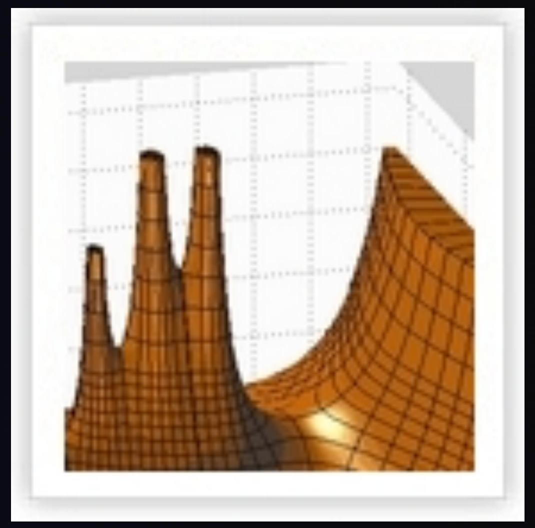

X Partial differential equations, optimal design and numerics

On Boundary Controllability and Synchronization for Coupled Wave Equations

Yue Wang, joint work with Tatsien Li and Günter Leugering

August 22, 2024

@ Thematic Session: Analysis and Control for Wave Phenomena

FAU Friedrich-Alexander-Universität Research Center for Mathematics of Data | MoD

Friedrich-Alexander-Universität DYNAMICS, CONTROL, MACHINE LEARNING AND NUMERICS

Motivation

- **Role of Hyperbolic Systems:** Characterize and predict waves with finite propagation speeds (e.g., vibrations in rigid bodies or fluid).

- **Applications:** Used in fluid and solid mechanics, electromagnetism, seismic waves, and optical transmission.

- Developing **control theories** (like controllability, stabilizability, synchronization) and implementing high-performance **numerical simulations and computational methods** is of profound significance for understanding natural phenomena and optimizing the performance of the system.

- To do so, you needs:
  nonlinear functional analysis, PDEs, networks and graph theory,
  control theory, optimal design, spectral analysis, numerical analysis, …

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In this talk, I aim to present some interesting models, controllability properties, numerical realization for coupled wave equations, with some key results and research perspectives, and to leave space for future discussion at Benasque!

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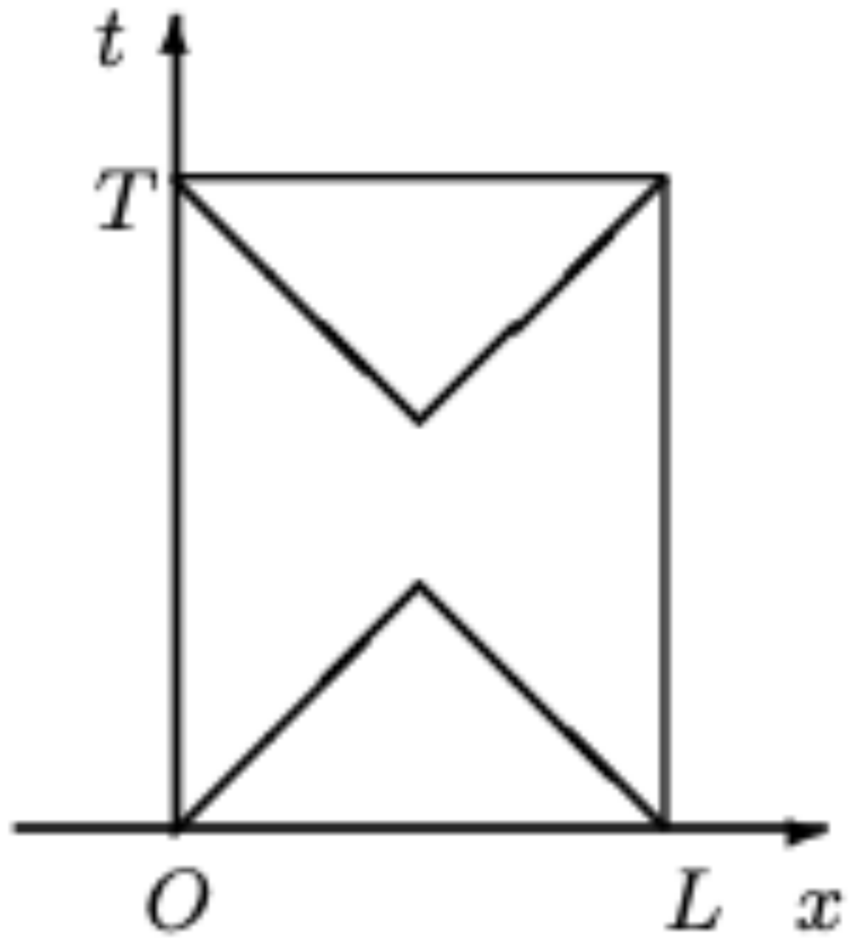
- Motivation
- **Boundary controllability for coupled wave equations (1D, quasilinear case)**
- **Synchronization for coupled wave equations (high dimensional, linear case)**

Boundary Control for hyperbolic wave

Some Key Properties.

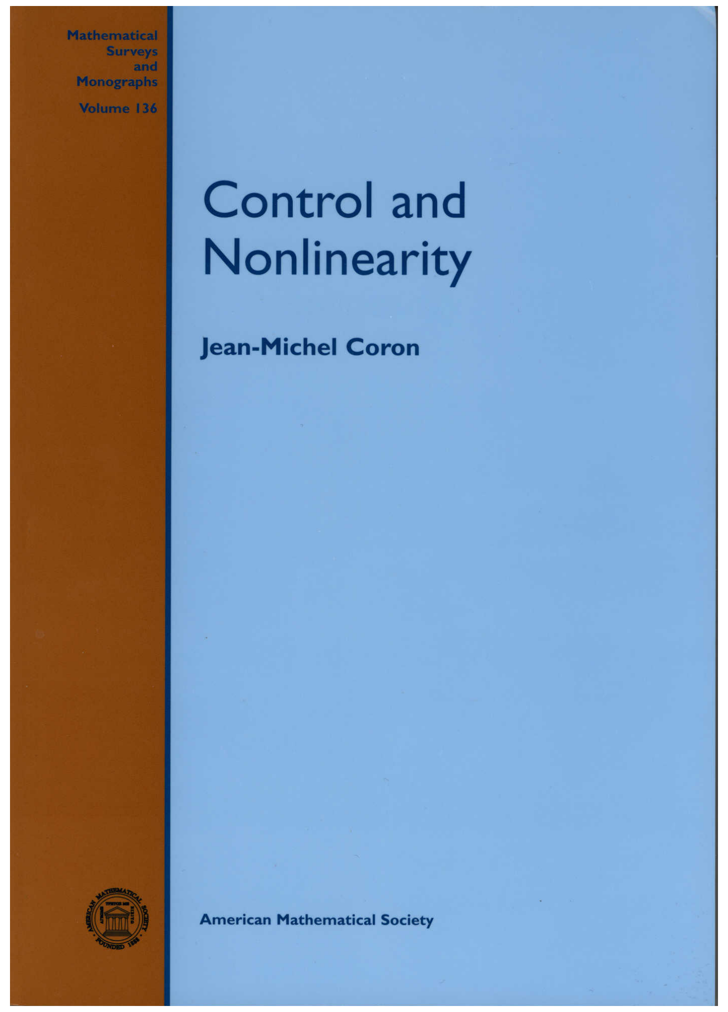
- Prescribe **suitable boundary conditions** for IBVP on a bounded domain
  - > local & global
  - > internal control & boundary control
- Controllability time $(T > 0)$
  - > a finite speed of propagation of the hyperbolic wave

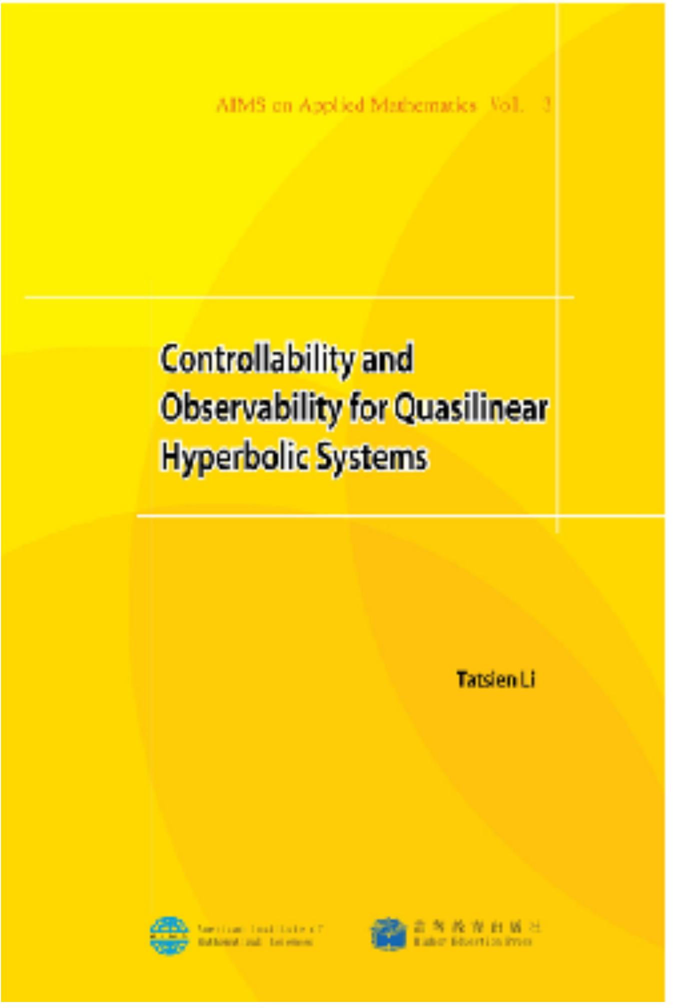


  - > maximum determinate domains
  - > $T(> 0)$ should be chosen as small as possible (optimal controllability time).

Difficulties (interests) may arise in ...

- **Nonlinearity.**
  - **Weak solutions**. [of quasilinear hyperbolic systems $\rightarrow$ shock waves $\rightarrow$ an irreversible process $\rightarrow$ Impossible to get exact boundary controllability for any arbitrarily given initial and final states [A. Bressan, G. M. Coclite, '02] $\rightarrow$ weaken the definition $\rightarrow$ case by case (the scalar convex conservation law [F. Ancona, A. Marson '98,'99, T. Horsin, '98], the p-system in isentropic gas dynamics [O. Glass, '07]].
  - **Classical solution** exists only locally in time (P. D. Lax, '64; F. John, '90; T. Li, '94) $\rightarrow$ semi-global classical solution ($T > 0$ might be suitably large) [M. Cirinà, '70, T.Li, Y.Jin, B.Rao, '00, '01] $\rightarrow$ Local exact controllability in the quasilinear case.





