Trajectory controllability for heat equations with memory

Yubiao Zhang[1,2]

[1]Center for Applied Mathematics, Tianjin Univeristy
[2]Chair for Dynamics, Control, Machine Learning and Numerics,
Friedrich-Alexander-Universität Erlangen-Nürnberg

(joint work with Zhaohao Yu and Enrique Zuazua)

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Overview

Introduction

PDEs with memory involve the past values of solutions. A typical example is as:

$$\partial_{tt}y(t,x) - \Delta y(t,x) - \int_{-\infty}^{t} M(t-s)\Delta y(s,x)ds = 0,\ t>0,\ x\in\mathbb{R}^n.$$

There are many other related terminologies such as integro-differential equations, functional differential equations...

The notion of memory traces back to

- Boltzmann. Math.-Naturw. Kl (1874), Ann. Phys. (1878);
- Maxwell. Phil. Trans. Roy. Soc. London (1867);
- Volterra. Acta Math. (1912), Gauthier-Villars (1913).

Why memory?

In the modeling for many applications, it is natural that

(1) one physical quantity induces time-delayed actions on others,

(2) and a large number of such actions form memory (a time integral).

Two typical models

(1) The Coleman-Gurtin model:

$$\partial_t y(t,x) - \alpha A y(t,x) - \int_0^t M(t-s) A y(s,x) ds = 0,$$

where $\alpha > 0$ and A is a positive elliptic differential operator of second order;

(2) The Gurtin-Pipkin model:

$$\partial_t y(t,x) - \int_0^t M(t-s) A y(s,x) ds = 0.$$

Challenge and actions

- **Challenge:** When memory is added to a model, it's not so clear what new things appear. Models with memory are far from being well understood.

- **Actions needed:**

  (1) investigate new phenomena in a model with memory;

  (2) develop new mathematical methods for these phenomena;

  (3) understand how these methods enhance applications in control problems...

Model in consideration

We focus on the following model with simple memory: let $\Omega \subset \mathbb{R}^n$ be a bounded domain (with a C^2-boundary). Define

$$Af := \Delta f, \quad \text{with its domain} \;\; D(A) := H^2(\Omega) \cap H_0^1(\Omega).$$

Write $y(\cdot; y_0)$ for the solution of the following heat equation with memory:

$$\begin{cases} y'(t) - Ay(t) + \displaystyle\int_0^t y(s)ds = 0, \; t > 0, \\ y(0) = y_0 \in L^2(\Omega). \end{cases}$$

Formally, the model is **heat $+$ integral-type perturbation** .

New phenomenon

Let's first take a look at the simple case:

$$y'(t) - Ay(t) + \int_0^t y(s)ds = 0.$$

Write $\{\lambda_j\}_{j\geq 1}$ for the eigenvalues of $-A$. Apply the spectral method to get a parameterized ODE with memory:

$$w'(t) + \lambda_j w(t) + \int_0^t w(s)ds = 0.$$

Solve it to get

$$w(t) = O(1) \cdot \exp\left(-\lambda_j t\right) + O(\lambda_j^{-2}) \cdot \exp\left(-\lambda_j^{-1} t\right), \ \forall j \in \mathbb{N}^+.$$

Two exponential functions have different decay rates in the eigenvalues.

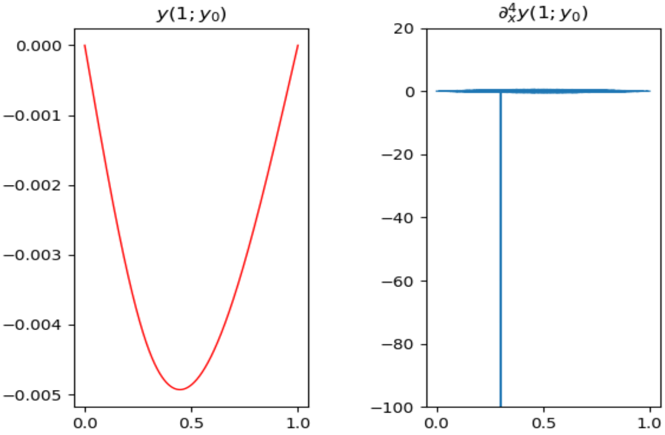
The first exponential function reflects “heat”, and what about the second one?

