# Exponential convergence to steady-states for damped adhesive strings

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# Adhesive strings and reversible decohesion ("unzipping")

## Aim

Modelling the (global) dynamics of an elastic string interacting with a rigid substrate through an adhesive layer and studying attachment–detachment regimes.

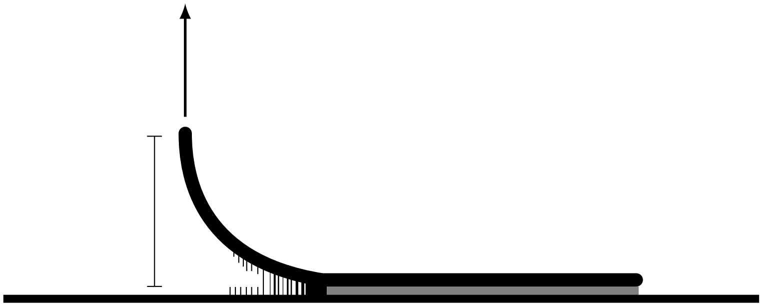

Figure: [Maddalena–Percivale–Puglisi–Truskinovsky, *Cont. Mech. Thermodyn.* 2009].

**Applications:** peeling of polymeric tapes, to rolling of cells, geckos' fibrillar structures, denaturation of DNA...

# The model under consideration

*Dynamics of a one-dimensional linearly elastic body, whose reference configuration is $(0, L)$, interacting with a rigid substrate through an adhesive material, acting through the force $\Phi'(u)$:*

$$
\text{(W)} \qquad \begin{cases} \partial^2_{tt} u - \partial^2_{xx} u + \partial_t u + \Phi'(u) = 0\,, & (t,x) \in (0,+\infty)\times(0,L)\,, \\ \partial_x u(t,0) = \partial_x u(t,L) = 0\,, & t \in (0,+\infty)\,, \\ u(0,x) = u_0(x)\,,\ \partial_t u(0,x) = v_0(x)\,, & x \in (0,L)\,. \end{cases}
$$

- *wave operator* describing the dynamics of an adhesive string;
- a *damping term*, $\partial_t u$, accounts for the effect of friction;
- *forcing term* $\Phi'$: if the displacement $u$ is small compared to $u_*$, the force is *purely elastic*; otherwise, if $|u| \geq u_*$, the adhesive material *ceases to act* on the elastic body;
- *Neumann boundary conditions*.

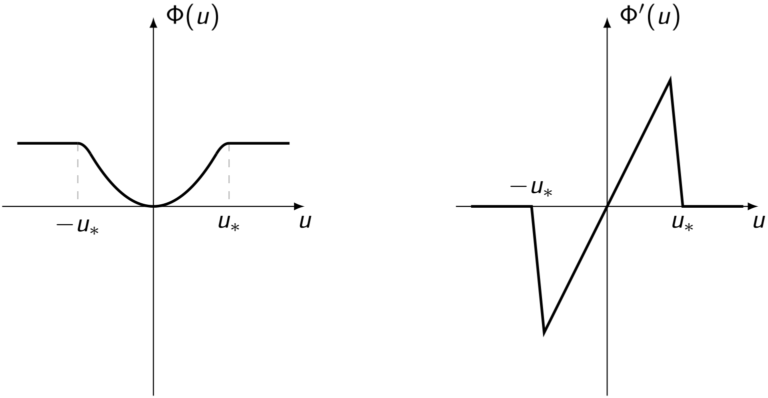


Figure: The potential $\Phi$ and the force $\Phi'$ as functions of $u$.

# Main problem

## Problem: long-time behavior

The attachment–detachment process ruled by the nonlinear force $\Phi'(u)$ induces the natural question about the *long-time behavior* of such dynamics:

> *Does the system converge towards a stationary state as* $t \to +\infty$*?*
> *Or does switching between the two states (attached/detached) persist?*

The problem contains two tasks:

1. study the global-in-time well-posedness of (W) *in a suitable function space* (more or less standard);
2. study the limit of $u(t)$ as $t \to +\infty$.

