Controllability results for KdV-type equations

Hugo Parada

X Partial differential equations, optimal design and numerics, Benasque

Ongoing works with R. Capistrano-Filho and S. Majumdar

August 22, 2024

The KdV equation

The Korteweg-de Vries (KdV) equation $\partial_t u + \partial_x u + \partial_x^3 u + u\partial_x u = 0$ was introduced by Diederik Korteweg and Gustav de Vries in 1985[1] to model the propagation of long water waves in a channel.

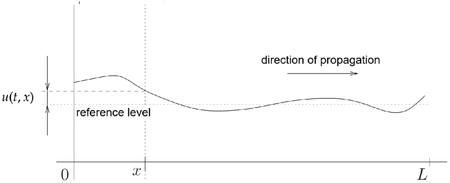

Figure: Solitary waves.

Model a variety of phenomena, including water waves, tsunamis, transmission of electrical signals in nerve fibers, plasma, cosmology, etc.

[1]D. Korteweg and G. de Vries, "On the change of form of long waves advancing in a rectangular channel, and a new type of long stationary wave", Phil. Mag **39**, 422–443 (1895)

Control of KdV

- First control and stabilization results Russel, Zhang[2] (Periodic framework).
- In the non-periodic framework we have the work of Rosier[3].
- Several other results, Coron, Crépeau, Cerpa, Nguyen, etc. Nonlinear system

[2]D. L. Russell and B. Y. Zhang, "Controllability and stabilizability of the third-order linear dispersion equation on a periodic domain", SIAM journal on control and optimization **31**, 659–676 (1993)

[3]L. Rosier, "Exact boundary controllability for the Korteweg-de Vries equation on a bounded domain", ESAIM: Control, Optimisation and Calculus of Variations **2**, 33–55 (1997)

Control of KdV

- First control and stabilization results Russel, Zhang[2] (Periodic framework).
- In the non-periodic framework we have the work of Rosier[3].
- Several other results, Coron, Crépeau, Cerpa, Nguyen, etc. Nonlinear system

Rosier considered the following control problem

$$\begin{cases} \partial_t y + \partial_x y + \partial_x^3 y + y\partial_x y = 0, & (t,x) \in (0,T)\times(0,L), \\ y(t,0) = y(t,L) = 0, & t \in (0,T), \\ \partial_x y(t,L) = h(t), & t \in (0,T), \\ y(0,x) = y_0(x), & x \in (0,L). \end{cases}$$

The linearized system is exactly controllable if and only if

$$L \notin \mathcal{N}_N = \left\{ 2\pi\sqrt{\frac{k^2+kl+l^2}{3}}; k,l \in \mathbb{N}^* \right\}$$

[2]D. L. Russell and B. Y. Zhang, "Controllability and stabilizability of the third-order linear dispersion equation on a periodic domain", SIAM journal on control and optimization **31**, 659–676 (1993)

[3]L. Rosier, "Exact boundary controllability for the Korteweg-de Vries equation on a bounded domain", ESAIM: Control, Optimisation and Calculus of Variations **2**, 33–55 (1997)

Rosier's strategy

Consider the linear equation

$$\begin{cases} \partial_t y + \partial_x y + \partial_x^3 y = 0, & (t,x) \in (0,T) \times (0,L), \\ y(t,0) = y(t,L) = 0, & t \in (0,T), \\ \partial_x y(t,L) = h(t), & t \in (0,T), \\ y(0,x) = y_0(x), & x \in (0,L). \end{cases}$$

Via H.U.M it is possible to show that exact controllability holds if and only if

$$\|z_0\|_{L^2(0,L)} \leq C \|\partial_x z(\cdot,0)\|_{L^2(0,T)}, \qquad \forall z_0 \in L^2(0,L) \tag{Obs}$$

and $z = S(\cdot)z_0 \in C([0,T]; L^2(0,L)) \cap L^2(0,T; H^1(0,L))$ solution of

$$\begin{cases} \partial_t z + \partial_x z + \partial_x^3 z = 0, & (t,x) \in (0,T) \times (0,L), \\ z(t,0) = z(t,L) = \partial_x z(t,L) = 0, & t \in (0,T), \\ z(0,x) = z_0(x), & x \in (0,L). \end{cases}$$

Overdeterminated system

Using compactness ideas we just focus on the stationary problem Exact controllability equivalent to study

$$\exists(\lambda,\psi)\in\mathbb{C}\times H^3(0,L)\setminus\{0\}\begin{cases}\lambda\psi+\psi'+\psi'''=0,\\ \psi(0)=\psi(L)=\psi'(0)=\psi'(L)=0\end{cases}\qquad(\mathcal{A})$$

(Paley-Wiener) Extend ψ to $\mathbb{R}$, then $\lambda\psi+\psi'+\psi'''=\underbrace{\psi''(0)}_{\alpha}\delta_0-\underbrace{\psi''(L)}_{\beta}\delta_L$.

Overdeterminated system

Using compactness ideas we just focus on the stationary problem Exact controllability equivalent to study

$$\exists(\lambda,\psi)\in\mathbb{C}\times H^3(0,L)\setminus\{0\}\begin{cases}\lambda\psi+\psi'+\psi'''=0,\\ \psi(0)=\psi(L)=\psi'(0)=\psi'(L)=0\end{cases}\qquad(\mathcal{A})$$

(Paley-Wiener) Extend ψ to $\mathbb{R}$, then $\lambda\psi+\psi'+\psi'''=\underbrace{\psi''(0)}_{\alpha}\,\delta_0-\underbrace{\psi''(L)}_{\beta}\,\delta_L.$

Overdeterminated system

Using compactness ideas we just focus on the stationary problem Exact controllability equivalent to study

$$\exists(\lambda,\psi)\in\mathbb{C}\times H^3(0,L)\setminus\{0\}\begin{cases}\lambda\psi+\psi'+\psi'''=0,\\ \psi(0)=\psi(L)=\psi'(0)=\psi'(L)=0\end{cases}\qquad(\mathcal{A})$$

(Paley-Wiener) Extend ψ to $\mathbb{R}$, then $\lambda\psi+\psi'+\psi'''=\underbrace{\psi''(0)}_{\alpha}\delta_0-\underbrace{\psi''(L)}_{\beta}\delta_L$.