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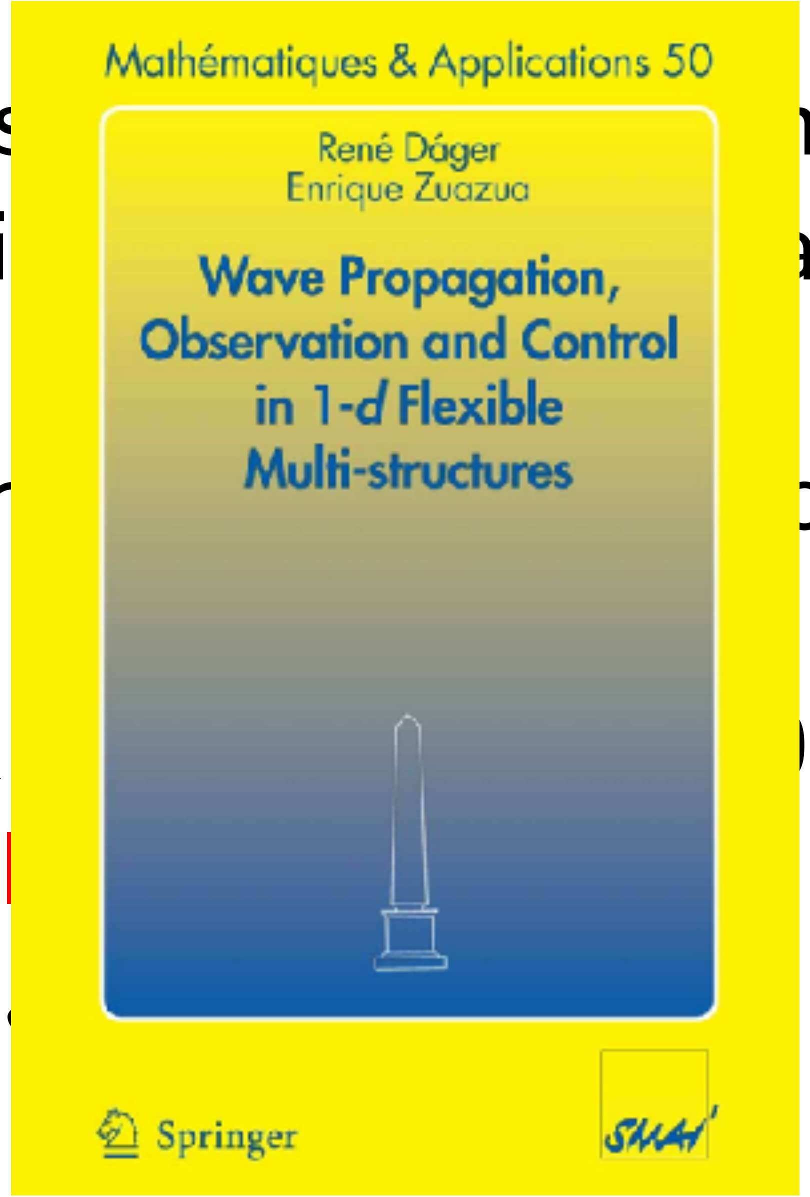
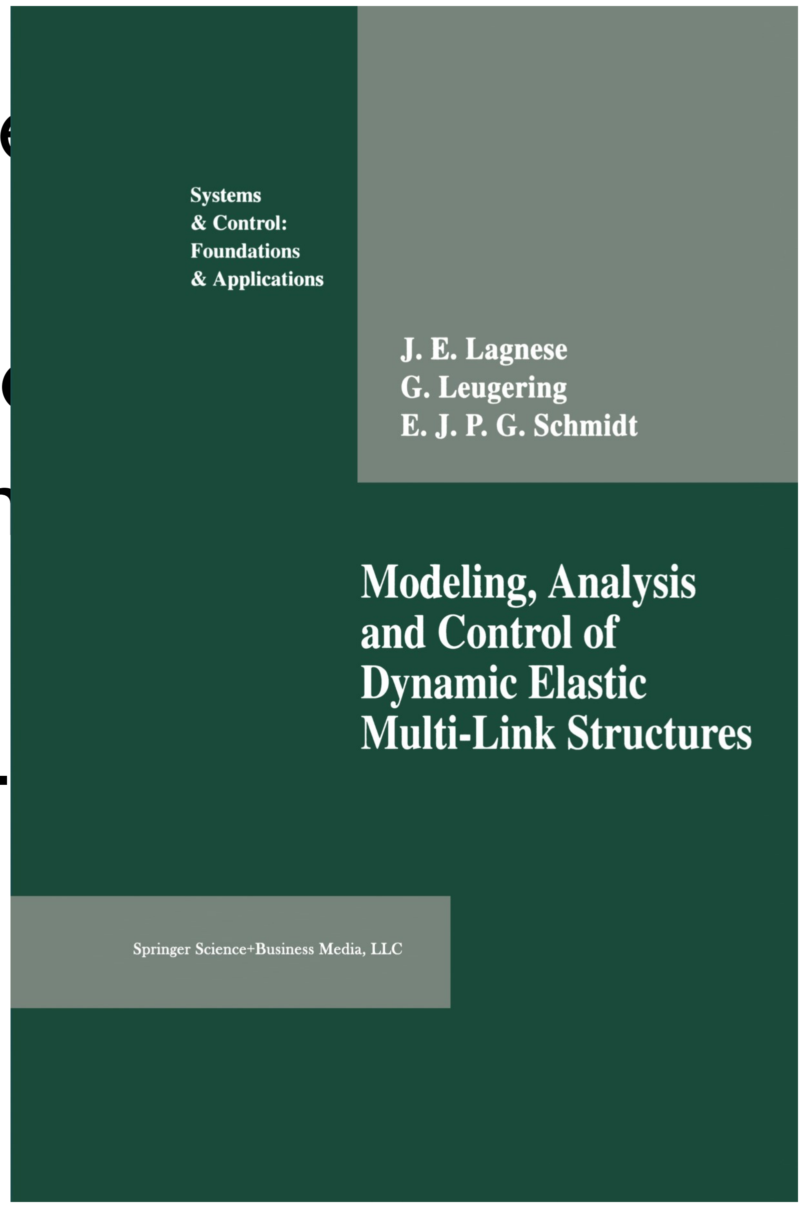
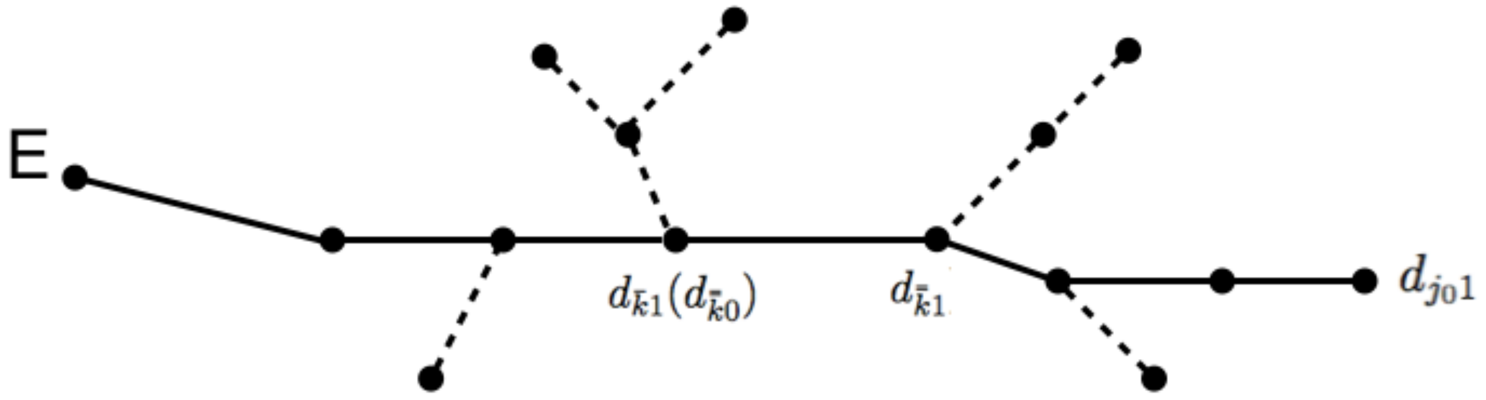
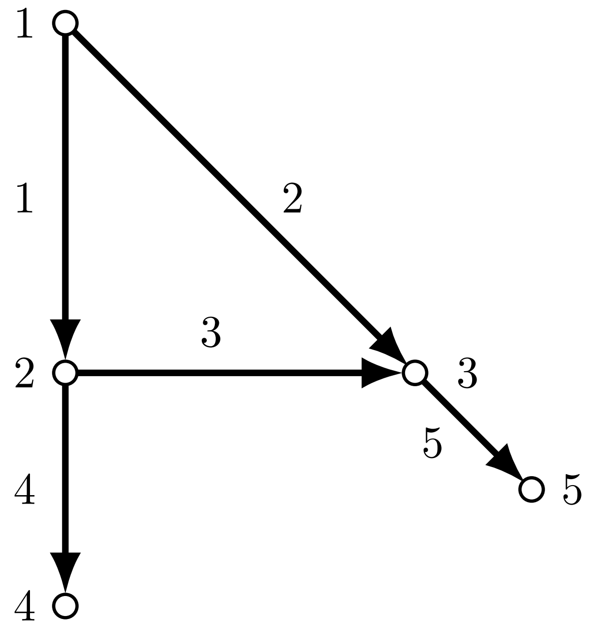




- **Networked Structure.**
  - > **Coupling at the junction**. Complexity and Nonlinearity in **interface conditions**.
  - > Complex topological structure of networks $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ may change the controllability results [Lagnese-Leugeing-Schmidt, '94]





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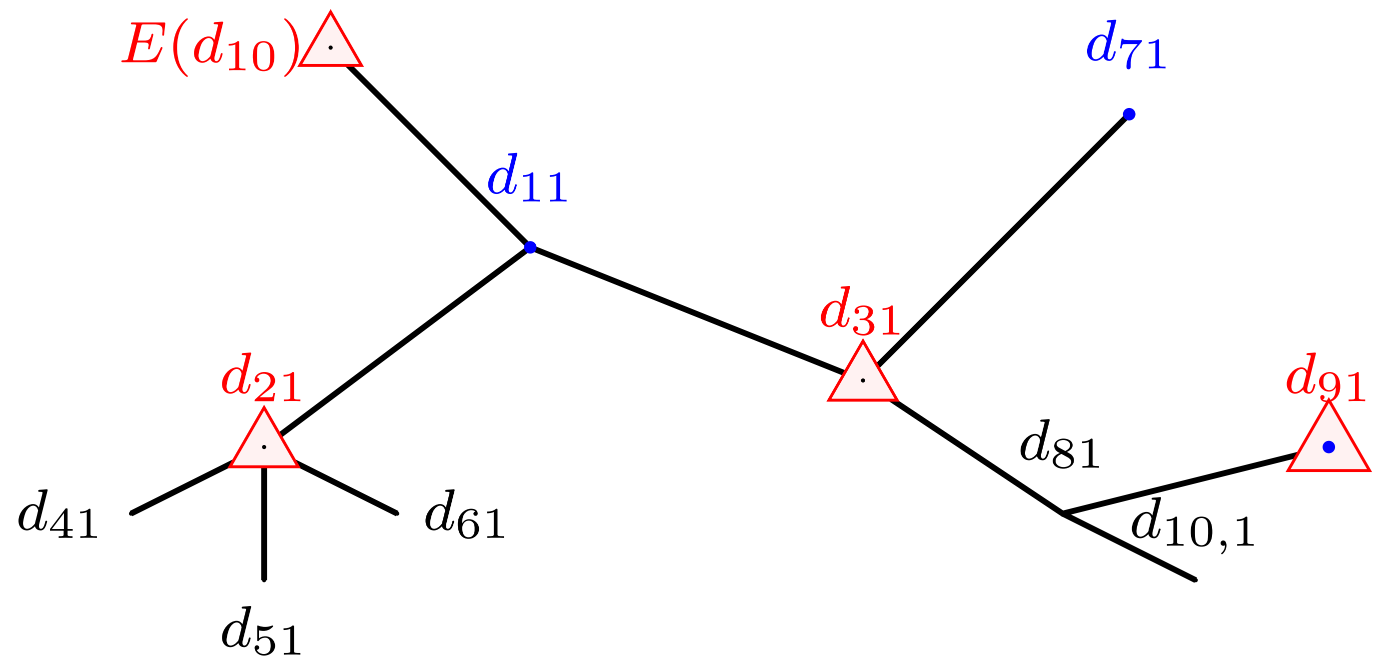