# Study of the long-time behavior of the dynamics

We expect that $u(t) \to u_\infty$ as $t \to +\infty$, where $u_\infty$ is a solution to the *stationary problem*:

(S) $$\begin{cases} -\partial_{xx}^2 u_\infty(x) + \Phi'(u_\infty(x)) = 0\,, \quad x \in (0, L)\,, \\ \partial_x u_\infty(0) = \partial_x u_\infty(L) = 0\,. \end{cases}$$

Continuous set of possible choices for the limit profiles:

$$\{\Phi' = 0\} = (-\infty, -u_*] \cup \{0\} \cup [u_*, +\infty).$$

**Part 1:** *characterization of the limit points*. We show that $\{u(t)\}_{t\geq 0}$ converges, as $t \to +\infty$ to a *uniquely determined* limit profile $u_\infty$ satisfying (S).

**Part 2:** *rate of convergence*. We show that the convergence $u(t) \to u_\infty$ as $t \to +\infty$ occurs in an *exponential* fashion: *i.e.*,

$$\|u(t) - u_\infty\|_{H^1((0,L))} + \|\partial_t u(t)\|_{L^2((0,L))} \leq Me^{-\kappa t}.$$

# Main results

## Theorem 1 (Well-posedness)

*Given initial data*

$$u_0 \in H^1(\Omega) \quad \textit{and} \quad v_0 \in L^2(\Omega),$$

*there exists a unique solution* $(u, \partial_t u) \in C\left([0, +\infty); H^1(\Omega) \times L^2(\Omega)\right)$ *to* (W).

## Theorem 2 (Long-time asymptotics)

*Given initial data*

$$u_0 \in H^1(\Omega) \quad \textit{and} \quad v_0 \in L^2(\Omega),$$

*let* $u$ *be the unique solution to* (W)*. Then there exists* $u_\infty$ *constant a.e. in* $\Omega$ *with* $u_\infty \in \{\Phi' = 0\}$ *such that* $u(t) \to u_\infty$ *in* $H^1(\Omega)$ *and* $\partial_t u(t) \to 0$ *in* $L^2(\Omega)$ *for* $t \to +\infty$*.*

**NB:** Both results hold for a bounded, open, and connected set $\Omega \subset \mathbb{R}^d$, with $d \geq 1$ (assuming also that the boundary is $C^2$ when $d \geq 2$).

## Theorem 3 (Exponential decay rate)

*Let* $\Omega = (0, L) \subset \mathbb{R}$. *Given initial data*

$$u_0 \in H^1(\Omega) \quad \textit{and} \quad v_0 \in L^2(\Omega),$$

*let* $u$ *be the unique solution to* (W).

*Let* $u_\infty$ *be as in Theorem 2. Let us assume that* $u_\infty = 0$ *or* $|u_\infty| > u_*$.

*Then*

$$\|u(t) - u_\infty\|_{H^1(\Omega)} + \|\partial_t u(t)\|_{L^2(\Omega)} \leq M_\Phi e^{-\kappa t}$$

*for some* $M_\Phi > 0$ *(possibly depending on the Lipschitz constant of* $\Phi'$ *) and* $\kappa > 0$.

# Main tool: energy balances

The *energy functional*

$$E(u,v) \coloneqq \frac{1}{2}\|v\|^2_{L^2(\Omega)} + \frac{1}{2}\|\nabla u\|^2_{L^2(\Omega)} + \|\Phi(u)\|_{L^1(\Omega)}$$

satisfies

$$E(u(t), \partial_t u(t)) + \int_0^t \|\partial_t u(s)\|^2_{L^2(\Omega)}\, ds = E(u_0, v_0)\,, \quad \text{for every } t \in [0, +\infty)\,.$$

In particular, $E$ is non-negative and decreasing on trajectories.

This suggests the appropriate function space (*the energy space*) to look for (global) solutions of (W).