Taking Fourier transform $(\mathcal{A})$ is equivalent to the existence of $(p,\alpha,\beta)\in\mathbb{C}^3$ $(\alpha,\beta\neq 0)$ such that

$$\hat{\psi}(\xi)=if(\xi),\quad f(\xi)=\frac{\alpha-\beta e^{-iL\xi}}{\xi^3-\xi+p},\qquad \lambda=-ip\in i\mathbb{R},$$

1. f is an entire function in $\mathbb{C}$;
2. $\int_{\mathbb{R}}|f(\xi)|^2(1+|\xi|^2)d\xi;$
3. $\forall\xi\in\mathbb{C},\ |f(\xi)|\leq((1+|\xi|)^N e^{L|Im(\xi)|}),\ C,N>0.$

Entire function

$$f(\xi) = \frac{\alpha - \beta e^{-iL\xi}}{\xi^3 - \xi + p}$$

The roots of $\alpha - \beta e^{-iL\xi}$ are simple and periodic. Then we must study the case

$$\xi_1 := \xi_0 + k\frac{2\pi}{L}, \quad \xi_2 := \xi_1 + l\frac{2\pi}{L},$$
$$\xi^3 - \xi + p = (\xi - \xi_0)(\xi - \xi_1)(\xi - \xi_2)$$

$$\begin{cases} \xi_0 + \xi_1 + \xi_2 = 0, \\ \xi_0\xi_1 + \xi_0\xi_2 + \xi_1\xi_2 = -1, \\ \xi_0\xi_1\xi_2 = -p. \end{cases}$$

Entire function

$$f(\xi) = \frac{\alpha - \beta e^{-iL\xi}}{\xi^3 - \xi + p}$$

The roots of $\alpha - \beta e^{-iL\xi}$ are simple and periodic. Then we must study the case

$$\xi_1 := \xi_0 + k\frac{2\pi}{L}, \quad \xi_2 := \xi_1 + l\frac{2\pi}{L},$$
$$\xi^3 - \xi + p = (\xi - \xi_0)(\xi - \xi_1)(\xi - \xi_2)$$

$$\begin{cases} \xi_0 + \xi_1 + \xi_2 = 0, \\ \xi_0\xi_1 + \xi_0\xi_2 + \xi_1\xi_2 = -1, \\ \xi_0\xi_1\xi_2 = -p. \end{cases}$$

After some calculations we get

$$L = 2\pi\sqrt{\frac{k^2 + kl + l^2}{3}},$$

Critical length phenomena

Entire function

$$f(\xi) = \frac{\alpha - \beta e^{-iL\xi}}{\xi^3 - \xi + p}$$

The roots of $\alpha - \beta e^{-iL\xi}$ are simple and periodic. Then we must study the case

$$\xi_1 := \xi_0 + k\frac{2\pi}{L}, \quad \xi_2 := \xi_1 + l\frac{2\pi}{L},$$
$$\xi^3 - \xi + p = (\xi - \xi_0)(\xi - \xi_1)(\xi - \xi_2)$$

$$\begin{cases} \xi_0 + \xi_1 + \xi_2 = 0, \\ \xi_0\xi_1 + \xi_0\xi_2 + \xi_1\xi_2 = -1, \\ \xi_0\xi_1\xi_2 = -p. \end{cases}$$

After some calculations we get

$$L = 2\pi\sqrt{\frac{k^2 + kl + l^2}{3}},$$

Critical length phenomena

In what follows we focus on the related overderminated systems appearing developing this strategy. We do not put emphasis in the regularity framework and nonlinear systems.

KdV on a star network

The KdV equation in a network, it was proposed to model the pressure on the arterial tree in[4]. We will study this equation on a star shaped network.

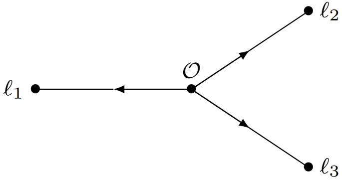

Figure: Star shaped network for $N = 3$.

$$\begin{cases} \partial_t y_j + \partial_x y_j + \partial_x^3 y_j + y_j \partial_x y_j = 0, & \text{(deflection)} \\ v_j = y_j - \frac{1}{6} y_j^2 + 2\partial_x^2 y_j, & \text{(velocity)} \end{cases}$$

Central node conditions

$$\begin{cases} y_j(t,0) = y_1(t,0), & \text{(continuity)} \\ \sum_{j=1}^{N} y_j(t,0) v_j(t,0) = 0, & \text{(null sum of the flux).} \end{cases}$$

[4]K. Ammari and E. Crépeau, "Feedback Stabilization and Boundary Controllability of the Korteweg–de Vries Equation on a Star-Shaped Network", SIAM Journal on Control and Optimization **56**, 1620–1639 (2018)

KdV on a star network

The KdV equation in a network, it was proposed to model the pressure on the arterial tree in[4]. We will study this equation on a star shaped network.

