An Insensitizing control problem involving tangential gradient terms for a reaction-diffusion equation with dynamic boundary conditions

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Joint work with

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X Partial differential equations, optimal design and numerics
Benasque, Spain.

Supported by Qualifica Grant QUAL21 005 USE.

August 22, 2024

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Outline

Let $\Omega \subset \mathbb{R}^d$ ($d = 2$ or $d = 3$) be a bounded domain with smooth boundary Γ. Consider the following problem

$$\begin{cases} L(y) + |y|^{p-1} y = f + \mathbb{1}_\omega v & \text{in } Q := \Omega \times (0,T), \\ L_\Gamma(y, y_\Gamma) + |y_\Gamma|^{q-1} y_\Gamma = f_\Gamma & \text{on } \Sigma := \partial\Omega \times (0,T), \\ y\big|_\Gamma = y_\Gamma & \text{on } \Sigma, \\ y(\cdot, 0) = y^0 + \tau_1 \hat{y}^0 & \text{in } \Omega, \\ y_\Gamma(\cdot, 0) = y_\Gamma^0 + \tau_2 \hat{y}_\Gamma^0 & \text{on } \Gamma \end{cases}$$

where

- $L(y) := \partial_t y - \Delta y + Ry$,
- $L_\Gamma(y, y_\Gamma) := \partial_t y_\Gamma + \partial_\nu y - \Delta_\Gamma y_\Gamma + R_\Gamma y_\Gamma$,
- ∂_ν is the outward normal derivative operator,
- Δ_Γ is the Laplace-Beltrami operator acting on Γ.

Given $\mathcal{O} \subseteq \Omega$ and $\mathcal{G} \subseteq \Gamma$ be two nonempty open subsets, we define the sentinel

$$\Phi(y, y_\Gamma) := \frac{1}{2}\int_0^T \int_{\mathcal{O}} |y(x,t;\tau_1,\tau_2,v)|^2 \, dx \, dt + \frac{1}{2}\int_0^T \int_{\mathcal{G}} |\nabla_\Gamma y_\Gamma(x,t;\tau_1,\tau_2,v)|^2 \, dS \, dt$$

Definition (Insensitizing controls)

We say that a control $v \in L^2(\omega \times (0,T))$ insensitizes Φ iff

$$\left.\frac{\partial \Phi}{\partial \tau_1}\right|_{\tau_1=\tau_2=0} = \left.\frac{\partial \Phi}{\partial \tau_2}\right|_{\tau_1=\tau_2=0} = 0 \quad \forall (\hat{y}^0, \hat{y}^0_\Gamma) \in L^2(\Omega) \times L^2(\Gamma),$$

with $\|(\hat{y}^0, \hat{y}^0_\Gamma)\|_{L^2(\Omega)\times L^2(\Gamma)} = 1$.

Insensitizing controls for parabolic equations

- J.-L. Lions (1990, 1992).
- O. Bodart, C. Fabre (1995).
- L. de Teresa (2000).
- O. Bodart, M. González-Burgos, R. Pérez-García. (2004).
- L. de Teresa, E. Zuazua (2009).

Heat equation with dynamic boundary conditions

- G. Goldstein (2005).
- A. Favini, G. Ruiz Goldstein, J. Goldstein, S. Romanelli (2006).
- C. Gal (2012, 2015).

Controllability of the heat equation with dynamic boundary conditions

- L. Maniar, M. Meyries, R. Schnaubelt (2017).
- L. Maniar et. al. (2020, 2021, 2022, 2023).
- R. Lecaros, R. Morales, A. Pérez, S. Zamorano (2023)

Insensitizing controls for heat equation with dbc

M. Zhang, J. Yin, H. Gao. (2019).

- For $1 \leq p \leq +\infty$, we define

$$\mathbb{L}^p := L^p(\Omega) \times L^p(\Gamma),$$

endowed by its natural norm.

- For $k \in \mathbb{N}$, we introduce the space

$$\mathbb{H}^k := \{(y, y_\Gamma) \in H^k(\Omega) \times H^k(\Gamma) \,:\, y\big|_\Gamma = y_\Gamma\},$$

where $H^k(\Omega)$ and $H^k(\Gamma)$ are the usual Sobolev spaces in Ω and Γ, respectively.

- For $T > 0$, we define

$$\mathbb{E} := H^1(0, T; \mathbb{L}^2) \cap L^2(0, T; \mathbb{H}^2)$$

and

$$\mathbb{D} := C^0([0, T]; \mathbb{H}^1) \cap L^2(0, T; \mathbb{H}^2).$$

Assumptions

(H1) $\omega \Subset \Omega$, $\mathcal{O} \subset \Omega$ are open sets s.t. $\omega \cap \mathcal{O} \neq \emptyset$ and $\mathcal{G} \subseteq \Gamma$ is an arbitrary open set.

