(Lack of) Local controllability of a water tank controlled by acceleration

In collaboration with Jean-Michel Coron and Hoai-Minh Nguyen

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X Partial Differential Equations, Optimal Design and Numerics

Introduction

Small-time Local Controllability (around 0)
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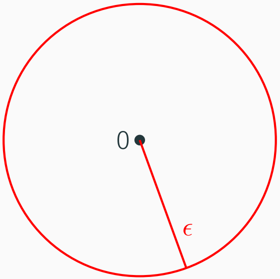

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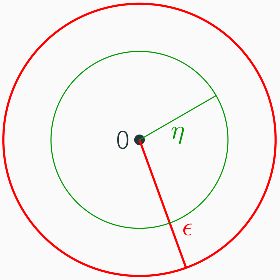

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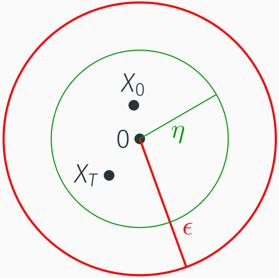
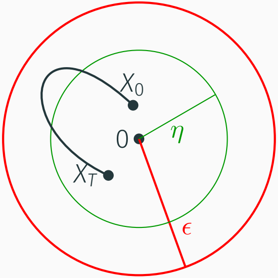
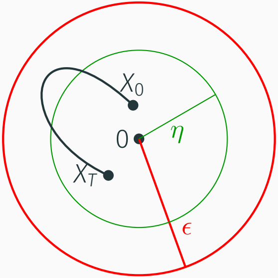




Small-time Local Controllability (around 0)
$\dot{X} = f(X, u)$ with $f(0,0) = 0$. For $\epsilon > 0$, does there exists $\eta > 0$ such that if $0 < T < \epsilon$, $|X_0| < \eta$, $|X_T| < \eta$, we can find $|u|_{L^\infty(0,T)} < \epsilon$ such that $X(T) = X_T$?



Small-time Local Controllability (around 0)
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Theorem (Linear test)
Small-time local controllability holds if the linearized equation is controllable.

Proof.

$$\dot{X} = L_1 X + L_2 u + NL(X,u)$$

$$\mathcal{F}: g \mapsto Y \text{ solution to } \dot{Y} = L_1 Y + g,\ \ Y(0) = 0$$

Banach fixed-point theorem to $(X,u) \mapsto (Y,v)$ where

$$\begin{cases} v := \text{Linear control}(X_0, X_1 - \mathcal{F} \circ NL(X,u)(T)) \\ Y := e^{TL_1} X_0 + \mathcal{F} \circ L_2 v + \mathcal{F} \circ NL(X,u) \end{cases}$$

□

A simple quadratic obstruction

$$\begin{cases} \dot{x}_1 = u \\ \dot{x}_2 = x_1^2 \end{cases}$$

$\dot{x}_2 \geq 0$: no controllability.

A quadratic obstruction in small time

$$\begin{cases} \dot{x}_1 = u \\ \dot{x}_2 = x_1 \\ \dot{x}_3 = x_1^2 - x_2^2 \end{cases}$$

If $x_2(0) = x_2(T) = 0$, $\int_0^T x_2^2 \leq (T/\pi)^2 \int_0^T \dot{x}_2^2$ (Poincaré). If T is small, $x_3(T) \geq x_3(0)$: no small-time controllabillity

Another small-time obstruction?

$$\begin{cases} \dot{x}_1 = u \\ \dot{x}_2 = x_1 \\ \dot{x}_3 = x_1^3 + x_2^2 \end{cases}$$

Small-time local controllability... but not if we ask $|u|_{W^{1,\infty}} \ll 1$!

[Beauchard-Marbach, Quadratic obstructions to small-time local controllability for scalar-input systems, 2018,...]

Outline

Previous examples of quadratic obstruction

Schrödinger equation

$$i\partial_t\psi(t,x) = -\partial_x^2\psi(t,x) + u(t)\mu(x)\psi(t,x), \quad x \in (0,1) \text{ with Dirchlet B.C.}$$

Theorem (Smallness of reachable space, Ball, Marsden & Slemrod 1982)

Let $\psi_0 \in L^2(0,1)$. The set

$$\{\psi(T,\cdot)\colon T>0,\ u \in L^2(0,T),\ \psi \text{ solution with } \psi(0,\cdot)=\psi_0\}$$

is contained in a countable union of compact subsets of $L^2(0,1)$.

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is contained in a countable union of compact subsets of $L^2(0,1)$.

Theorem (Local controllability in H^3 around the ground state, Beauchard & Laurent 2010)

$(\varphi_k)_k$ eigenfunctions of $-\partial_x^2$. If $|\langle \mu\varphi_1, \varphi_k\rangle_{L^2}| \geq ck^{-3}$, for every $T > 0$, for every ψ_0, ψ_1 with appropriate boundary conditions and

$$\|\psi_0 - \varphi_1\|_{H^3} + \|\psi_1 - e^{-i\lambda_1 T}\varphi_1\|_{H^3} \text{ small enough,}$$

there exists $u \in L^2(0,T)$ such that the associated solution satisfies $\psi(T,\cdot) = \psi_1$.