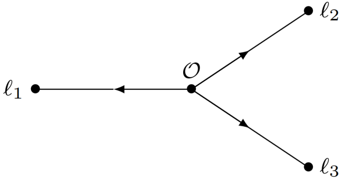

Figure: Star shaped network for $N = 3$.

$$\begin{cases} \partial_t y_j + \partial_x y_j + \partial_x^3 y_j + y_j \partial_x y_j = 0, & \text{(deflection)} \\ v_j = y_j - \frac{1}{6} y_j^2 + 2\partial_x^2 y_j, & \text{(velocity)} \end{cases}$$

Central node conditions

$$\begin{cases} y_j(t,0) = y_1(t,0), & \text{(continuity)} \\ \sum_{j=1}^{N} y_j(t,0) v_j(t,0) = 0, & \text{(null sum of the flux)}. \end{cases}$$

[4]K. Ammari and E. Crépeau, "Feedback Stabilization and Boundary Controllability of the Korteweg–de Vries Equation on a Star-Shaped Network", SIAM Journal on Control and Optimization **56**, 1620–1639 (2018)

LKdV on a star network

We study the linearization around $\underline{0}$

$$\begin{cases} \partial_t y_j(t,x) + \partial_x y_j(t,x) + \partial_x^3 y_j(t,x) = 0, & t \in (0,T),\ x \in (0,\ell_j),\ j = 1,\dots N, \\ y_j(t,0) = y_1(t,0), & t \in (0,T),\ \forall j = 2,\dots N, \\ \displaystyle\sum_{j=1}^{N} \partial_x^2 y_j(t,0) = -\alpha y_1(t,0) + g_0(t), & t \in (0,T), \\ y_j(t,\ell_j) = p_j(t), \quad \partial_x y_j(t,\ell_j) = g_j(t), & t \in (0,T),\ j = 1,\dots N, \\ y_j(0,x) = y_j^0(x), & x \in I_j, \end{cases}$$

Control on network

Controllability is equivalent to an observability inequality for the adjoint system

$$\begin{cases} -\partial_t \varphi_j - \partial_x \varphi_j - \partial_x^3 \varphi_j = 0, & \\ \varphi_j(t,0) = \varphi_1(t,0), & \forall j = 2, \dots N, \\ \displaystyle\sum_{j=1}^{N} \partial_x^2 \varphi_j(t,0) = (\alpha - N)\varphi_1(t,0), & t \in (0,T), \\ \varphi_j(t,\ell_j) = \partial_x \varphi_j(t,0) = 0, & \\ \varphi_j(T,x) = \varphi_j^T(x). & \end{cases}$$

$$\|\underline{\varphi}(T,x)\|^2_{\mathbb{L}^2(\mathcal{T})} \leq C \left(\underbrace{\sum_{j=1}^{N} \|\partial_x^2 \varphi_j(t,\ell_j)\|^2_{L^2(0,T)}}_{Dirichlet} + \underbrace{\sum_{j=1}^{N} \|\partial_x \varphi_j(t,\ell_j)\|^2_{L^2(0,T)}}_{Neumann} + \underbrace{\|\varphi_1(t,0)\|^2_{L^2(0,T)}}_{Central\ \ node} \right).$$

Neumann controls

Observability inequality reads[4]

$$\|\underline{\varphi}(T,x)\|^2_{\mathbb{L}^2(\mathcal{T})} \leq C\left(\sum_{j=1}^{N}\|\partial_x\varphi_j(t,\ell_j)\|^2_{L^2(0,T)} + \|\varphi_1(t,0)\|^2_{L^2(0,T)}\right).$$

Theorem (Controllability with $N+1$ controls)

$\#\{\ell_j \in \mathcal{N}_N\} \leq 1$. *Controls* $g_0, g_1, \ldots g_N$.

[4]K. Ammari and E. Crépeau, "Feedback Stabilization and Boundary Controllability of the Korteweg–de Vries Equation on a Star-Shaped Network", SIAM Journal on Control and Optimization **56**, 1620–1639 (2018)

Idea of the proof

This is related to study

$$\begin{cases} \lambda_j\varphi_j + \varphi_j' + \varphi_j''' = 0, & j = 1, \dots N \\ \varphi_j(\ell_j) = \varphi_j'(\ell_j) = 0, & j = 1, \dots N \\ \varphi_j(0) = \varphi_j'(0) = 0, & j = 1, \dots N \\ \sum_{j=1}^{N} \varphi_j''(0) = 0. \end{cases}$$

- If $\ell_j \notin \mathcal{N}_N$, wr have $\varphi_j(0) = \varphi_j'(0) = \varphi_j(\ell_j) = \varphi_j'(\ell_j) = 0$, thus $\varphi_j \equiv 0$.
- If $\ell_1 \in \mathcal{N}_N$. As $\varphi_j = 0$, for $j = 2, \dots N$, then

$$\begin{cases} \lambda_1\varphi_1 + \varphi_1' + \varphi_1''' = 0, \\ \varphi_1(\ell_1) = \varphi_1'(\ell_1) = 0, \\ \varphi_1(0) = \varphi_1'(0) = \varphi_1''(0) = 0. \end{cases}$$

Then $\varphi_1 = 0$.