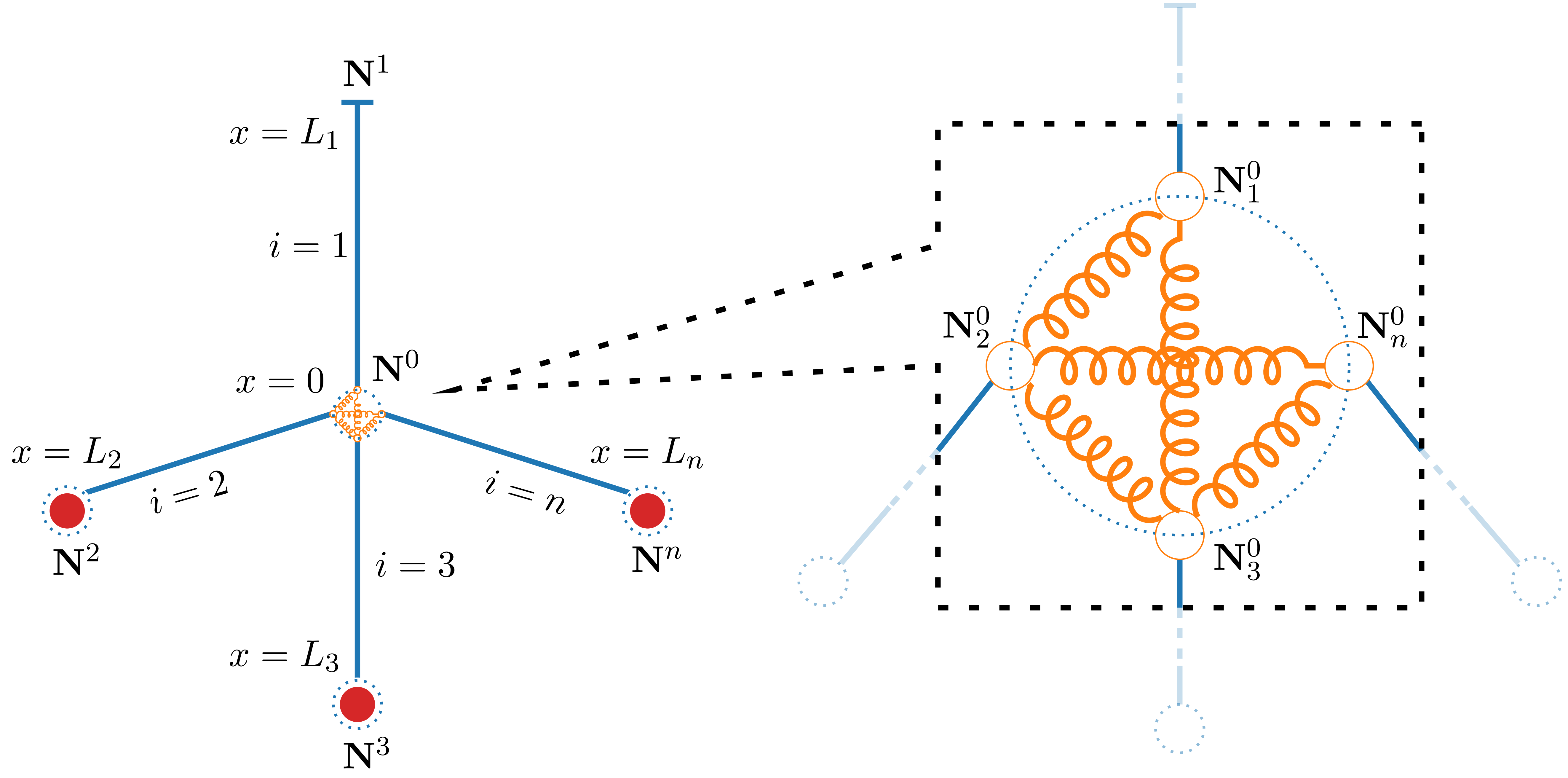
Controllability and
Control Design Problem

* Optimal controllability time T^*.
* Minimum number of controls.
* Placement of controls.
* Calculation of controls.

- Nodal Profile Control: Our aim is to fit (a part of) the boundary traces to a given profile after a suitably long time $t = T$ by means of boundary controls. **[Project: Control theory on planar or spacial string networks: controllability and partial nodal control for quasilinear hyperbolic systems. (Individual funding & NSFC-1121101.Joint work with T.Li.]**

Difficulties (interests) may arise in …

- Nonlinearity
- Networked Structure
- New boundary/interface conditions

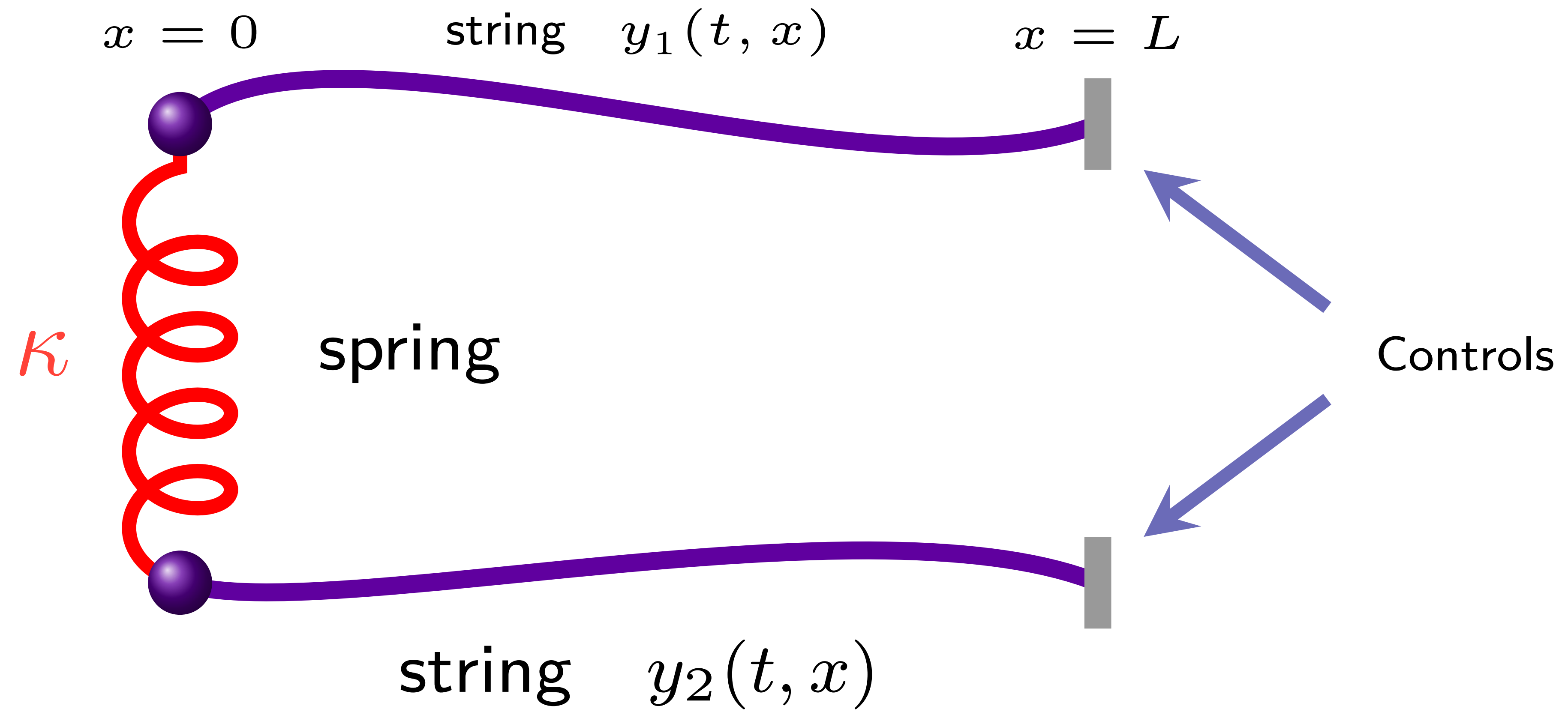


Difficulties (interests) may arise in ...

- **New boundary/interface conditions**

$$
\begin{cases}
y^i_{tt} - K_i(y^i_x)_x = 0, \quad 0 \le x \le L, t > 0, \qquad i = 1, 2, \\
x = 0 : y^1_{tt}(0,t) = K_1(y^1_x(0,t)) - \kappa(y^1(0,t) - y^2(0,t)), \\
\qquad\quad y^2_{tt}(0,t) = K_2(y^2_x(0,t)) + \kappa(y^1(0,t) - y^2(0,t)), \\
x = L : y^i = u^i(t), \qquad i = 1, 2.
\end{cases}
$$

Dynamical transmission conditions



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$$\begin{cases} y^i_{tt} - K_i(y^i_x)_x = 0, \quad 0 \le x \le L, t > 0, \qquad i = 1,2, \\ x = 0 : y^1_{tt}(0,t) = K_1(y^1_x(0,t)) - \kappa(y^1(0,t) - y^2(0,t)), \\ \qquad y^2_{tt}(0,t) = K_2(y^2_x(0,t)) + \kappa(y^1(0,t) - y^2(0,t)), \\ x = L : y^i = u^i(t), \qquad i = 1,2. \end{cases}$$

- If the spring stiffness tends to infinity, formally the system tends to the classical string-mass problem. [1]
- For spring-mass system it is known that the mass smoothens the waves while crossing the mass-point.[2]
- If the spring stiffness tends to zero, the strings become uncoupled.
- The spring coupling can be seen as a weakening of the classical transmission conditions at a multiple joint.[3]

---

[1] G. Leugering, 1998; F. Almusallams, 2015; Y.Wang, T.Li, 2018

[2] S. Hansen, E.Zuazua 1995

[3] G.Leugering,S.Micu, I.Roventa, Y.Wang, 2022

Example: Networks of vibrating strings

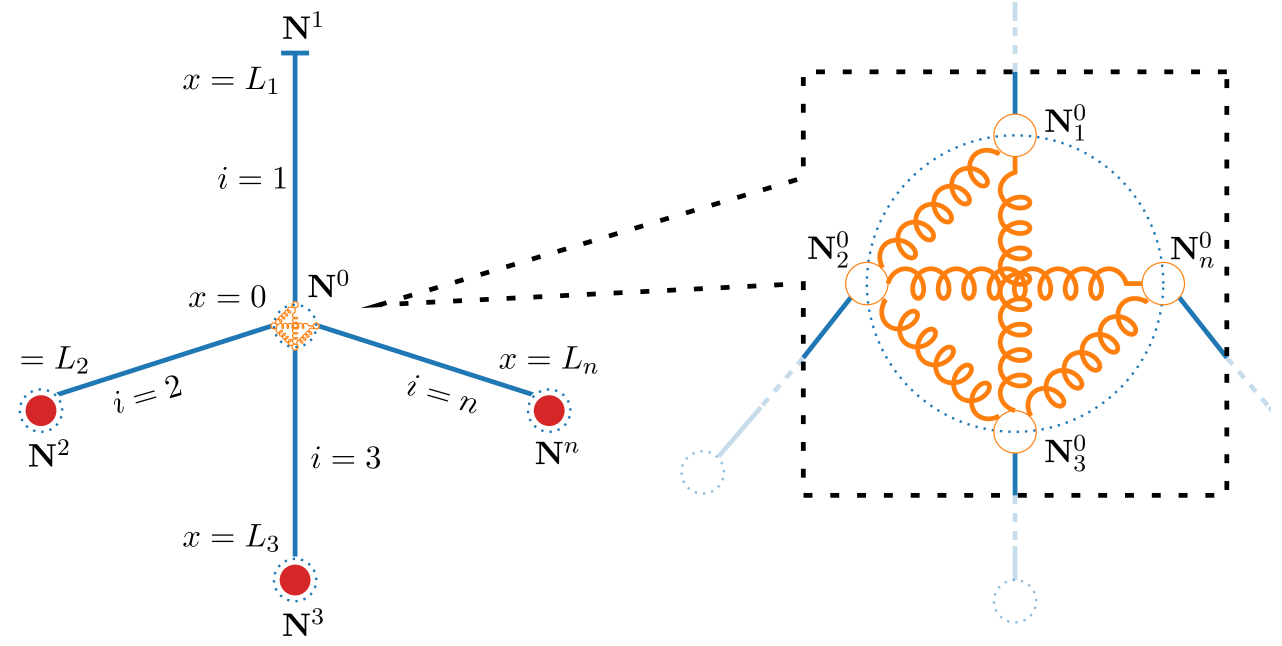
New boundary conditions + coupling

Consider the following **coupled** system of 1-D **quasilinear** wave equations $(i = 1, ..., n)$:

$$
(\mathbf{E}) \begin{cases} y^i_{tt} - (K^i(y^i, y^i_x))_x = F(\mathbf{y}, \mathbf{y}_x, \mathbf{y}_t), \quad x \in [0, L_i], t \in [0, T] \\ y^i_{tt}(t, 0) = G^i(t, \mathbf{y}(t,0), \mathbf{y}_x(t,0), \mathbf{y}_t(t,0)) \\ \qquad\qquad + \int_0^t H^i(t, s, \mathbf{y}(s, 0)) \mathrm{d}s, \ t \in [0, T] \\ y^i(t, L_i) = u^i(t), \ t \in [0, T] \\ (y^i, y^i_t)(0, x) = (\phi^i(x), \psi^i(x)), \quad x \in [0, L_i]. \end{cases}
$$

Second-order differential operators

(temporal) non-locality



where

- $\mathbf{y} = (y^1, ..., y^n)^{\mathrm{T}}$ is an unknown vector function of (t, x),
- $K^i = K^i(y^i, y^i_x)$ are given C^2 functions of y^i and y^i_x,
- $\frac{\partial}{\partial y^i_x} K^i(y^i, y^i_x) > 0$,
- F^i, G^i, H^i are given C^1 functions of their arguments and 0 value at null state (i.e. 0 is an equiblium).

Exact boundary controllability

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$$
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$$

The system (E) is locally exact controllable

- with n controls [G.Leugering, T.Li, Y.Wang, '18,'19].