The *auxiliary functional*

$$J(u,v) := \frac{1}{2}\|u\|^2_{L^2(\Omega)} + \langle u, v\rangle_{L^2(\Omega)}$$

(*cf.* [Haraux, *Bol. Soc. Bras. Mat.* 1986]) satisfies

$$\begin{aligned} J(u(t),\partial_t u(t)) &+ \int_0^t \left(\|\nabla u(s)\|^2_{L^2(\Omega)} + \langle u(s), \Phi'(u(s))\rangle_{L^2(\Omega)}\right) ds \\ &= J(u_0,v_0) + \int_0^t \|\partial_t u(s)\|^2_{L^2(\Omega)}\, ds\,, \quad \text{for every } t\in[0,+\infty)\,. \end{aligned}$$

In particular,

$$J(u(t),\partial_t u(t)) + \int_t^{+\infty} \|\partial_t u(s)\|^2_{L^2(\Omega)}\, ds$$

is non-negative and decreasing on trajectories.

# Outline of the proof of the qualitative convergence result

1. **Compactness argument** (via energy functional)**.** Accumulation points of the trajectories $\{u(t)\}_{t\geq 0}$ satisfy (S).

**!!** The compactness argument alone *does not suffice* to infer convergence of the whole trajectory as $t \to +\infty$ to a uniquely determined limit profile.

*Why?* Because there is a *continuous set* of possible choices for the limit profiles: the set of solutions to (S) is given by *constant functions* valued in

$$\{\Phi' = 0\} = (-\infty, -u_*] \cup \{0\} \cup [u_*, +\infty).$$

2. **Selection of a unique limit profile.** The auxiliary functional $J$ is a *Lyapunov functional* for the system and allows us to conclude the convergence $u(t) \to u_\infty$ as $t \to +\infty$.

   **NB:** the initial datum enforces the selection of a unique limit profile $u_\infty \in \{\Phi' = 0\}$.

# Outline of the proof of the exponential decay rate

1. **Compact embedding** $H^1((0,L)) \subset\subset C([0,L])$ (true only in 1D).

   This allows us to single out only one of the two possible attachment–detachment regimes (for time large enough) and study separately

   $$\partial_{tt}^2 u + \partial_t u - \partial_{xx}^2 u = 0, \qquad (t,x) \in (0,+\infty)\times(0,L)\,,$$

   or

   $$\partial_{tt}^2 u + \partial_t u - \partial_{xx}^2 u + 2u = 0, \qquad (t,x) \in (0,+\infty)\times(0,L)\,.$$

2. **Grönwall-type inequalities for perturbed energy functionals.**

   [Haraux–Zuazua, *ARMA* 1988]:

   $$G_\lambda(u,v) := \frac{1}{2}\|v\|_{L^2(\Omega)}^2 + \frac{1}{2}\|\nabla u\|_{L^2(\Omega)}^2 + \int_\Omega (\Phi(u) - \Phi(u_\infty))\,dx + \lambda\langle u - u_\infty, v\rangle_{L^2(\Omega)}$$

   (perturbation, depending on $\lambda \in (0,1)$, of $E(u,v) - E(u_\infty, 0)$).

# Further questions

- **Abrupt attachment–detachment.** If the rate of the exponential decay $Me^{-\kappa t}$ is *uniform* w.r.t. the slope of the decreasing part of the force $\Phi'(u)$, then we may consider a *discontinuous force*:

$$\Phi'(u) := \begin{cases} 2u & \text{for } |u| < u_* \,, \\ 0 & \text{for } |u| \geq u_* \,. \end{cases}$$

  - We already have positive results for an ODE model!

- **System of thermoelasticity.** The temperature gradient acts as a force on the wave equation governing the elastic component and the pressure waves act as a heat-source on the diffusion equation governing the temperature:

$$\text{(T)} \qquad \begin{cases} \partial_{tt}^2 u - \partial_{xx}^2 u - \partial_x \theta + \Phi'(u) = 0 \,, & (t,x) \in (0,+\infty) \times (0,L) \,, \\ \partial_t \theta - \partial_{xx}^2 \theta - \partial_{tx}^2 u = 0 \,, & (t,x) \in (0,+\infty) \times (0,L) \,. \end{cases}$$

# Thank you for your attention!

## References

Giuseppe Maria Coclite, Nicola De Nitti, Francesco Maddalena, Gianluca Orlando, Enrique Zuazua. Exponential convergence to steady-states for trajectories of a damped dynamical system modelling adhesive strings. *Mathematical Models and Methods in Applied Sciences* 34, No. 08, 1445–1482 (2024). .