Idea of the proof

This is related to study

$$\begin{cases} \lambda_j\varphi_j + \varphi_j' + \varphi_j''' = 0, & j = 1, \dots N \\ \varphi_j(\ell_j) = \varphi_j'(\ell_j) = 0, & j = 1, \dots N \\ \varphi_j(0) = \varphi_j'(0) = 0, & j = 1, \dots N \\ \sum_{j=1}^{N} \varphi_j''(0) = 0. \end{cases}$$

- If $\ell_j \notin \mathcal{N}_N$, wr have $\varphi_j(0) = \varphi_j'(0) = \varphi_j(\ell_j) = \varphi_j'(\ell_j) = 0$, thus $\varphi_j \equiv 0$.
- If $\ell_1 \in \mathcal{N}_N$. As $\varphi_j = 0$, for $j = 2, \dots N$, then

$$\begin{cases} \lambda_1\varphi_1 + \varphi_1' + \varphi_1''' = 0, \\ \varphi_1(\ell_1) = \varphi_1'(\ell_1) = 0, \\ \varphi_1(0) = \varphi_1'(0) = \varphi_1''(0) = 0. \end{cases}$$

Then $\varphi_1 = 0$.

N Neumann controls.

Theorem (Controllability with N controls.)

$\ell_j \notin \mathcal{N}_N$, for $j = 2, \dots N$. Controls g_0, $g_2, \dots g_N$.

What about no central node control?

Exact controllability with N Neumann controls if $\forall j$, $\ell_j \notin \mathcal{N}_N$?

N Neumann controls.

Theorem (Controllability with N controls.)

$\ell_j \notin \mathcal{N}_N$, for $j = 2, \dots N$. Controls g_0, $g_2, \dots g_N$.

What about no central node control?
This case was considered in[5], and the following result was proved.

Theorem (Controllability with N controls.)

There exist $L_0, T_0 > 0$, such that if

$$\max_{j=1,\dots N} \ell_j < L_0, \quad \text{and } T > T_0.$$

Exact controllability with controls $g_1, \dots g_N$.

Exact controllability with N Neumann controls if $\forall j$, $\ell_j \notin \mathcal{N}_N$?

[5]E. Cerpa et al., "On the boundary controllability of the Korteweg–de Vries equation on a star-shaped network", IMA Journal of Mathematical Control and Information **37**, 226–240 (2020)

N Neumann controls.

This ask us to study the solutions of

$$\begin{cases} \lambda_j\varphi_j + \varphi_j' + \varphi_j''' = 0, & j = 1, \dots N \\ \varphi_j(0) = \varphi_1(0), & j = 2, \dots N \\ \varphi_j(\ell_j) = \varphi_j'(\ell_j) = \varphi_j'(0) = 0, & j = 1, \dots N \\ \displaystyle\sum_{j=1}^{N} \varphi_j''(0) = (\alpha - N)\varphi_1. \end{cases}$$

Rosier's approach arises

$$i\hat{\varphi}_j(\xi) = \frac{\kappa\xi^2 + \beta_j e^{-i\xi\ell_j} + \gamma_j}{\xi^3 - \xi - p_j}, \dots$$

N Neumann controls.

I pass now to an easier case, $\ell_j = L, \forall j$. Now, we can define $\psi = \sum_{j=1}^{N} \varphi_j$.

Proposition

Let $L \notin \mathcal{N}_N$. Then $\psi = 0$ if and only if $\varphi_j = 0$ for all $j = 1, \dots N$.

N Neumann controls.

I pass now to an easier case, $\ell_j = L, \forall j$. Now, we can define $\psi = \sum_{j=1}^{N} \varphi_j$.

Proposition

Let $L \notin \mathcal{N}_N$. Then $\psi = 0$ if and only if $\varphi_j = 0$ for all $j = 1, \ldots N$.

If $\psi = 0$, we have $\psi(0) = N\varphi_j(0) = 0$ then φ_j solves

$$\begin{cases} \lambda\varphi_j + \varphi_j' + \varphi_j''' = 0, \\ \varphi_j(L) = \varphi_j'(L) = 0, \\ \varphi_j(0) = \varphi_j'(0) = 0. \end{cases}$$

As $L \notin \mathcal{N}_N$, $\varphi_j = 0$.

N Neumann controls.

By definition ψ solves

$$\begin{cases} \lambda\psi + \psi' + \psi''' = 0, \\ \psi(L) = \psi'(L) = \psi'(0) = 0, \\ \psi''(0) = \dfrac{\alpha - N}{N}\psi(0). \end{cases}$$

Consider the case $\alpha = N$

N Neumann controls.

By definition ψ solves

$$\begin{cases} \lambda\psi + \psi' + \psi''' = 0, \\ \psi(L) = \psi'(L) = \psi'(0) = 0, \\ \psi''(0) = \dfrac{\alpha - N}{N}\psi(0). \end{cases}$$

Consider the case $\alpha = N$

$$\begin{cases} \lambda\psi + \psi' + \psi''' = 0, \\ \psi(L) = \psi'(L) = 0, \\ \psi'(0) = \psi''(0) = 0. \end{cases} \qquad \begin{aligned} &\theta(x) = \psi(L - x), \\ &\tilde{\lambda} = -\lambda \end{aligned} \qquad \begin{cases} \tilde{\lambda}\theta + \theta' + \theta''' = 0, \\ \theta(0) = \theta'(0) = 0, \\ \theta'(L) = \theta''(L) = 0. \end{cases}$$

N Neumann controls.

By definition ψ solves

$$\begin{cases} \lambda\psi + \psi' + \psi''' = 0, \\ \psi(L) = \psi'(L) = \psi'(0) = 0, \\ \psi''(0) = \dfrac{\alpha - N}{N}\psi(0). \end{cases}$$

Consider the case $\alpha = N$

$$\begin{cases} \lambda\psi + \psi' + \psi''' = 0, \\ \psi(L) = \psi'(L) = 0, \\ \psi'(0) = \psi''(0) = 0. \end{cases} \qquad \begin{array}{l} \theta(x) = \psi(L - x), \\ \tilde{\lambda} = -\lambda \end{array} \qquad \begin{cases} \tilde{\lambda}\theta + \theta' + \theta''' = 0, \\ \theta(0) = \theta'(0) = 0, \\ \theta'(L) = \theta''(L) = 0. \end{cases}$$

$$\theta \equiv 0 \iff L \notin \mathcal{N}_D$$

Exact controllability with N controls if $L \notin \mathcal{N}_N \cup \mathcal{N}_D$.