Controllability Time (sharp): $T^* = \max_{i=1,\dots n} \dfrac{2L_i}{\sqrt{K^i_{y_x}(0,0)}}$

Exact boundary controllability

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$$

The system (E) is locally exact controllable

- with n controls [G.Leugering, T.Li, Y.Wang, '18,'19].
- This result can be improved by **reducing the number of controls to** $n-1$, but the space of controlled initial data is **asymmetric** [G.Leugering, S.Micu, I.Robenta, Y.Wang, '22] [G.Leugering, C.Rodriguez, Y.Wang, '24, submitted]

Key techniques for wellposedness and boundary controllability:
1. Characteristics
2. Explicit constructive method with modular structure

We introduce $\mathbf{w}^i = (w_1^i, w_2^i, w_3^i)^T := (y^i, y_x^i, y_t^i)^T$. Then we get

$$\frac{\partial}{\partial t}\begin{pmatrix} w_1^i \\ w_2^i \\ w_3^i \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -K_{w_2^i}^i & 0 \end{pmatrix} \frac{\partial}{\partial x}\begin{pmatrix} w_1^i \\ w_2^i \\ w_3^i \end{pmatrix} = \begin{pmatrix} w_2^i \\ 0 \\ F^i(\mathbf{w}^i) + K_{w_1^i}^i w_2^i \end{pmatrix}$$

with $(t,x) \in [0,T] \times [0,L_i]$. This, in turn, can be rewritten in the form of a quasilinear hyperbolic system

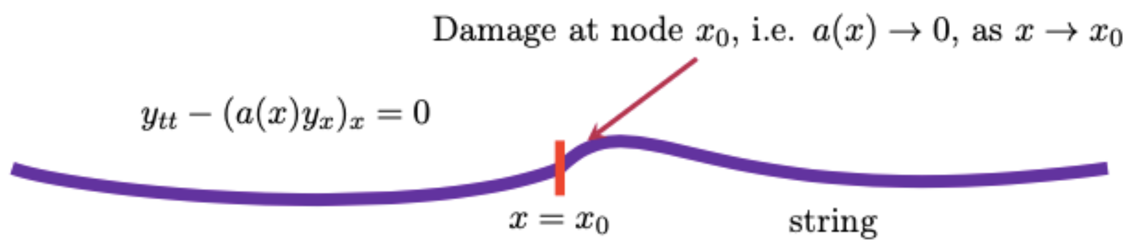
$$\mathbf{w}_t^i + A^i(x, \mathbf{w}^i)\mathbf{w}_x^i = \tilde{F}(\mathbf{w}^i),$$

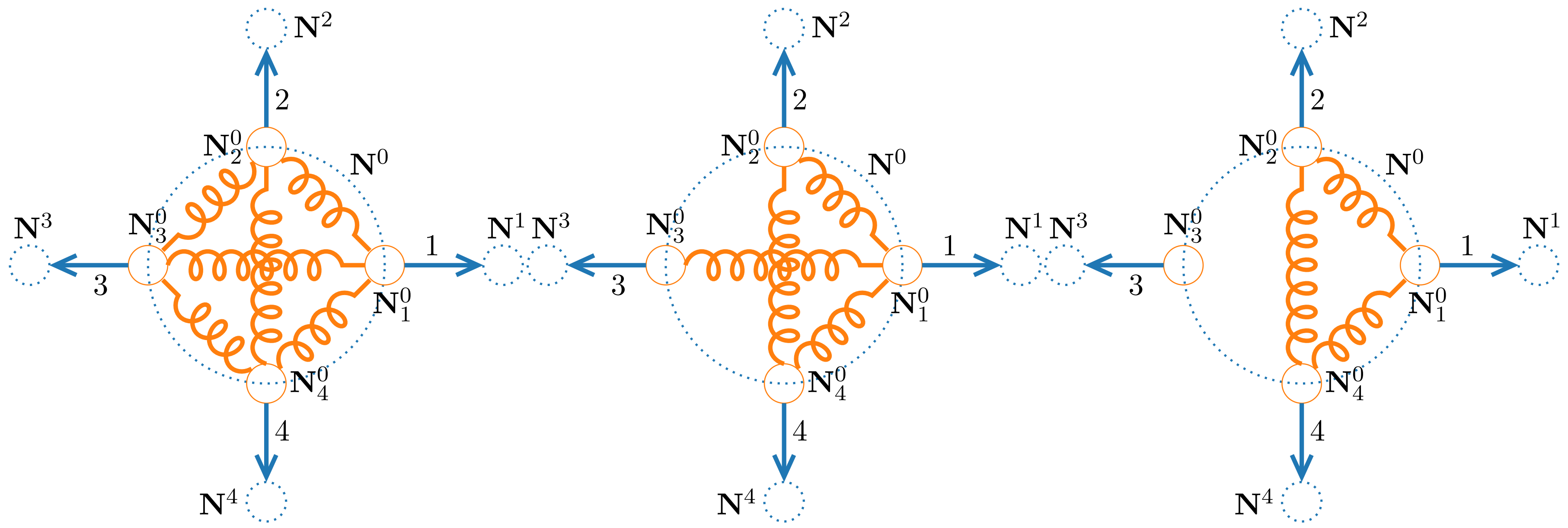
where A^i has 3 distinct real eigenvalues:

$$\lambda_i^- = -\sqrt{K_{w_2^i}^i(w_1^i, w_2^i)}, \quad \lambda_i^0 = 0, \quad \lambda_i^+ = \sqrt{K_{w_2^i}^i(w_1^i, w_2^i)}.$$

Difficulties (interests) may arise in ...

- **Nonlinearity**
- **Network structures**
- **New boundary/interface conditions**
- **Degeneration**
  1. Lack of controllability/observability for wave equations with degeneration.
     ◇ Lack of one-sided exact controllability in $y_{tt} - (a(x)y_x)_x = 0$ when $a(x) = x^\alpha, \alpha \geq 2$.[F. Alabau-Boussouira, P. Cannarsa and G. Leugering, '17] [Macia F, Zuazua E. '02]
  2. Constrained Optimal Control for wave equation with dynamical degeneration (hybrid system)
     ◇ $\min J(y, \alpha, u)$ s.t $0 \leq \alpha \leq 1$ and $y_{tt} - (x^{\alpha(t)}y_x)_x = 0, \dot{\alpha}(t) = \nu\alpha(t) + (y_x^2(t,0) - \gamma)_+$
  3. Some relaxed version of the damage problem?
     ◇ Missing springs in the coupling [joint work with G. Leugering, C. Rodriguez].





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- Nonlinearity
- Network structures
- New boundary/interface conditions
- Degeneration
- Control design and how to compute the control?

Difficulties (interests) may arise in ...

- **Control design and how to compute the control?**

Internship at FAU-MoD (Center for Mathematics of Data)

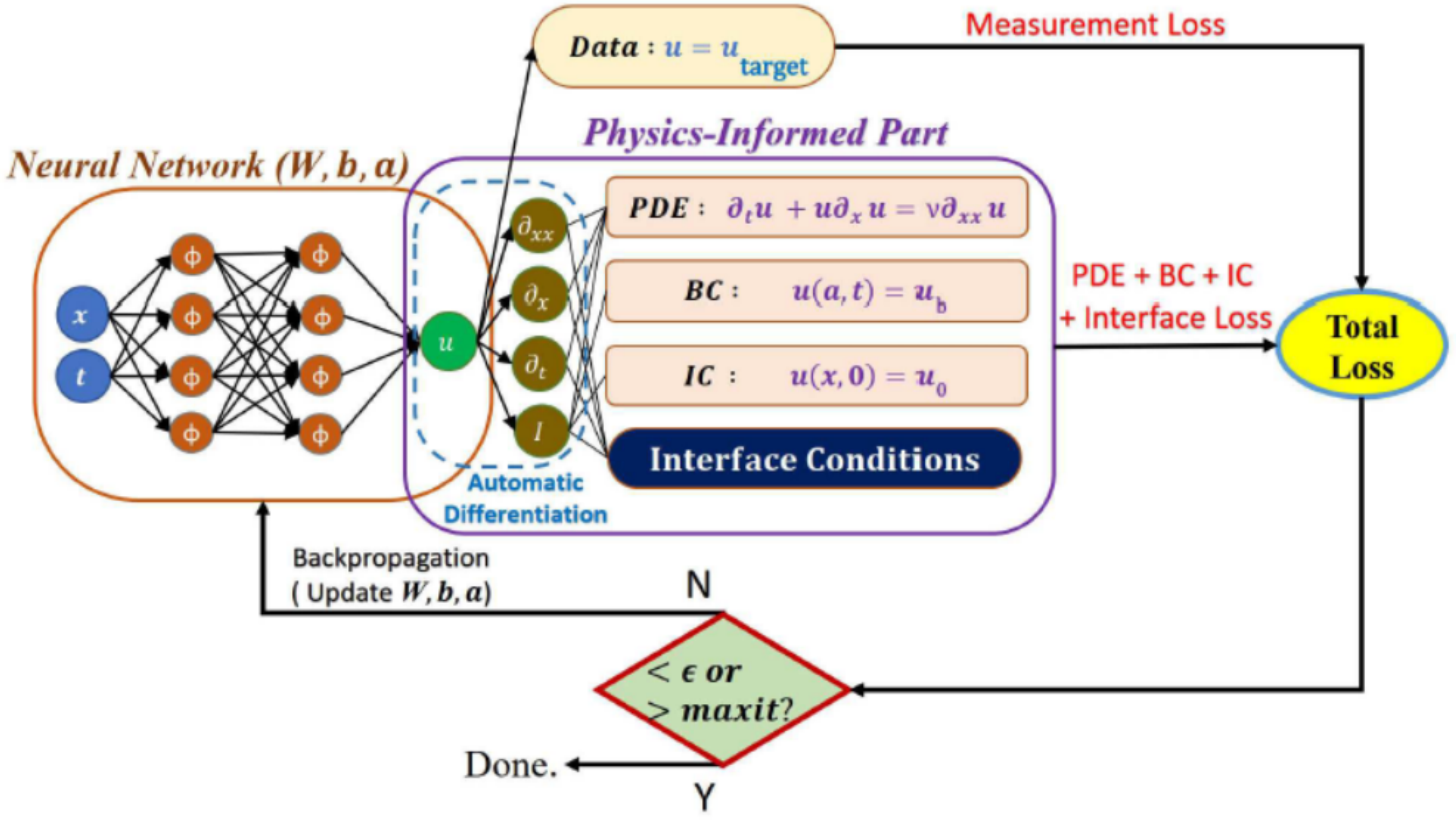
- Dania Sana (June - September 2022)
  Approximating Partial Differential Equations via Physical-Informed Neural Networks
  Supervisors: Yue Wang, Enrique Zuazua
- https://github.com/DCN-FAU-AvH/PINNs_wave_equation