The solution and its fourth derivative:

$\Omega = (0,1)$,

$y_0 = \delta_{0.3}$,

$t = 1$.



Conjecture

Wave (with the null velocity)! Though it is not so intuitive.

We will explain later that the solutions, corresponding to the second exponential function, have the propagation of singularities along the time direction.

Hybrid parabolic–hyperbolic effect

Decomposition (Wang–Zhang–Zuazua, JMPA, 2022)

Each solution can be decomposed into two parts:

$$y(\cdot; y_0) = \text{parabolic component} + \text{hyperbolic component}.$$

Here, the parabolic and hyperbolic components are understood in the following way:

(1) the parabolic component $y_p(t; y_0)$ has the infinite–order smoothing effect, i.e.,

$$y_p(t; y_0) \in \cap_{k\in\mathbb{N}^+} D(A^k) \text{ for each } t > 0; \tag{1}$$

Hybrid parabolic–hyperbolic effect

(2) the hyperbolic component $y_h(t; y_0)$ has the leading term $-A^{-2}y_0$ and thus the propagation of singularities along the time direction. More precisely,

Propagation of singularities

$\forall s \in \mathbb{R}, \forall t_0 > 0, \forall x_0 \in \Omega,$

$$y_h(\cdot; y_0) \notin H^{s+4}_{loc}(t_0, x_0) \Leftrightarrow y_0 \notin H^s_{loc}(x_0).$$

Here, $H^s_{loc}(p)$ means the space of functions each of which is in $H^s(\mathcal{U}_p)$ for an open neighborhood U_p of the point p.

This allows us to simply call it the "wave" with the null velocity or "static wave".

Characteristic lines: $\{(x_0, t) \ : \ t \geq 0\}, x_0 \in \Omega.$

The original solution $y(\cdot; y_0)$ also has the same propagation of singularities.

Hybrid parabolic-hyperbolic effect

The hybrid property of the model here can be summarized as follows:

The model here behaves more like "heat" around the initial time, and more like "wave" at positive time.

(*) The results here can be extended to general memory kernels rather than the constant kernel.

Trajectory controllability

In what follows, χ_E denotes the characteristic function of a set E.

Take a time-varying measurable subset $Q \subset (0, +\infty) \times \Omega$ as the "control region". Write $y(\cdot; y_0, u)$ for the solution of the following heat equation with memory:

$$\begin{cases} y'(t) - Ay(t) + \int_0^t y(s)ds = \chi_Q u, \ t > 0, \\ y(0) = y_0 \in L^2(\Omega). \end{cases}$$

Here, $u \in L^\infty(\mathbb{R}^+; L^2(\Omega))$ is the control.

What controllability?

The classical null controllability has very limited values for the current model: one cannot expect that a solution will always remain in equilibrium under the null control once it reaches that state, i.e., $y(T; y_0, \chi_{(0,T)}u) = 0$ cannot guarantee

$$y(t; y_0, \chi_{(0,T)}u) = 0, \ \forall\, t \geq T.$$

Definition (Null trajectory controllability)

The system is said to have the null trajectory controllability after time T if $\forall\, y_0 \in L^2(\Omega)$, $\exists$ a control $u \in L^\infty(\mathbb{R}^+; L^2(\Omega))$ s.t.

$$y(t; y_0, u) = 0, \ \forall\, t \geq T.$$

Aim: characterize the geometry of such "control region" Q.