Dirichlet controls

Theorem (Controllability with $N+1$ controls.)

$\#\{\ell_j \in \mathcal{N}_D\} \leq 1$. *Controls* g_0, $p_1, \dots p_N$.

Theorem (Controllability with N controls.)

$\ell_j \notin \mathcal{N}_D$, *for* $j = 2, \dots N$. *Controls* g_0, $p_2, \dots p_N$

No central node control ?

Proposition

$\ell_j = L \notin \mathcal{N}_D$. *Then* $\psi = \sum_{j=1}^{N} \varphi_j = 0$ *if and only if* $\varphi_j = 0$ *for all* $j = 1, \dots N$.

Dirichlet controls

Theorem (Controllability with $N+1$ controls.)

$\#\{\ell_j \in \mathcal{N}_D\} \leq 1$. *Controls* g_0, $p_1, \dots p_N$.

Theorem (Controllability with N controls.)

$\ell_j \notin \mathcal{N}_D$, *for* $j = 2, \dots N$. *Controls* g_0, $p_2, \dots p_N$

No central node control ?

Proposition

$\ell_j = L \notin \mathcal{N}_D$. *Then* $\psi = \sum_{j=1}^{N} \varphi_j = 0$ *if and only if* $\varphi_j = 0$ *for all* $j = 1, \dots N$.

Dirichlet controls

Theorem (Controllability with $N+1$ controls.)

$\#\{\ell_j \in \mathcal{N}_D\} \leq 1$. *Controls* g_0, $p_1, \dots p_N$.

Theorem (Controllability with N controls.)

$\ell_j \notin \mathcal{N}_D$, *for* $j = 2, \dots N$. *Controls* g_0, $p_2, \dots p_N$

No central node control ?

Proposition

$\ell_j = L \notin \mathcal{N}_D$. *Then* $\psi = \sum_{j=1}^{N} \varphi_j = 0$ *if and only if* $\varphi_j = 0$ *for all* $j = 1, \dots N$.

N Dirichlet controls.

For $\alpha = N$, ψ solves

$$\begin{cases} \lambda\psi + \psi' + \psi''' = 0, \\ \psi(L) = \psi''(L) = 0, \\ \psi'(0) = \psi''(0) = 0. \end{cases}, \quad \text{Change of variables} \quad \begin{cases} \lambda\psi + \psi' + \psi''' = 0, \\ \psi(0) = \psi''(0) = 0, \\ \psi'(L) = \psi''(L) = 0. \end{cases}$$

Following Glass-Guerrero[6] (Dirichlet) and Cerpa[7] (Neumann).

$$\mathcal{N}_{new} = \left\{ L > 0, a, b \in \mathbb{C}, \quad L^2 = -(a^2 + ab + b^2); \quad a^2 e^a = b^2 e^b = (a+b)^2 e^{-(a+b)} \right\}.$$

---

[6] O. Glass and S. Guerrero, "Controllability of the Korteweg–de Vries equation from the right dirichlet boundary condition", Systems & Control Letters **59**, 390–395 (2010)

[7] E. Cerpa, "Control of a Korteweg-de Vries equation: a tutorial", Mathematical Control & Related Fields **4**, 45–99 (2014)

N Dirichlet controls.

For $\alpha = N$, ψ solves

$$\begin{cases} \lambda\psi + \psi' + \psi''' = 0, \\ \psi(L) = \psi''(L) = 0, \\ \psi'(0) = \psi''(0) = 0. \end{cases} \quad , \quad \text{Change of variables} \quad \begin{cases} \lambda\psi + \psi' + \psi''' = 0, \\ \psi(0) = \psi''(0) = 0, \\ \psi'(L) = \psi''(L) = 0. \end{cases}$$

Following Glass-Guerrero[6] (Dirichlet) and Cerpa[7] (Neumann).

$$\mathcal{N}_{new} = \left\{ L > 0, a, b \in \mathbb{C}, \quad L^2 = -(a^2 + ab + b^2); \quad a^2e^a = b^2e^b = (a+b)^2e^{-(a+b)} \right\}.$$

---

[6] O. Glass and S. Guerrero, "Controllability of the Korteweg–de Vries equation from the right dirichlet boundary condition", Systems & Control Letters **59**, 390–395 (2010)

[7] E. Cerpa, "Control of a Korteweg-de Vries equation: a tutorial", Mathematical Control & Related Fields **4**, 45–99 (2014)

N Dirichlet controls.

For $\alpha = N$, ψ solves

$$\begin{cases} \lambda\psi + \psi' + \psi''' = 0, \\ \psi(L) = \psi''(L) = 0, \\ \psi'(0) = \psi''(0) = 0. \end{cases}, \quad \text{Change of variables} \quad \begin{cases} \lambda\psi + \psi' + \psi''' = 0, \\ \psi(0) = \psi''(0) = 0, \\ \psi'(L) = \psi''(L) = 0. \end{cases}$$

In this case the critical lengths are not known (yet).Following Glass-Guerrero[6] (Dirichlet) and Cerpa[7] (Neumann).

$$\mathcal{N}_{new} = \left\{ L > 0, a, b \in \mathbb{C}, \quad L^2 = -(a^2 + ab + b^2); \quad a^2 e^a = b^2 e^b = (a+b)^2 e^{-(a+b)} \right\}.$$

[6]O. Glass and S. Guerrero, "Controllability of the Korteweg–de Vries equation from the right dirichlet boundary condition", Systems & Control Letters **59**, 390–395 (2010)

[7]E. Cerpa, "Control of a Korteweg-de Vries equation: a tutorial", Mathematical Control & Related Fields **4**, 45–99 (2014)

N Dirichlet controls.