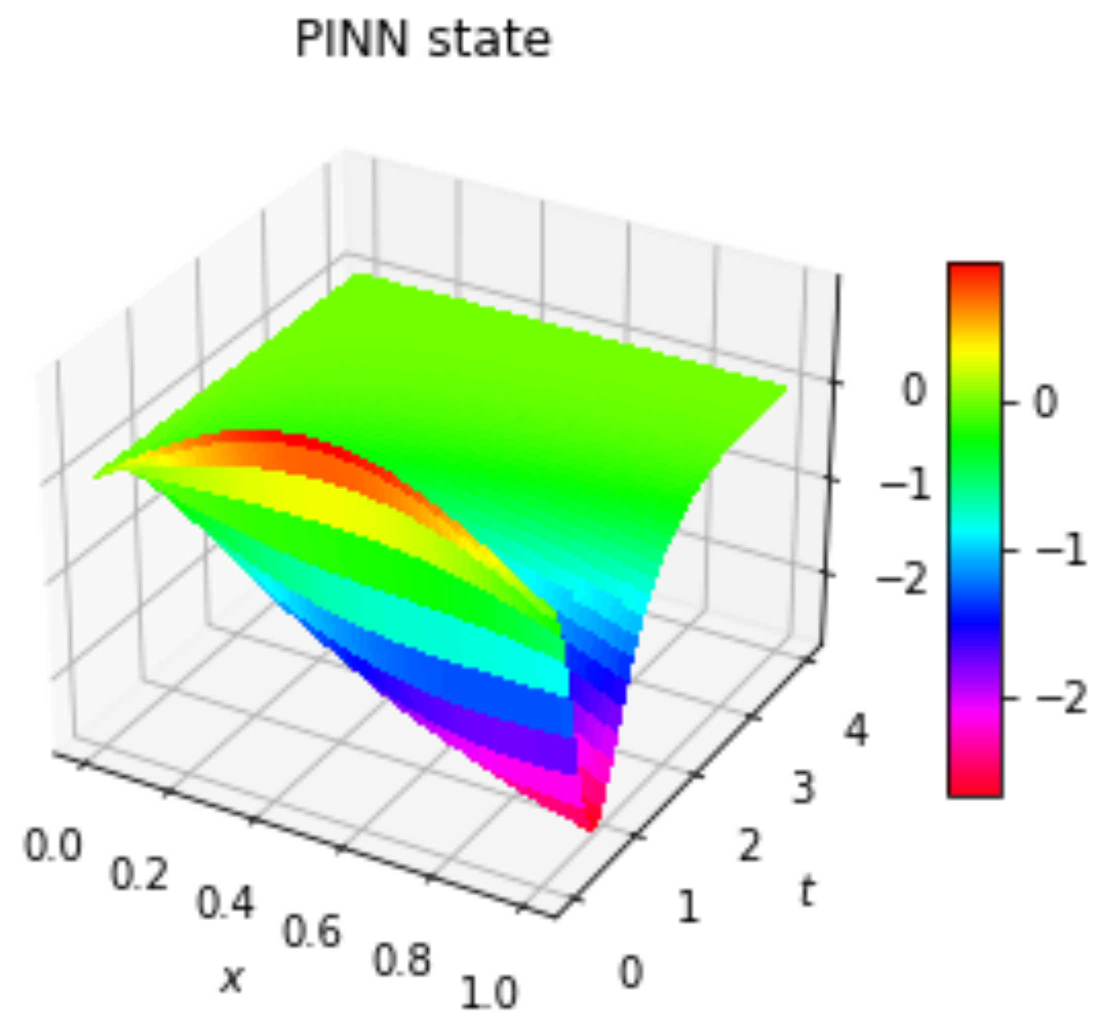
Simulation, inverse problems, and control for (degenerate) 1-D wave equations using PINNs

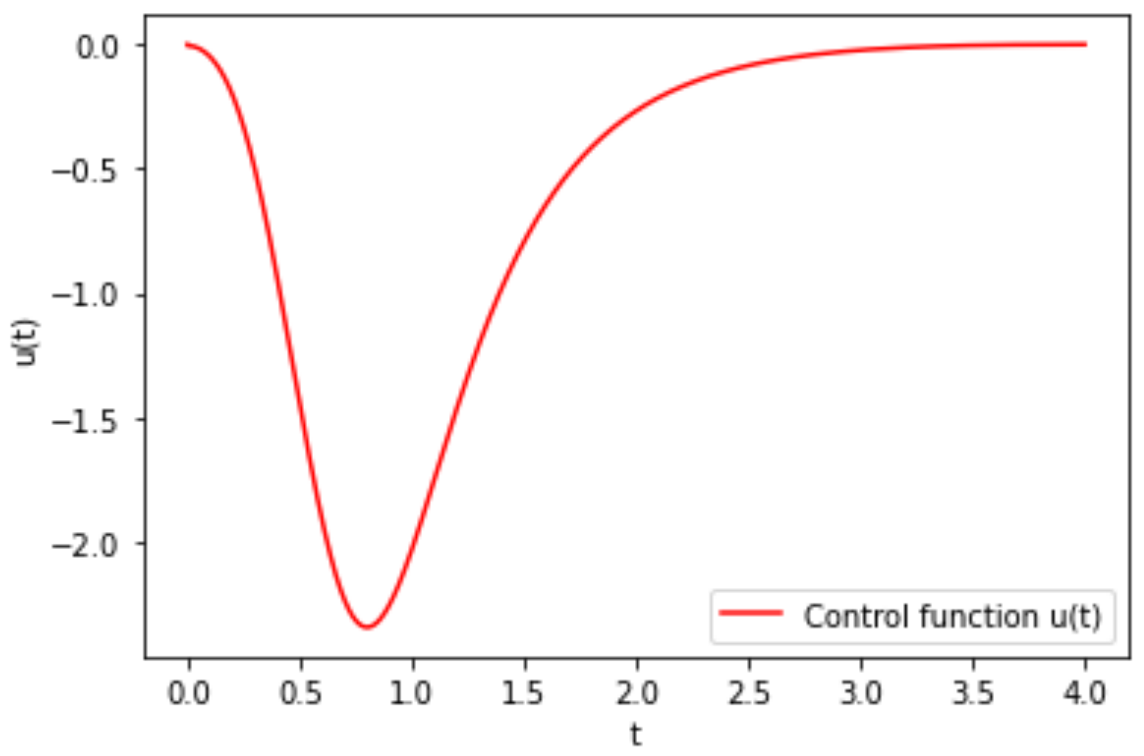
Difficulties (interests) may arise in …

- **Control design and how to compute the control?**

Physics Informed Neural Networks (PINNs)





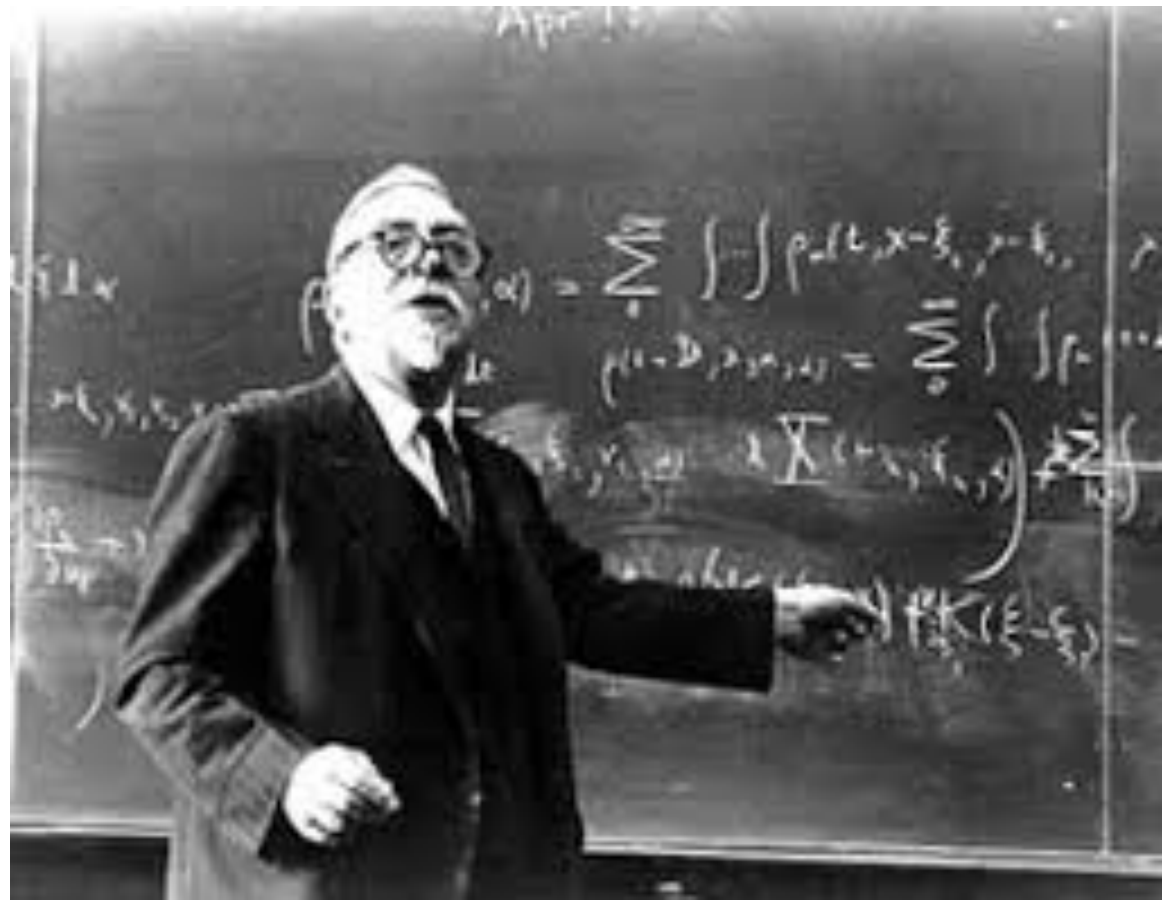


Difficulties (interests) may arise in …

- Nonlinearity
- Network structures
- New boundary/interface conditions
- Degeneration
- Control Design: How to compute the control?
- Lack of exact controllability, what else we could expect?

Back to Origins of Control Theory

To control means to act, to put things in order to guarantee that the system behaves as desired.



In 1948, Norbert Wiener defined Cybernetics (or Control Theory) as **the science of control and communication in animals and machines.**

"...In a desirable future, engines would obey and imitate human beings.."
Cybernetics by N. Wiener (1894-1964)

Back to Origins of Control Theory

NORBERT WIENER

1894–1964

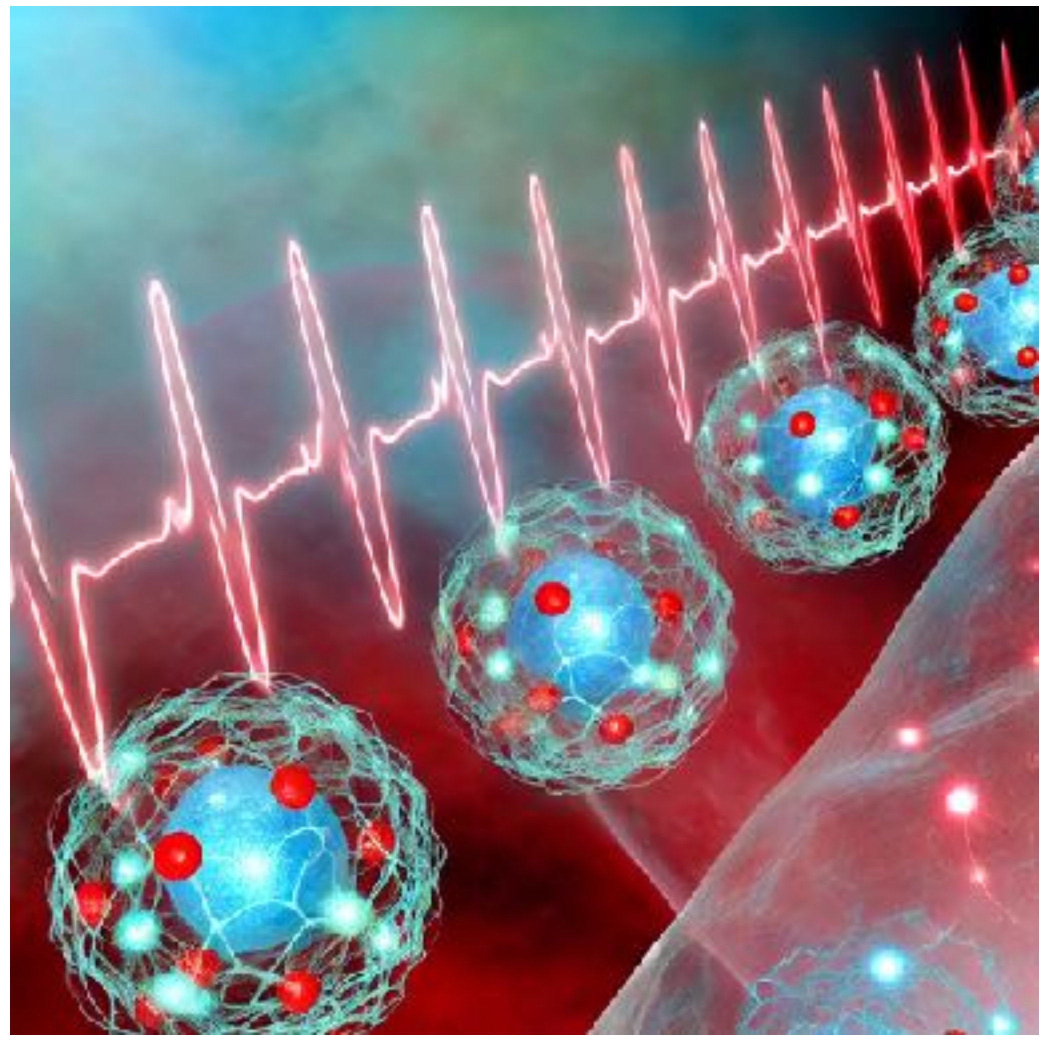
Brain Waves and conditions for synchronization

What is synchronization?

Synchronization is a common phenomenon and has been studied vastly in many subjects, including biology, physics, engineering, and mathematics.