Related references

- Barbu and Iannelli. Differential Integral Equations. 13 (2000);
- Fu, Yong and Zhang. J. Differential Equations. 247 (2009);
- Ivanov and Pandolfi. J. Math. Anal. Appl. 355 (2009);

  Guerrero and Imanuvilov. ESAIM Control Optim. Calc. Var. 19 (2013);

  Gao and Zhou. Computers and Mathematics with Applications 67 (2014);
- Chaves-Silva, Zhang and Zuazua. SIAM J. Control Optim. 55 (2017). A kind of memory-type controllability is considered there;
- Pandolfi. arXiv: 1804.01865v1;
- Wang, Zhang and Zuazua. arXiv: 2101.10615v3.

  ...

Moving observation condition

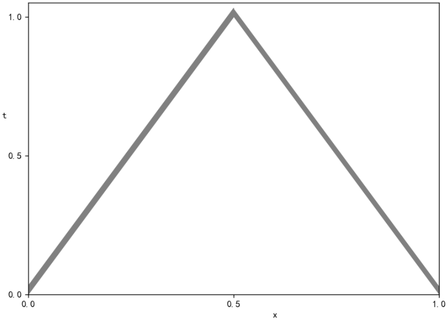
Definition (Wang-Zhang-Zuazua, arXiv, 2023)

Let $T > 0$. The pair (Q, T) is said to satisfy the moving observation condition (MOC for simplicity) if

$$\mathcal{T}_\Omega(Q, T) := \operatorname*{ess-inf}_{x\in\Omega} \int_0^T \chi_Q(t, x)dt > 0.$$

This concept is originally induced by the following paper

- F. Chaves-Silva, X. Zhang and E. Zuazua. Controllability of evolution equations with memory. SIAM J. Control Optim. 55 (2017) 2437-2459.



In plain language, the MOC says that each characteristic line goes through the observation set $Q \cap ((0, T) \times \Omega)$ within a lower limited elapsed time.

Null trajectory controllability

Theorem (Zhao-Zhang-Zuazua, being prepared)

Let $T > 0$. The following two statements are equivalent:

(i) (Q, T) *satisfies the MOC.*

(ii) *The system has the null trajectory controllability after time T.*

- The hyperbolic effect determines the geometry of the control region Q.
- The null trajectory controllability cannot hold for a large class of analytic memory kernels. More things need to be clarified.

More comments

- The theorem remains true with $L_t^p L_x^2$-controls ($1 < p \leq +\infty$).
- The MOC can ensure a kind of exact trajectory controllability for the same system, and can be used to characterize the structure of attainable sets of controlled trajectories.
- The results can extend to general autonomous parabolic equations with constant memory.

Main ideas: ONE

By the duality argument, the null trajectory controllability is equivalent to the following observability inequality:

$$\|\varphi(0; z, h)\|_{L^2(\Omega)} \lesssim \int_0^T \|\chi_Q(t,\cdot)\varphi(t; z, h)\|_{L^2(\Omega)} dt, \ \forall\, z, h \in L^2(\Omega) \tag{2}$$

for any solution $\varphi(\cdot; z, h)$ to the following dual equation

$$-\varphi'(t) - A\varphi(t) + \int_t^T \varphi(s) ds = h, \ t \in (0, T); \ \varphi(T) = z.$$

Challenges in the proof of (2):

- a nonhomogeneous term h in the dual equation;
- due to the hybrid parabolic-hyperbolic effect, the methods of observability for pure heat equations fail in the current situation.

Main ideas: TWO

Use the hybrid parabolic-hyperbolic effect: the following decomposition holds:

$$\varphi(\cdot; z, h) = \text{hyperbolic } \varphi_h + \text{parabolic } \varphi_p.$$

Recover the hyperbolic component φ_h from partial measurements, adapting the method of the observability for wave equations, then the parabolic component φ_p.

(*) Most challenge: decouple the parabolic and hyperbolic components from their linear superposition. This needs very delicate analysis!

Summary

(1) Memory is a huge perturbation and brings a heavy influence on the nature of the model.

(2) Memory is “static waves” (with the null velocity). A solution always remembers its past singularities.

(3) The hyperbolic nature of the memory can help study control problems such as the controllability/observability problems...

Thanks for your attention