For $\alpha = N$, ψ solves

$$\begin{cases} \lambda\psi + \psi' + \psi''' = 0, \\ \psi(L) = \psi''(L) = 0, \\ \psi'(0) = \psi''(0) = 0. \end{cases}, \quad \text{Change of variables} \quad \begin{cases} \lambda\psi + \psi' + \psi''' = 0, \\ \psi(0) = \psi''(0) = 0, \\ \psi'(L) = \psi''(L) = 0. \end{cases}$$

Following Glass-Guerrero[6] (Dirichlet) and Cerpa[7] (Neumann).

$$\mathcal{N}_{new} = \left\{ L > 0, a, b \in \mathbb{C}, \quad L^2 = -(a^2 + ab + b^2); \quad a^2 e^a = b^2 e^b = (a+b)^2 e^{-(a+b)} \right\}.$$

[6] O. Glass and S. Guerrero, "Controllability of the Korteweg–de Vries equation from the right dirichlet boundary condition", Systems & Control Letters **59**, 390–395 (2010)

[7] E. Cerpa, "Control of a Korteweg-de Vries equation: a tutorial", Mathematical Control & Related Fields **4**, 45–99 (2014)

Critical lengths for Kawahara (with S. Majumdar)

Generalization of KdV (high order dispersive effects).

$$\begin{cases} \partial_t u - \partial_x u + \partial_x^3 u - \partial_x^5 u = 0, & (t,x) \in (0,T) \times (0,L), \\ u(t,0) = u(t,L) = \partial_x u(t,0) = \partial_x u(t,L) = 0, & t \in (0,T), \\ \partial_x^2 u(t,L) = \partial_x^2 u(t,0) = 0, & t \in (0,T). \end{cases}$$

There exists ψ not trivial such that $\psi' - \psi''' - \psi''''' = 0$ and $\psi(0) = \psi(L) = \psi'(0) = \psi'(L) = \psi''(0) = \psi''(L) = 0$. If $L \in \mathcal{N}^*$.

For our system there is no critical length phenomena.

Proposition

If $\lambda, \psi \in \mathbb{C} \times H^5(0,L)$ satisfies

$$\begin{cases} \lambda\psi - \psi' + \psi''' - \psi''''' = 0, & x \in (0,L) \\ \psi(0) = \psi(L) = \psi'(0) = \psi'(L) = \psi''(0) = \psi''(L) = 0, \end{cases}$$

then $\psi \equiv 0$.

Critical lengths for Kawahara (with S. Majumdar)

Generalization of KdV (high order dispersive effects).

$$\begin{cases} \partial_t u - \partial_x u + \partial_x^3 u - \partial_x^5 u = 0, & (t,x) \in (0,T) \times (0,L), \\ u(t,0) = u(t,L) = \partial_x u(t,0) = \partial_x u(t,L) = 0, & t \in (0,T), \\ \partial_x^2 u(t,L) = \partial_x^2 u(t,0) = 0, & t \in (0,T). \end{cases}$$

There exists ψ not trivial such that $\psi' - \psi''' - \psi''''' = 0$ and $\psi(0) = \psi(L) = \psi'(0) = \psi'(L) = \psi''(0) = \psi''(L) = 0$. If $L \in \mathcal{N}^*$.

For our system there is no critical length phenomena.

Proposition

If $\lambda, \psi \in \mathbb{C} \times H^5(0,L)$ *satisfies*

$$\begin{cases} \lambda\psi - \psi' + \psi''' - \psi''''' = 0, & x \in (0,L) \\ \psi(0) = \psi(L) = \psi'(0) = \psi'(L) = \psi''(0) = \psi''(L) = 0, \end{cases}$$

then $\psi \equiv 0$.

Critical lengths for Kawahara (with S. Majumdar)

Generalization of KdV (high order dispersive effects).

$$\begin{cases} \partial_t u - \partial_x u + \partial_x^3 u - \partial_x^5 u = 0, & (t,x) \in (0,T) \times (0,L), \\ u(t,0) = u(t,L) = \partial_x u(t,0) = \partial_x u(t,L) = 0, & t \in (0,T), \\ \partial_x^2 u(t,L) = \partial_x^2 u(t,0) = 0, & t \in (0,T). \end{cases}$$

There exists ψ not trivial such that $\psi' - \psi''' - \psi''''' = 0$ and $\psi(0) = \psi(L) = \psi'(0) = \psi'(L) = \psi''(0) = \psi''(L) = 0$. If $L \in \mathcal{N}^*$.

For our system there is no critical length phenomena.

Proposition

If $\lambda, \psi \in \mathbb{C} \times H^5(0,L)$ *satisfies*

$$\begin{cases} \lambda\psi - \psi' + \psi''' - \psi''''' = 0, & x \in (0,L) \\ \psi(0) = \psi(L) = \psi'(0) = \psi'(L) = \psi''(0) = \psi''(L) = 0, \end{cases}$$

then $\psi \equiv 0$.