Thousands of fireflies may twinkle at the same time

Audiences in the theater can applaud with a rhythmic beat



Pacemaker cells of the heart function simultaneously

Alpha-waves in Brain

Introduction of Synchronization

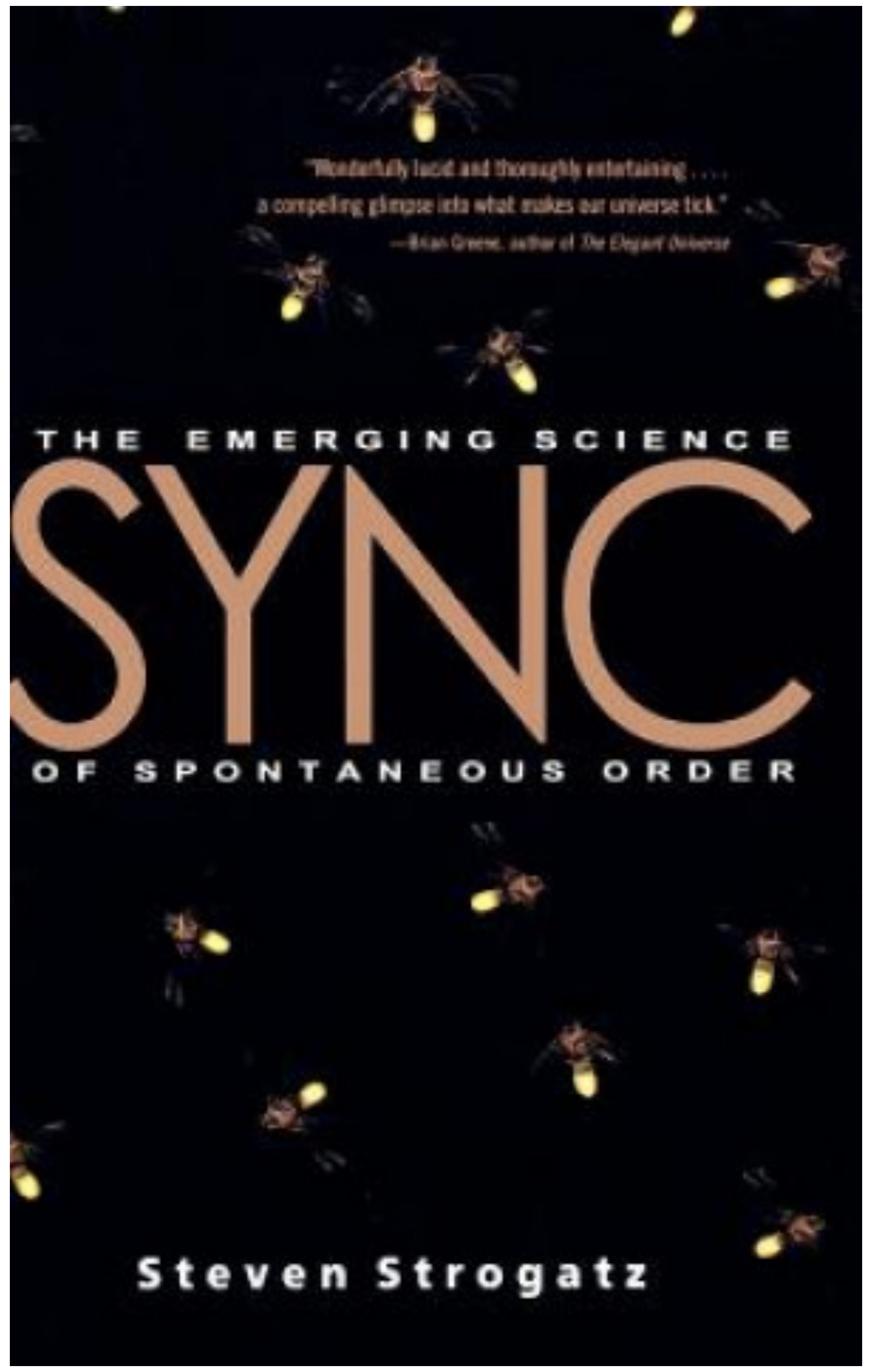
Why? From Randomness to an Order?

Early studies:

- In 1665, Ch. Huygens, two pendulums
- In 1961, N. Wiener, systematically studies

Related books:

- A. Pikovsky, M. Rosenblum, J. Kurths, Synchronization: A Universal Concept in Nonlinear Sciences, 2001
- S. Steven, SYNC—How Order Emerges from Chaos in the Universe, Nature, and Daily Life, 2004



■ Ch. Huygens, Oeuvres Compl`etes, Vol.15, Swets & Zeitlinger B.V., Amsterdam, 1967.

■ N. Wiener, Cybernetics, or Control and Communication in the Animal and the Machine, 2nd ed.. The M.I.T. Press/John Wiley & Sons, Inc., Cambridge, Mass./New York, London,1961.

Synchronization for ODEs

In principle, synchronization happens when different individuals possess likeness in nature, that is, they conform essentially to the same governing equation, and meanwhile, the individuals should bear a certain coupled relation.

- The previous studies focused on systems described by ordinary differential equations (ODEs), such as

$$X_i' = \boxed{f(t, X_i)} + \boxed{\sum_{j=1}^{N} A_{ij} X_j} \quad (i = 1, \cdots, N),$$

- Synchronization in the **consensus** sense

$$X_i(t) - X_j(t) \to 0 \quad (i, j = 1, \cdots, N) \ \text{ as } t \to +\infty,$$

- Synchronization in the **pinning** sense (with a priori unknown state a)

$$X_i(t) \to a \quad (i = 1, \cdots, N) \ \text{ as } t \to +\infty,$$

Synchronization for PDEs

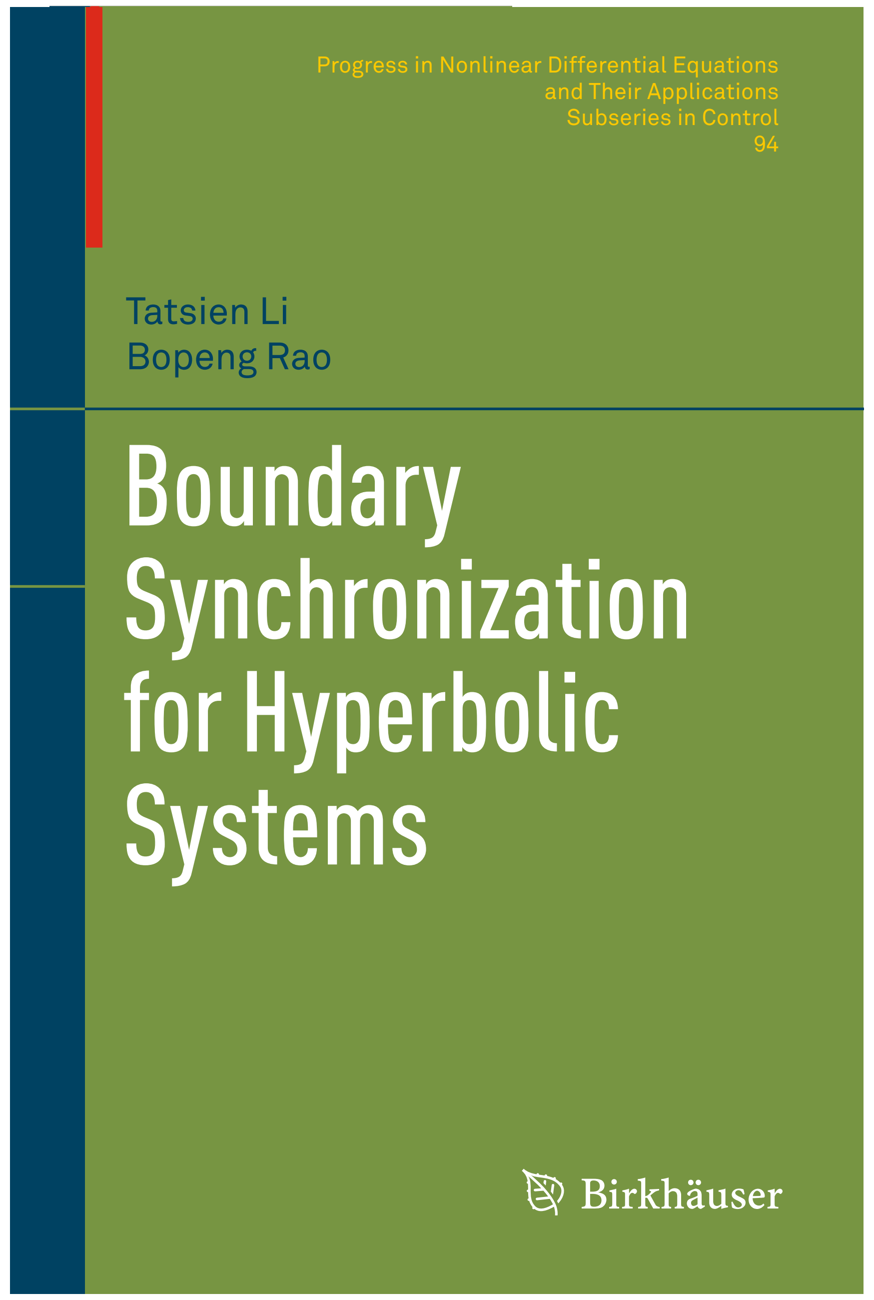
- Since 2012, Li, Rao,... Synchronization for hyperbolic systems

  Finite time:

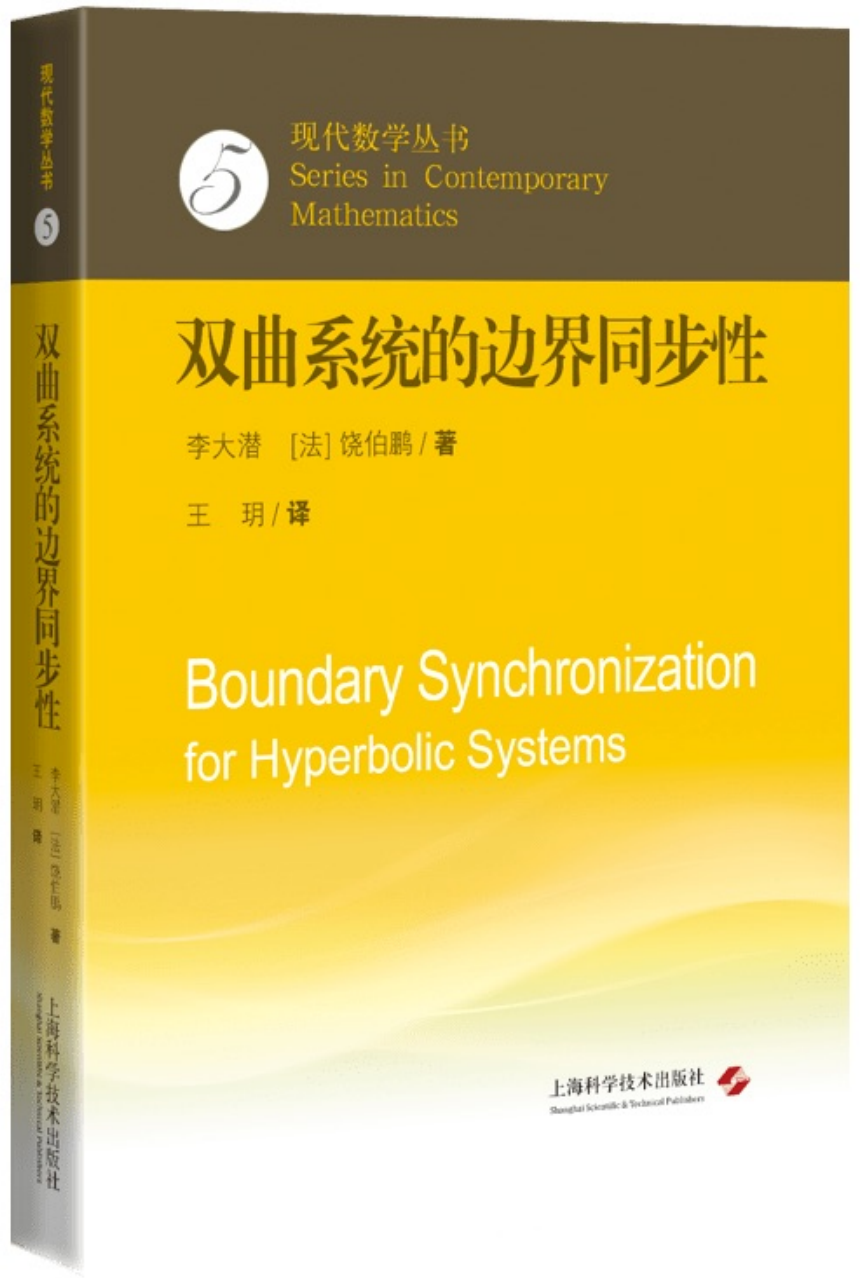
  - Exact boundary synchronization
  - Approximate boundary synchronization

