X Partial differential equations, optimal design and numerics

Energy decay for strongly damped wave equations

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23 Aug 2024

Joint work with A. Arnal, J. Royer and P. Siegl

This project has received funding from the European Union's Horizon 2020
research and innovation programme under the
Marie Skłodowska-Curie Grant Agreement No. 101034413.

Strong damping

Damped wave equation

$$\textbf{(DWE)} \quad : \quad \begin{cases} u_{tt}(t,x) + 2a(x)u_t(t,x) = (\Delta_x - q(x))u(t,x), & x \in \Omega \subseteq \mathbb{R}^d, \quad t \geq 0 \\ u(0,\cdot) = v_1 \\ u_t(0,\cdot) = v_2 \end{cases}$$

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- write as **first order** IVP in **time**

$$\textbf{(ACP)} \quad : \quad \partial_t \mathbf{u} = \overbrace{\begin{pmatrix} 0 & I \\ \Delta - q & -2a \end{pmatrix}}^{= G} \mathbf{u}, \qquad \mathbf{u}(0,\cdot) = \mathbf{v}$$

- implement G as **generator** of C_0-**semigroup** in

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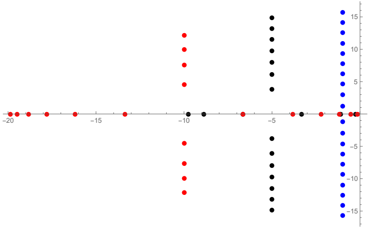


$\Omega = (-1,1), \quad q = 0, \quad a = a_0 \geq 0$

Strong damping

$a, q \geq 0$ **locally integrable**

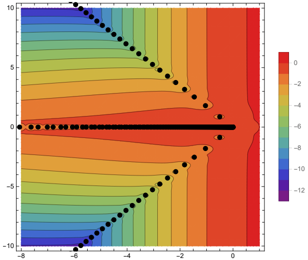
- **unbounded** damping at **infinity** [Freitas-Siegl-Tretter 2018]

$$\sigma_{\mathrm{ess}}(G) = (-\infty, 0], \qquad \sigma(G) \setminus (-\infty, 0] \subseteq \sigma_{\mathrm{disc}}(G)$$

- **NO** (exponential) **decay** of semigroup **norm**

- uniform **decay** on **subspace** $E \subseteq \mathcal{H}$

$$\forall\, \mathbf{v} \in E \ : \ \|e^{tG}\mathbf{v}\|_{\mathcal{H}} \leq \rho(t)\, \|\mathbf{v}\|_E, \qquad \rho(t) \xrightarrow{t\to\infty} 0$$



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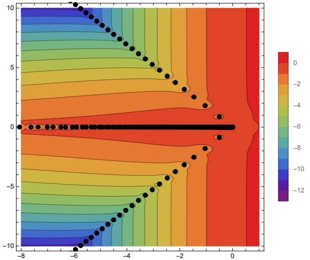

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Theorem $\Omega = \mathbb{R}^d, \quad d \geq 3, \quad q = 0, \quad a \in C(\mathbb{R}^d), \quad a \geq a_0 > 0$ **[Ikehata-Takeda 2020]**

If $v_1 \in H^1(\mathbb{R}^d)$ and $av_1, v_2 \in L^2(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ then unique **weak solution** of **(DWE)** satisfies

$$\|\nabla u(t,\cdot)\| + \|u_t(t,\cdot)\| \lesssim \langle t\rangle^{-1} C^{\mathrm{IT}}(\mathbf{v}), \qquad \|u(t,\cdot)\| \lesssim \langle t\rangle^{-\frac{1}{2}} C^{\mathrm{IT}}(\mathbf{v})$$

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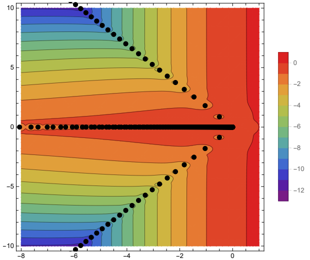

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[Freitas-Hefti-Siegl 2020, Arnal 2022, Sobajima-Wakasugi 2018, Kleinhenz et al.]

Non-uniform energy decay

Main result

$a \geq a_0 > 0$

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- improvement for **unbounded damping**

$$a \approx |x|^{\beta}, \quad |x| \to \infty \qquad \Longrightarrow \qquad \|e^{tG}\mathbf{v}\|_{\mathcal{H}_1} \lesssim \langle t \rangle^{-\frac{1}{2} - \frac{\beta}{2(2+\beta)}} \|\mathbf{v}\|_E$$

Strategy

$a \geq a_0 > 0$ on $\Omega \setminus B_R(0)$ + resolvent bdd at $\pm i\infty$

- **resolvent** behaviour **around zero** determines **semigroup decay** [Batty-Chill-Tomilov 2016]

$\lambda \to 0$ in $\overline{\mathbb{D}_+}$

$$\|(G-\lambda)^{-1}\|_{\mathcal{H}\to\mathcal{H}} \longrightarrow \|e^{tG}G(G-1)^{-1}\|$$

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Schur complement

$$\mathbf{t}_\lambda[u] = \|\nabla u\|^2 + \|q^{\frac{1}{2}}u\|^2 + 2\lambda\|a^{\frac{1}{2}}u\|^2 + \lambda^2\|u\|^2, \qquad u \in \mathcal{D}_{\mathbf{t}} = H_0^1(\Omega) \cap \operatorname{dom} q^{\frac{1}{2}} \cap \operatorname{dom} a^{\frac{1}{2}}$$

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- lower bound for **self-adjoint operator**

$$H_\lambda = -\Delta + |\lambda| a \gtrsim |\lambda|, \qquad \operatorname{dom} \mathbf{h}_\lambda = H_0^1(\Omega) \cap \operatorname{dom} a^{\frac{1}{2}}$$

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Neumann bracketing

$$H_\lambda \geq H_\lambda^+ \oplus H_\lambda^-, \qquad \operatorname{dom} \mathbf{h}_\lambda^\pm = H_1(\Omega^\pm) \cap \operatorname{dom} a^{\frac{1}{2}}, \qquad \inf \sigma(H_\lambda^\pm) \gtrsim |\lambda|$$

- follows from **uniform positivity** on Ω^-
- on Ω^+ by **asymptotic perturbation** theory

Dropping uniform positivity

Resolvent at $\pm i\infty$

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Unbounded with (GCC) :

$$\|(G-\lambda)^{-1}\| \lesssim 1, \qquad \lambda \to \pm i\infty$$

- $\Omega = \mathbb{R}$: damping **unbounded** [Arnal 2022]

  (**smooth** data + **control** on **derivatives**)

- $\Omega = \mathbb{R}^d$: **power-like** damping $a = |x|^\beta$

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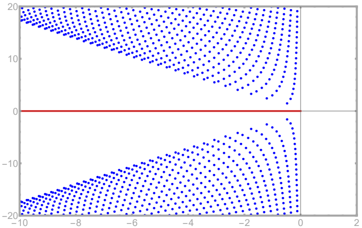

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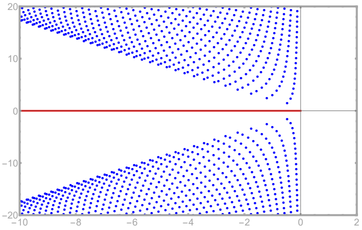

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- for **bounded** semigroup [Batty-Chill-Tomilov 2016]

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Thank you for your attention!

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