further generalizations

  [Shkalikov, 2016; Motovilov and Shkalikov, 2017; Motovilov and Shkalikov, 2019]
- essential needed ingredient in applications: asymptotics of eigenfunctions of A

Assumptions (the case $\kappa = 1$)

1. $\mu_{n+1} - \mu_n \gtrsim 1, \quad n \to \infty,$ [e.g. $\mu_n = n$]

2. $|\langle B\psi_m, \psi_n\rangle| \leq \omega_m \omega_n, \quad m, n \in \mathbb{N}$

where

$$\omega_n \geq 0, \quad \frac{\omega_n^2}{\mu_n} \in \ell^1(\mathbb{N}) \quad \text{and} \quad \sum_{j \neq n} \frac{\omega_j^2}{|\mu_n - \mu_j|} = o(1), \quad n \to \infty.$$

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- the previous assumption $\omega_n \lesssim \frac{1}{n^\alpha}$ with $\alpha > 0$ guarantees

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- newly possible:
  - log-decay $\frac{\omega_n^2}{\mu_n} \in \ell^1(\mathbb{N})$ and $\omega_n^2 = o\left(\frac{1}{\log n}\right)$, $n \to \infty$
  - arbitrarily slow decay and gaps, e.g.

$$\omega_n = o(1) \quad \text{and} \quad \omega_n = 0, \quad n \neq m^2, \; m \in \mathbb{N}$$