(H2) p and q satisfy:

- If $d = 2$, then $\frac{5}{2} < p, q < +\infty$.
- If $d = 3$, then $\frac{5}{2} < p, q \leq 4$.

Theorem (Existence of insensitizing controls)

Assume that $d = 2$ or $d = 3$ and suppose that **(H1)** and **(H2)** hold. Moreover, $(y^0, y^0_\Gamma) = (0,0)$. Then, there exist constants $M_1 > 0$ and $\delta > 0$ such that for any $(f, f_\Gamma) \in L^2(0,T;\mathbb{L}^2)$ satisfying

$$\|e^{\frac{M_1}{t}}(f, f_\Gamma)\|_{L^2(0,T;\mathbb{L}^2)} \leq \delta,$$

there is a control $v \in L^2(Q_\omega)$ that *insensitizes* Φ.[a]

[a] M. C. Santos, N. Carreño, R. Morales. An Insensitizing control problem involving tangential gradient terms for a reaction-diffusion equation with dynamic boundary conditions. (2024) ArXiv:2407:09882.

Our problem is equivalent to find a control $v \in L^2(\omega \times (0,T))$ such that the solution of the coupled system

$$\begin{cases} L(y) + |y|^{p-1}y = f + \mathbb{1}_\omega v & \text{in } Q, \\ L_\Gamma(y, y_\Gamma) + |y_\Gamma|^{q-1}y_\Gamma = f_\Gamma & \text{on } \Sigma, \\ y\big|_\Gamma = y_\Gamma & \text{on } \Sigma, \\ (y(\cdot,0), y_\Gamma(\cdot,0)) = (y^0, y^0_\Gamma) & \text{in } \Omega \times \Gamma, \end{cases} \tag{1}$$

$$\begin{cases} L^*(z) + p|y|^{p-2}yz = \mathbb{1}_{\mathcal{O}} y & \text{in } Q, \\ L^*_\Gamma(z, z_\Gamma) + q|y_\Gamma|^{q-2}y_\Gamma z_\Gamma = \nabla_\Gamma \cdot (\mathbb{1}_{\mathcal{G}} \nabla_\Gamma y_\Gamma) & \text{on } \Sigma, \\ z\big|_\Gamma = z_\Gamma & \text{on } \Sigma, \\ (z(\cdot,T), z_\Gamma(\cdot,T)) = (0,0) & \text{in } \Omega \times \Gamma. \end{cases} \tag{2}$$

satisfies

$$(z, z_\Gamma)(\cdot,0) = (0,0) \quad \text{in } \Omega \times \Gamma.$$

To do this, we will use a local inverse mapping theorem in Banach spaces. Thus, we shall consider a linear system of the form

$$\begin{cases} L(y) = f^0 + \mathbb{1}_{\omega} v & \text{in } Q, \\ L(y, y_\Gamma) = f^0_\Gamma & \text{on } \Sigma, \\ y\big|_\Gamma = y_\Gamma & \text{on } \Sigma, \\ (y(\cdot,0), y_\Gamma(\cdot,0)) = (y^0, y^0_\Gamma) & \text{in } \Omega \times \Gamma, \end{cases}$$

$$\begin{cases} L^*(z) = f^1 + \mathbb{1}_{\mathcal{O}} y & \text{in } Q, \\ L^*_\Gamma(z, z_\Gamma) = f^1_\Gamma + \nabla_\Gamma \cdot (\mathbb{1}_{\mathcal{G}} \nabla_\Gamma y_\Gamma) & \text{on } \Sigma, \\ z\big|_\Gamma = z_\Gamma & \text{on } \Sigma, \\ (z(\cdot,T), z_\Gamma(\cdot,T)) = (0,0) & \text{in } \Omega \times \Gamma, \end{cases}$$

and study the null controllability of this system.

- By duality, the (relaxed) null controllability is equivalent to proving a suitable observability inequality for the corresponding adjoint system.
- Such observability inequality can be proved using Carleman estimates.
- With this estimate at hand, we can characterize the null controllability using classical methods.
- After that, using a suitable Local Inverse Mapping Theorem, we can deduce the existence of insensitizing controls for our problem.