  Infinite time:

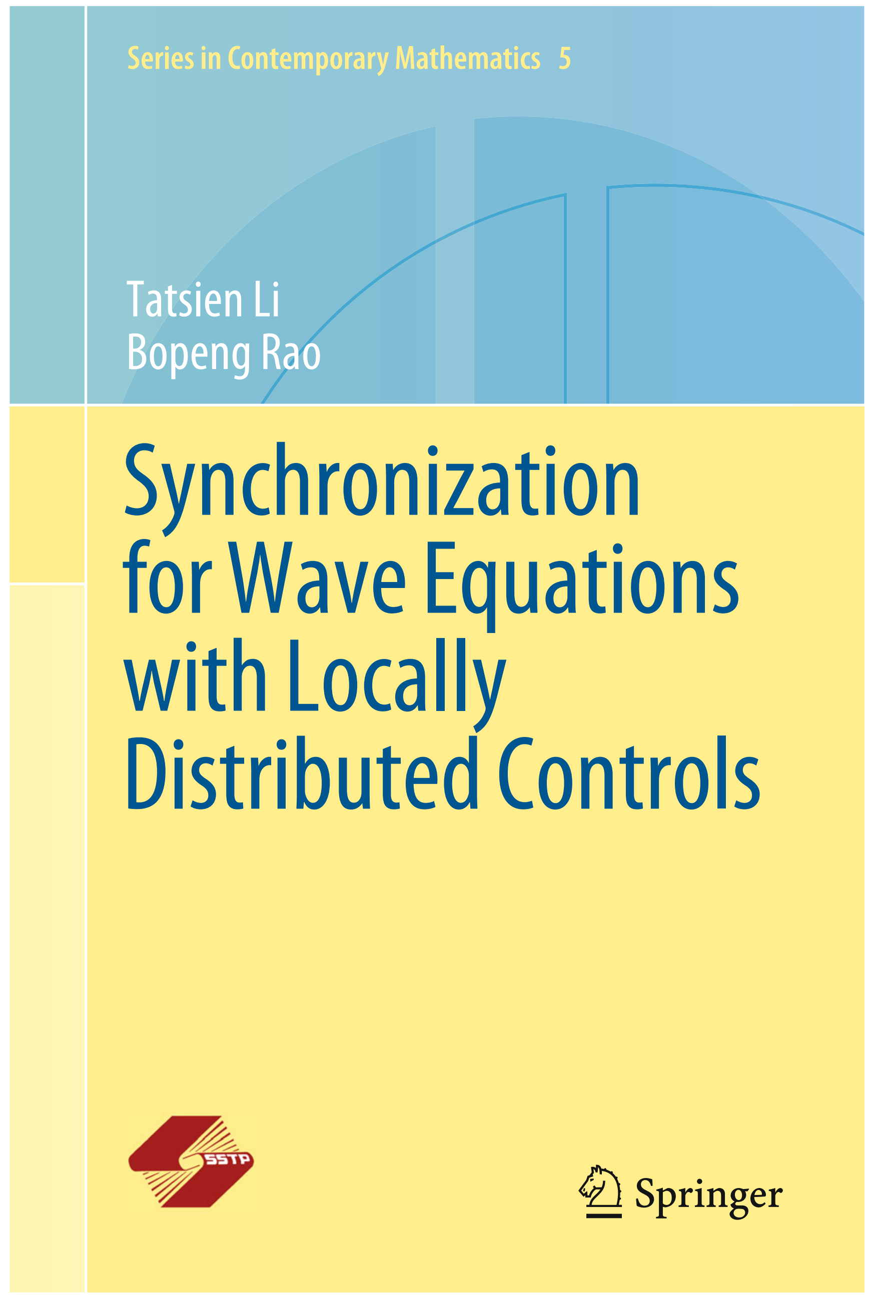
  - Asymptotic synchronization
  - Uniform (exponential) synchronization



2019

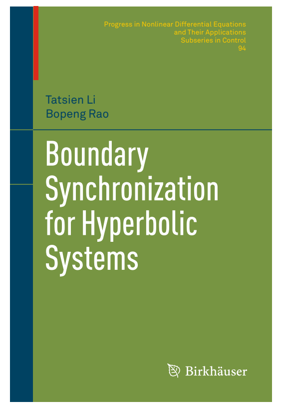


2021

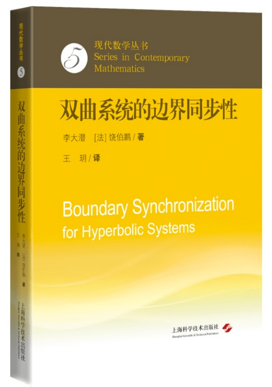


2024

Boundary sync. for wave equations



2019



2021

Consider the following coupled system of wave equations:

$$\begin{cases} U'' - \Delta U + AU = 0 & \text{in } (0,+\infty)\times\Omega, \\ U = 0 & \text{on } (0,+\infty)\times\Gamma_0, \\ U = DH & \text{on } (0,+\infty)\times\Gamma_1 \end{cases}$$

with the initial condition

$$t=0:\ U=\hat{U}_0,\ U'=\hat{U}_1 \ \text{ in } \Omega,$$

where $U=(u^{(1)},\cdots,u^{(N)})^T$ is the state variable, $H=(h^{(1)},\ldots,h^{(M)})^T$ denotes the applied boundary control $(M\le N)$, $A\in\mathbb{M}^{N\times N}(\mathbb{R})$ is the coupling matrix, and $D\in\mathbb{M}^{N\times M}(\mathbb{R})$ is the boundary control matrix;
Ω is a bounded domain, with smooth boundary $\Gamma=\Gamma_1\cup\Gamma_0$ satisfying $\overline{\Gamma}_1\cap\overline{\Gamma}_0=\emptyset$ and $\mathrm{mes}(\Gamma_1)>0$.

Def. Exact boundary synchronization for $t\ge T$

$$u^{(1)}(t,\cdot)\equiv u^{(2)}(t,\cdot)\equiv\cdots\equiv u^{(N)}(t,\cdot):=u(t,\cdot),$$

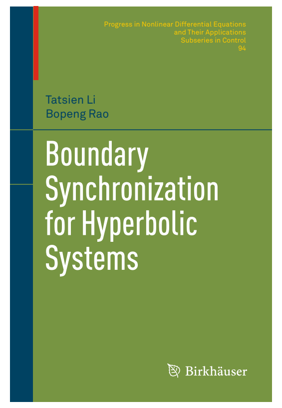
while $u=u(t,x)$ is called the corresponding exactly synchronizable state which is unknown beforehand. This final condition is equivalent to

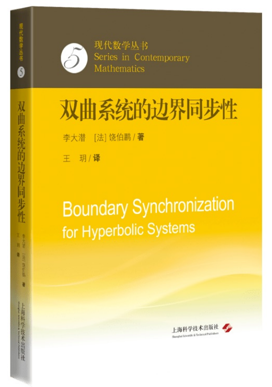
$$t\ge T:\ C_1U\equiv 0,$$

$$\text{where } C_1=\begin{pmatrix} 1 & -1 & & \\ & \ddots & \ddots & \\ & & 1 & -1 \end{pmatrix}_{(N-1)\times N}.$$

Initial Data: $(\widehat{U}_0,\widehat{U}_1)\in(L^2(\Omega))^N\times(H^{-1}(\Omega))^N$
Null Controllability: when $M=N$ and the usual multiplier geometrical condition
Question: in the case of **partial lack of boundary controls, which kind of controllability in a weaker sense** can be realized by means of fewer boundary controls?

Boundary sync. for wave equations





2019 **2021**

$$C_1 = \begin{pmatrix} 1 & -1 & & \\ & \ddots & \ddots & \\ & & 1 & -1 \end{pmatrix}_{(N-1)\times N}$$

Consider the following coupled system of wave equations:

$$\begin{cases} U'' - \Delta U + AU = 0 & \text{in } (0,+\infty)\times\Omega, \\ U = 0 & \text{on } (0,+\infty)\times\Gamma_0, \\ U = DH & \text{on } (0,+\infty)\times\Gamma_1 \end{cases}$$

with the initial condition

$$t = 0: \ U = \hat{U}_0, \ U' = \hat{U}_1 \ \text{ in } \Omega,$$

where $U = (u^{(1)}, \cdots, u^{(N)})^T$ is the state variable, $H = (h^{(1)}, \ldots, h^{(M)})^T$ denotes the applied boundary control $(M \le N)$, $A \in \mathbb{M}^{N\times N}(\mathbb{R})$ is the coupling matrix, and $D \in \mathbb{M}^{N\times M}(\mathbb{R})$ is the boundary control matrix;

Ω is a bounded domain, with smooth boundary $\Gamma = \Gamma_1 \cup \Gamma_0$ satisfying $\overline{\Gamma}_1 \cap \overline{\Gamma}_0 = \emptyset$ and $\mathrm{mes}(\Gamma_1) > 0$.

This system is exactly synchronizable iff $A = (a_{ij})$ satisfies **the row-sum condition**

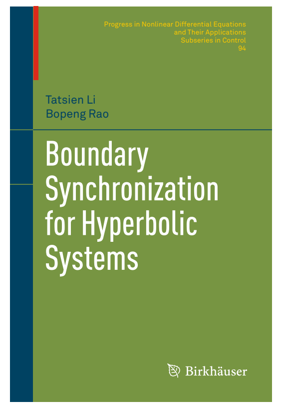
$$\sum_{p=1}^{N} a_{kp} = a \quad (k = 1, \cdots, N),$$

which is equivalent to $e_1 = (1,\ldots,1)^T$ is an eigenvalue of A, corresponding to the eigenvalue a.

Table 1: The exact boundary synchronization by p-groups

	Condition of C_p-compatibility	Minimal number of boundary controls
Exact boundary null controllability		N
Exact boundary synchronization	$C_1A = \overline{A}_1C_1$	$N-1$
Exact boundary synchronization by 2-groups	$C_2A = \overline{A}_2C_2$	$N-2$
....		
Exact boundary synchronization by p-groups	$C_pA = \overline{A}_pC_p$	$N-p$

Boundary sync. for wave equations



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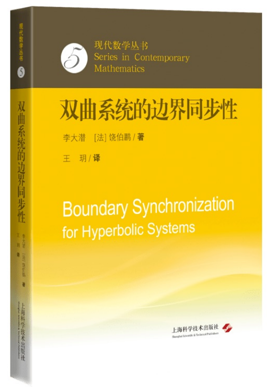

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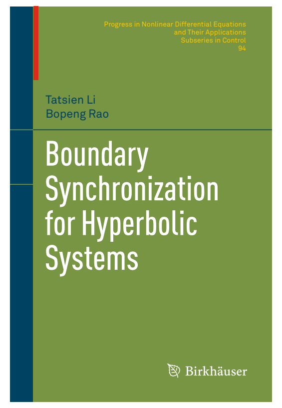
Null Controllability for Reduced System

$$\begin{cases} W_1'' - \Delta W_1 + \overline{A}_1 W_1 = 0 & \text{in } (0,+\infty)\times\Omega, \\ W_1 = 0 & \text{on } (0,+\infty)\times\Gamma_0, \\ W_1 = C_1 DH & \text{on } (0,+\infty)\times\Gamma_1. \end{cases}$$

where

$$W_1 = (w^{(1)}, \cdots, w^{(N-1)})^{\mathrm{T}} = C_1 U.$$

Boundary sync. for wave equations



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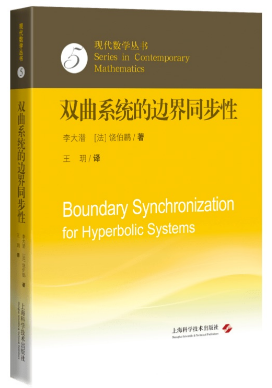