Critical lengths Kawahara

Taking Fourier

$$-i\hat{\psi}(\xi) = f(\xi, L) = \frac{\alpha_1 i\xi - \alpha_2 i\xi e^{-i\xi L} + \alpha_3 - \alpha_4 e^{-i\xi L}}{\xi^5 + \xi^3 + \xi + r} = \frac{N(\xi, L)}{q(\xi)}, \qquad r \in \mathbb{R}.$$

Denote by ξ_i, $i = 1, \dots, 5$ the roots of $q(\cdot)$, then $\frac{\alpha_1 i\xi_i + \alpha_3}{\alpha_2 i\xi_i + \alpha_4} = e^{-i\xi_i L}$. We can introduce the Möbius transformation $M(\xi_i) = e^{-iL\xi_i}$. From[8] we have

Lemma

- *Let* $\vec{\alpha} \in \mathbb{C}^4 \setminus \{0\}$ *with* $d(\vec{\alpha}) = \alpha_1\alpha_3 - \alpha_2\alpha_4 = 0$ *and* $L > 0$. *Then, the set of the imaginary parts of the zeros of* $N(\cdot, L)$ *has at most two elements.*
- *For* $L > 0$, *there is no Möbius transformation* M, *such that*

$$M(\xi) = e^{-iL\xi}, \quad \xi \in \{\xi_1, \xi_2, \overline{\xi_1}, \overline{\xi_2}\}, \qquad \xi_1, \xi_2, \overline{\xi_1}, \overline{\xi_2} \text{ different.}$$

It is enough to prove that the set of imaginary parts of zeros of $q(\cdot)$ has three elements.

[8] A. L. C. d. Santos et al., "Entire functions related to stationary solutions of the kawahara equation", (2016)

Critical lengths Kawahara

Taking Fourier

$$-i\hat{\psi}(\xi) = f(\xi, L) = \frac{\alpha_1 i\xi - \alpha_2 i\xi e^{-i\xi L} + \alpha_3 - \alpha_4 e^{-i\xi L}}{\xi^5 + \xi^3 + \xi + r} = \frac{N(\xi, L)}{q(\xi)}, \qquad r \in \mathbb{R}.$$

Denote by ξ_i, $i = 1, \ldots, 5$ the roots of $q(\cdot)$, then $\frac{\alpha_1 i\xi_i + \alpha_3}{\alpha_2 i\xi_i + \alpha_4} = e^{-i\xi_i L}$. We can introduce the Möbius transformation $M(\xi_i) = e^{-iL\xi_i}$. From[8] we have

Lemma

- *Let* $\vec{\alpha} \in \mathbb{C}^4 \setminus \{0\}$ *with* $d(\vec{\alpha}) = \alpha_1\alpha_3 - \alpha_2\alpha_4 = 0$ *and* $L > 0$. *Then, the set of the imaginary parts of the zeros of* $N(\cdot, L)$ *has at most two elements.*
- *For* $L > 0$, *there is no Möbius transformation* M, *such that*

$$M(\xi) = e^{-iL\xi}, \quad \xi \in \{\xi_1, \xi_2, \overline{\xi_1}, \overline{\xi_2}\}, \qquad \xi_1, \xi_2, \overline{\xi_1}, \overline{\xi_2} \text{ different.}$$

It is enough to prove that the set of imaginary parts of zeros of $q(\cdot)$ has three elements.

[8] A. L. C. d. Santos et al., "Entire functions related to stationary solutions of the kawahara equation", (2016)

Some perspectives

Take $\alpha \neq N$ in the controllability results for the network?

Neumann case

$$\begin{cases} \lambda\psi + \psi' + \psi''' = 0, \\ \psi(L) = \psi'(L) = \psi'(0) = 0, \\ \psi''(0) = \dfrac{\alpha - N}{N}\psi(0), \end{cases} \qquad \begin{cases} \lambda\psi + \psi' + \psi''' = 0, \\ \psi(L) = \psi'(L) = \psi'(0) = 0, \\ \psi''(0) + \tau\psi(0) = 0. \end{cases} \longrightarrow \begin{cases} \tau = 0, \; L \notin \mathcal{N}_D, \\ \tau = 1, \; L \notin \mathcal{F}. \end{cases}$$

Recover $\mathcal{N}_D$ via Fourier? ($\tau = 0$).

$$i\hat{\psi}(\xi) = \frac{\alpha\xi^2 + \beta e^{-i\xi L}}{\xi^3 - \xi - p}, \qquad \frac{-\alpha\xi - \alpha p}{\beta\xi} = e^{-i\xi L}.$$

Möbius transform uniquely defined by the values $\xi_j \to e^{-i\xi_j L}$, $j = 0, 1, 2$.

Some perspectives

Take $\alpha \neq N$ in the controllability results for the network?

Neumann case

$$\begin{cases} \lambda\psi + \psi' + \psi''' = 0, \\ \psi(L) = \psi'(L) = \psi'(0) = 0, \\ \psi''(0) = \dfrac{\alpha - N}{N}\psi(0), \end{cases} \qquad \begin{cases} \lambda\psi + \psi' + \psi''' = 0, \\ \psi(L) = \psi'(L) = \psi'(0) = 0, \\ \psi''(0) + \tau\psi(0) = 0. \end{cases} \longrightarrow \begin{cases} \tau = 0, \ L \notin \mathcal{N}_D, \\ \tau = 1, \ L \notin \mathcal{F}. \end{cases}$$

Recover $\mathcal{N}_D$ via Fourier? ($\tau = 0$).

$$i\hat{\psi}(\xi) = \frac{\alpha\xi^2 + \beta e^{-i\xi L}}{\xi^3 - \xi - p}, \qquad \frac{-\alpha\xi - \alpha p}{\beta\xi} = e^{-i\xi L}.$$

Möbius transform uniquely defined by the values $\xi_j \to e^{-i\xi_j L}$, $j = 0, 1, 2$.