We introduce the adjoint system

$$\begin{cases} L(\psi) = g^1 & \text{in } Q, \\ L_\Gamma(\psi, \psi_\Gamma) = g_\Gamma^1 & \text{on } \Sigma, \\ \psi\big|_\Gamma = \psi_\Gamma & \text{on } \Sigma, \\ (\psi(\cdot,0), \psi_\Gamma(\cdot,0)) = (\psi^0, \psi_\Gamma^0) & \text{in } \Omega \times \Gamma, \end{cases}$$

$$\begin{cases} L^*(\varphi) = g^0 + \mathbb{1}_{\mathcal{O}}\psi & \text{in } Q, \\ L_\Gamma^*(\varphi, \varphi_\Gamma) = g_\Gamma^0 + \nabla_\Gamma \cdot (\mathbb{1}_{\mathcal{G}} \nabla_\Gamma \psi_\Gamma) & \text{in } \Sigma \\ \varphi\big|_\Gamma = \varphi_\Gamma & \text{on } \Sigma, \\ (\varphi(\cdot,T), \varphi_\Gamma(\cdot,T)) = (0,0) & \text{in } \Omega \times \Gamma. \end{cases}$$

Then, the main task is to prove an observability inequality of the form

$$
\begin{aligned}
&\iint_Q \rho_1(|\varphi|^2 + |\nabla\varphi|^2 + |\psi|^2 + |\nabla\psi|^2)\,dx\,dt \\
&+ \iint_\Sigma \rho_2(|\varphi_\Gamma|^2 + |\nabla_\Gamma\varphi_\Gamma|^2 + |\psi_\Gamma|^2 + |\nabla_\Gamma\varphi|^2)\,dS\,dt \\
\leq & C\iint_Q \rho_3(|g^0|^2 + |g^1|^2)\,dx\,dt + C\iint_\Sigma \rho_4(|g^0_\Gamma|^2 + |g^1_\Gamma|^2)\,dS\,dt \\
&+ C\iint_{Q_\omega} \rho_5|\varphi|^2\,dx\,dt,
\end{aligned}
$$

where ρ_i $(i = 1, \ldots, 5)$ are weight functions which blow up at $t = 0$ and $C > 0$ only depends on $\Omega, \omega, \mathcal{O}, \mathcal{G}$ and T.

A sketch of the proof

- The functions ρ_i $(i = 1, \ldots, 5)$ are related to the Carleman estimates in $\mathbb{L}^2$ and $(\mathbb{H}^1)'$ obtained by L. Maniar et. al.[a] [b]
- The associated weight functions are based on those used for the heat equation with Dirichlet boundary conditions by Fursikov and Imanuvilov.
- We combine the estimates and absorb the lower-order terms. After that, we eliminate the singularities of the weight functions at $t = T$ and deduce the desired observability inequality.
- Thus, we can deduce a null controllability result for the linear controlled problem in a suitable space $\mathcal{C}_{p,q}$.

[a] Null controllability for parabolic equations with dynamic boundary conditions. *Evol. Equ. Control Theory*, 6(3):381-407, 2017 .

[b] I. Boutaayamou, S.-E. Chorfi, L. Maniar, and O. Oukdach. The cost of approximate controllability of heat equation with general dynamical boundary conditions. *Port. Math.*, 78(1):65-99, 2021

Consider the operator $\mathcal{A} : \mathcal{C}_{p,q} \to \mathcal{B}_2$ defined by

$$\begin{aligned}
&\mathcal{A}(y, y_\Gamma, z, z_\Gamma, v)\\
:= &(L(y) + |y|^{p-1}y - f - \mathbb{1}_\omega v, L_\Gamma(y, y_\Gamma) + |y_\Gamma|^{q-1}y_\Gamma - f_\Gamma,\\
&L^*(z) + p|y|^{p-2}yz - \mathbb{1}_\mathcal{O} y, L^*_\Gamma(z, z_\Gamma) + q|y_\Gamma|^{q-2}y_\Gamma z_\Gamma - \nabla_\Gamma \cdot (\mathbb{1}_\mathcal{G} \nabla_\Gamma y_\Gamma),\\
&y(\cdot, 0), y_\Gamma(\cdot, 0), z(\cdot, T), z_\Gamma(\cdot, T))
\end{aligned}$$

Then, it is sufficient to check the following two assertions:

(a) The mapping $\mathcal{A}'(0, 0, 0, 0, 0) : \mathcal{C}_{p,q} \to \mathcal{B}_2$ is surjective,
(Strongly related with the null controllability of the linear system.)

(b) $\mathcal{A}$ is an operator of class C^1 from $\mathcal{C}_{p,q}$ to $\mathcal{B}_2$.
We use the fact that $\mathbb{D} \hookrightarrow L^8(0, T; \mathbb{L}^8)$ continuously and **(H2)**.

Thank you for your attention!
Comments are welcome!

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