2021

Consider the following coupled system of wave equations:

$$(1.5) \qquad \begin{cases} U'' - \Delta U + AU = 0 & \text{in } (0, +\infty) \times \Omega, \\ U = 0 & \text{on } (0, +\infty) \times \Gamma_0, \\ U = DH & \text{on } (0, +\infty) \times \Gamma_1 \end{cases}$$

with the initial condition

$$(1.6) \qquad t = 0: \ U = \hat{U}_0, \ U' = \hat{U}_1 \ \text{ in } \Omega,$$

where $U = (u^{(1)}, \cdots, u^{(N)})^T$ is the state variable, $H = (h^{(1)}, \ldots, h^{(M)})^T$ denotes the applied boundary control ($M \leq N$), $A \in \mathbb{M}^{N \times N}(\mathbb{R})$ is the coupling matrix, and $D \in \mathbb{M}^{N \times M}(\mathbb{R})$ is the boundary control matrix;

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Def. Approximate boundary synchronization for $t \geq T$

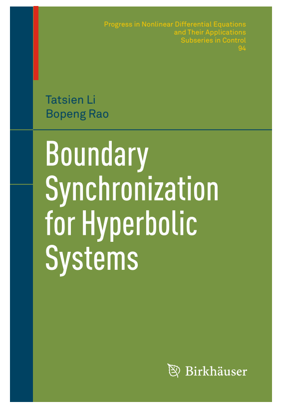
System (1.5) possesses the **approximate boundary synchronization** at the time $T > 0$ iffor any given initial data $(\widehat{U}_0, \widehat{U}_1) \in (L^2(\Omega))^N \times (H^{-1}(\Omega))^N$, there exist a sequence $\{H_n\}$ of boundary controls, $H_n \in L^2_{loc}(0, +\infty; (L^2(\Gamma_1))^M)$ with compact support in $[0, T]$, such that the corresponding sequence $\{U_n\} = \{(u_n^{(1)}, \cdots, u_n^{(N)})^T\}$ of solutions to problem (1.5)-(1.6) satisfies

$$u_n^{(k)} - u_n^{(l)} \to 0 \quad \text{as } n \to +\infty$$

for all $1 \leqslant k, l \leqslant N$ in the space

$$C^0_{loc}([T, +\infty); L^2(\Omega)) \cap C^1_{loc}([T, +\infty); H^{-1}(\Omega)).$$

Boundary sync. for wave equations



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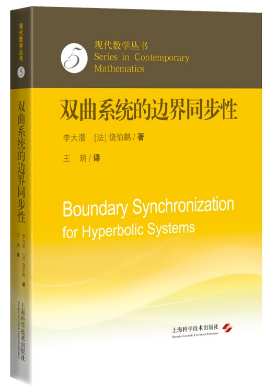

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Approximate Null Controllability for Reduced System

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where

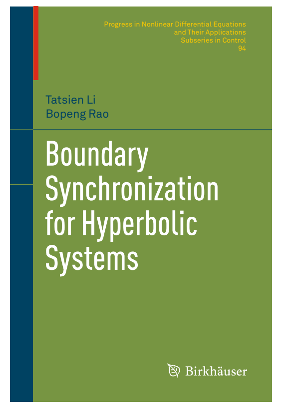
$$W_1 = (w^{(1)}, \cdots, w^{(N-1)})^{\mathrm{T}} = C_1 U.$$

Approximate C_1D-Observability for Reduced Adjoint Problem

$$\begin{cases} \Psi_1'' - \Delta\Psi_1 + \overline{A}_1^{\mathrm{T}}\Psi_1 = 0 & \text{in } (0,+\infty)\times\Omega, \\ \Psi_1 = 0 & \text{on } (0,+\infty)\times\Gamma, \\ t = 0: \ \Psi_1 = \widehat{\Psi}_0, \quad \Psi_1' = \widehat{\Psi}_1 & \text{in } \Omega. \end{cases}$$

$$(C_1D)^{\mathrm{T}}\partial_\nu\Psi \equiv 0 \text{ on } [0,T]\times\Gamma_1 \Longrightarrow (\widehat{\Psi}_0, \widehat{\Psi}_1) \equiv 0, \ i.e., \Psi \equiv 0.$$

Boundary sync. for wave equations



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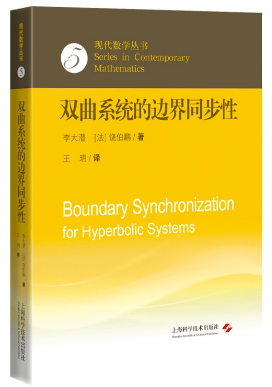

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where $U = (u^{(1)},\cdots,u^{(N)})^T$ is the state variable, $H = (h^{(1)},\dots,h^{(M)})^T$ denotes the applied boundary control $(M \le N)$, $A \in \mathbb{M}^{N\times N}(\mathbb{R})$ is the coupling matrix, and $D \in \mathbb{M}^{N\times M}(\mathbb{R})$ is the boundary control matrix;
Ω is a bounded domain, with smooth boundary $\Gamma = \Gamma_1 \cup \Gamma_0$ satisfying $\overline{\Gamma}_1 \cap \overline{\Gamma}_0 = \emptyset$ and $\mathrm{mes}(\Gamma_1) > 0$.

Main results: A **necessary** but **not sufficient** **rank condition** for approximate boundary synchronization,

1. with the condition of C_1 -compatibility condition :

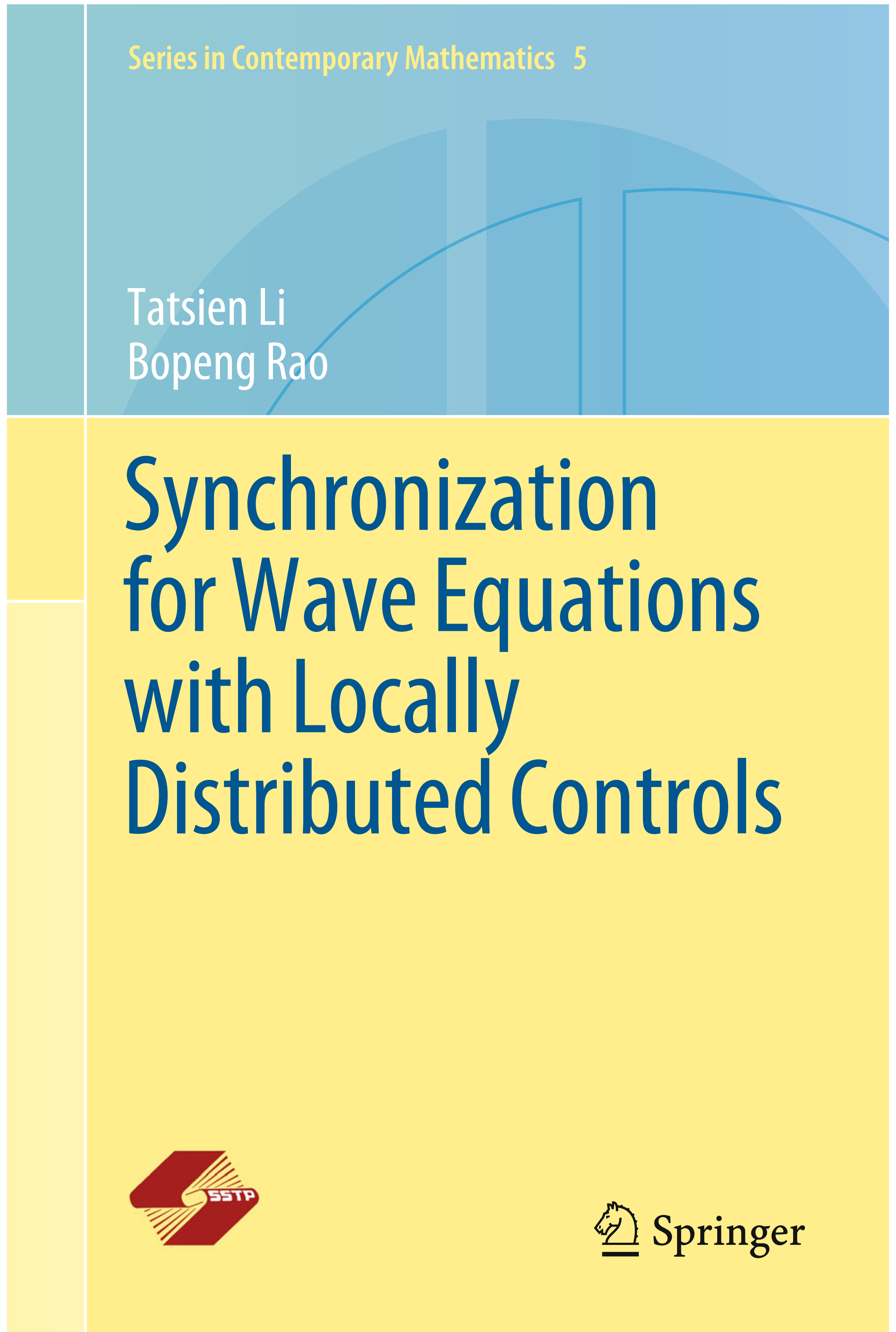
$$\mathrm{rank}(C_1 D, C_1 AD, \cdots, C_1 A^{N-1} D) = N - 1.$$

2. with/without the condition of C_1 -compatibility condition :

$$\mathrm{rank}(D, AD, \cdots, A^{N-1} D) \geqslant N - 1$$

More discussion of the approximately synchronizable state, and of Neumann, Robin boundary controls can be found in the book.

Internal SYNC for wave equations with locally distributed controls



2024

Part I
System of Wave Equations with Internal Controls

Let $\Omega \subset \mathbb{R}^m$ be a bounded domain with smooth boundary Γ and ω be a subdomain of Ω. Let A be a matrix of order N and D be a full column-rank matrix of order $N \times M (M \leqslant N)$, both with constant elements. Consider the following system with the state variable $U = (u^{(1)}, \dots, u^{(N)})^T$ and the internal control $H = (h^{(1)}, \dots, h^{(M)})^T$:

$$\begin{cases} U'' - \Delta U + AU = \boxed{D\chi_\omega H} & \text{in } (0, +\infty) \times \Omega, \\ U = 0 & \text{on } (0, +\infty) \times \Gamma \end{cases} \tag{I}$$

associated with the initial condition:

$$t = 0: \ U = \widehat{U}_0, \ U' = \widehat{U}_1 \ \text{in } \Omega, \tag{I_0}$$

where χ_ω denotes the characteristic function of ω, the symbol $'$ stands for the time-derivative, and $\Delta = \sum_{k=1}^m \frac{\partial^2}{\partial x_k^2}$ is the Laplacian operator.

A **necessary** and **sufficient** **rank condition** for approximate internal synchronization.

Part II
System of Wave Equations with Mixed Internal and Boundary Controls

Let $\Omega \subset \mathbb{R}^m$ be a bounded domain with smooth boundary Γ and $\omega \subset \Omega$ be a neighborhood of Γ. Let A be a matrix of order N; D_1 and D_2 be full column-rank matrices of order $N \times M_1$ and $N \times M_2$, respectively; and all the matrices are of constant elements. Consider the following system for the state variable $U = (u^{(1)}, \dots, u^{(N)})^T$, the internal control $H = (h^{(1)}, \dots, h^{(M_1)})^T$, and the boundary control $G = (g^{(1)}, \dots, g^{(M_2)})^T$:

$$\begin{cases} U'' - \Delta U + AU = D_1\chi_\omega H & \text{in } (0, +\infty) \times \Omega, \\ U = D_2 G & \text{on } (0, +\infty) \times \Gamma \end{cases} \tag{II}$$

Perspectives in Sync for PDEs

- Nonlinear case.
  - L. Hu; T. T. Li; P. Qu, Exact boundary synchronization for a coupled system of 1-D quasilinear wave equations, ESAIM: Control, Optimisation and Calculus of Variations, 22 (2016), 1136-1183.
    X. Lu, Local exact boundary synchronization for a kind of first order quasilinear hyperbolic systems, Chin. Ann. Math., Ser. B, 40 (2019), 79-96.
- The exact and approximate boundary synchronizations of nodal profile and on networks
- The phenomena of synchronization through coupling among individuals with possibly different motion laws (governing equations), whose nature is yet to be explored. The research on the existence of the exactly synchronizable state for a coupled system of wave equations with different wave speeds has been initiated.
- Generalized exact boundary synchronization.
- Other linear or nonlinear evolution equations (such as beam equations, plate equations, heat equations, etc.) .
- To extend the concept of synchronization to the case of components with different time delay will be more challenging and may expose quite different features.

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Summary

- Motivation
- **Boundary controllability for coupled wave equations (1D, quasilinear case)**
- **Synchronization for coupled wave equations (high dimensional, linear case)**

Thank you!

Benasque, August 22, 2024

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