Some perspectives

Take $\alpha \neq N$ in the controllability results for the network?

Neumann case

$$\begin{cases} \lambda\psi + \psi' + \psi''' = 0, \\ \psi(L) = \psi'(L) = \psi'(0) = 0, \\ \psi''(0) = \dfrac{\alpha - N}{N}\psi(0), \end{cases} \qquad \begin{cases} \lambda\psi + \psi' + \psi''' = 0, \\ \psi(L) = \psi'(L) = \psi'(0) = 0, \\ \psi''(0) + \tau\psi(0) = 0. \end{cases} \longrightarrow \begin{cases} \tau = 0, \ L \notin \mathcal{N}_D, \\ \tau = 1, \ L \notin \mathcal{F}. \end{cases}$$

Recover $\mathcal{N}_D$ via Fourier? ($\tau = 0$).

$$i\hat{\psi}(\xi) = \frac{\alpha\xi^2 + \beta e^{-i\xi L}}{\xi^3 - \xi - p}, \qquad \frac{-\alpha\xi - \alpha p}{\beta\xi} = e^{-i\xi L}.$$

Möbius transform uniquely defined by the values $\xi_j \to e^{-i\xi_j L}$, $j = 0, 1, 2$.

Some perspectives

Relation with delay differential equation??

$$y^{(n)}(t) + \sum_{j=0}^{n-1} a_j y^{(j)}(t) + \sum_{j=0}^{n-1} \alpha_j y^{(j)}(t-\tau) = 0$$

Characteristic function

$$\Delta(\xi) = \xi^n + \sum_{j=0}^{n-1} a_j \xi^j + e^{-\tau\xi} \sum_{j=0}^{n-1} \alpha_j \xi^j, \qquad \xi \in \mathbb{C}$$

Asymptotic behavior related to roots of Δ.......

Some perspectives

Relation with delay differential equation??

$$y^{(n)}(t)+\sum_{j=0}^{n-1}a_j y^{(j)}(t)+\sum_{j=0}^{n-1}\alpha_j y^{(j)}(t-\tau)=0$$

Characteristic function

$$\Delta(\xi)=\xi^n+\sum_{j=0}^{n-1}a_j\xi^j+e^{-\tau\xi}\sum_{j=0}^{n-1}\alpha_j\xi^j, \qquad \xi\in\mathbb{C}$$

Asymptotic behavior related to roots of Δ.......

A funny situation

Consider the systems

$$\begin{cases} \partial_t y + (1+\beta)\partial_x y + \partial_x^3 y = 0 \\ y(t,0) = y(t,L) = 0, \\ \partial_x y(t,L) = h(t), \\ y(0,x) = y_0(x) \end{cases} \quad (A) \qquad \begin{cases} \partial_t y + (1+\beta)\partial_x y + \partial_x^3 y = 0 \\ \partial_x^2 y(t,0) = \partial_x^2 y(t,L) = 0, \\ \partial_x y(t,L) = h(t), \\ y(0,x) = y_0(x) \end{cases} \quad (B)$$

Critical lengths

- $\beta = 0$, (A) critical lengths $\mathcal{N}_N$
- $\beta = 0$, (B) critical lengths $\mathcal{N}_N \cup \{k\pi, k \in \mathbb{N}^*\}$.
- $\beta = -1$, (A) critical lengths $\emptyset$.
- $\beta = -1$, (B) critical lengths $\mathbb{R}_+$.

A funny situation

Consider the systems

$$\begin{cases} \partial_t y + (1+\beta)\partial_x y + \partial_x^3 y = 0 \\ y(t,0) = y(t,L) = 0, \\ \partial_x y(t,L) = h(t), \\ y(0,x) = y_0(x) \end{cases} \quad (A) \qquad \begin{cases} \partial_t y + (1+\beta)\partial_x y + \partial_x^3 y = 0 \\ \partial_x^2 y(t,0) = \partial_x^2 y(t,L) = 0, \\ \partial_x y(t,L) = h(t), \\ y(0,x) = y_0(x) \end{cases} \quad (B)$$

Critical lengths

- $\beta = 0$, (A) critical lengths $\mathcal{N}_N$
- $\beta = 0$, (B) critical lengths $\mathcal{N}_N \cup \{k\pi, k \in \mathbb{N}^*\}$.
- $\beta = -1$, (A) critical lengths $\emptyset$.
- $\beta = -1$, (B) critical lengths $\mathbb{R}_+$.

A funny situation

Consider the systems

$$\begin{cases} \partial_t y + (1+\beta)\partial_x y + \partial_x^3 y = 0 \\ y(t,0) = y(t,L) = 0, \\ \partial_x y(t,L) = h(t), \\ y(0,x) = y_0(x) \end{cases} \quad (A) \qquad \begin{cases} \partial_t y + (1+\beta)\partial_x y + \partial_x^3 y = 0 \\ \partial_x^2 y(t,0) = \partial_x^2 y(t,L) = 0, \\ \partial_x y(t,L) = h(t), \\ y(0,x) = y_0(x) \end{cases} \quad (B)$$

Critical lengths

- $\beta = 0$, (A) critical lengths $\mathcal{N}_N$
- $\beta = 0$, (B) critical lengths $\mathcal{N}_N \cup \{k\pi, k \in \mathbb{N}^*\}$.
- $\beta = -1$, (A) critical lengths $\emptyset$.
- $\beta = -1$, (B) critical lengths $\mathbb{R}_+$.

Thank you for your attention