Proof.
Variant of the linear test □

Theorem (Quadratic obstruction for small-time local controllability, Coron, Beauchard, Morancey, Bournissou)

If $\langle \mu\varphi_1, \varphi_K \rangle = 0$, under some assumptions on μ, there exists $A > 0$, $T > 0$ and $\eta > 0$ such that for every u with $\|u\|_{H^3(0,T)} < \eta$,

$$\pm\Im\langle \psi(T), \varphi_K e^{-i\lambda_1 T}\rangle \geq A\|u_3\|_{L^2}^2 - C\|\psi(T) - \varphi_1 e^{-i\lambda_1 T}\|_{L^2}^2$$

where $u_0 = u$, $u_{k+1}(t) := \int_0^t u_k(s)\,\mathrm{d}s$.

Theorem (Small-time local controllability with oscillating controls, Bournissou 2022)

Under more assumptions on μ, the Schrödinger equation with bilinear controls is small-time locally controllable around $\varphi_1 e^{-i\lambda_1 T}$ with targets in $D((-\partial_x^2)^{11/2})$ and controls small in $H_0^2(0,T)$.

Proofs.
$\psi(t,x) = \varphi_1 e^{-i\lambda_1 T} + \psi_{\text{lin}}(u) + \psi_{\text{quad}}(u) + \psi_{\text{cub}}(u) + \text{error}.$ □

Theorem (Viscuous Burgers equation, Marbach 2018)

If $y(0,x) = 0$ *and*

$$\partial_t y(t,x) - \partial_x^2 y(t,x) + y(t,x)\partial_x y(t,x) = u(t), \quad x \in (0,1) \text{ with Dirichlet B.C.},$$

for some test function ρ, $T > 0$ *small enough, and* $u_1(t) := \int_0^t u(s)\,\mathrm{d}s$,

$$\langle \rho, y(T,\cdot)\rangle \geq k\|u_1\|_{H^{-1/4}}^2.$$

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Theorem (Nonlinear heat equation, Beauchard et Marbach 2020)

If $\langle \Gamma[0], \varphi_0\rangle = 0$, *under some assumptions on* $\Gamma \in C^2(H^1_{\mathrm{N}}; H^{-1}_{\mathrm{N}})$, *there exists* $A \neq 0$ *such that for every* $\epsilon > 0$, *there exist* $T > 0$ *and* $\eta > 0$ *such that for every* $\delta \in [-1,1]$ *and* $\|u\|_{H^{2n+2}} < \eta$, *if*

$$\partial_t z(t,x) - \partial_x^2 z(t,x) = u(t)\Gamma(z(t))(x), \quad x \in (0,1) \text{ with Neuman B.C.},$$

and $z(0) = \delta\varphi_0$ *and for* $j \geq 1$, $\langle z(T), \varphi_j\rangle \neq 0$,

$$\left|\langle z(T), \varphi_0\rangle - \delta + A\|u_n\|_{L^2}^2\right| \leq \epsilon(|\delta| + \|u_n\|_{L^2}^2).$$

where $u_0 = u$, $u_{k+1}(t) := \int_0^t u_k(s)\,\mathrm{d}s$.

KdV equation

$$\begin{cases} \partial_t y + \partial_x y + \partial_x^3 y + y\partial_x y = 0, & (t,x) \in (0,T) \times (0,L) \\ y(t,0) = y(t,L) = 0, \partial_x y(t,L) = u(t) & t \in (0,T) \end{cases}$$

KdV equation linearized around 0

$$\begin{cases} \partial_t y_1 + \partial_x y_1 + \partial_x^3 y_1 = 0, & (t,x) \in (0,T) \times (0,L) \\ y_1(t,0) = y_1(t,L) = 0, \partial_x y_1(t,L) = u(t) & t \in (0,T) \end{cases}$$

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Theorem (Rosier 1997)

The linearized KdV equation is controllable in some time (equivalently in arbitrarily small time) iff $L \notin \mathcal{N} := \left\{ 2\pi\sqrt{\frac{k^2+kl+l^2}{3}}, (k,l) \in (\mathbb{N}^*)^2 \right\}$.

If $L \in \mathcal{N}$*, there is some finite dimensional unreachable space* $\mathcal{M}$.

Control of the nonlinear equation

Theorem (Rosier 1997)

If $L \notin \mathcal{N}$, the nonlinear KdV equation is small-time locally controllable.

Theorem (Coron and Crépeau 2004)

If L can be written in a unique way as $L = 2\pi\sqrt{\frac{k^2+kl+l^2}{3}}$ and that $k = l$, the nonlinear KdV equation is small-time locally controllable.

Theorem (Cerpa 2007, Crépeau and Cerpa 2009)

If $L \in \mathcal{N}$, there exists $T > 0$ such that the nonlinear KdV equation is locally controllable in time T.

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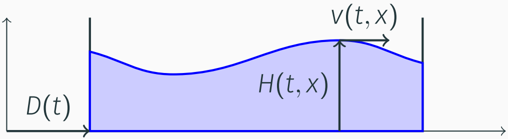
Theorem (Coron K Nguyen 2020)

If $k \neq l \in \mathbb{N}^*$, $L = 2\pi\sqrt{\frac{k^2+kl+l^2}{3}}$ and $2k + l \notin 3\mathbb{N}$, lack of small-time local controllable of the nonlinear KdV equation for H^3 initial conditions with controls small in $H^1(0, T)$.

Control of a Water-Tank

The water-tank system

$$\begin{cases} \partial_t H + \partial_x(vH) = 0, & (t,x) \in (0,T) \times (0,L) \\ \partial_t v + \partial_x(gH + v^2/2) = -u(t), & (t,x) \in (0,T) \times (0,L) \\ v(t,0) = v(t,L) = 0 & t \in (0,T) \\ \ddot{D}(t) = u(t) & t \in (0,T) \end{cases}$$



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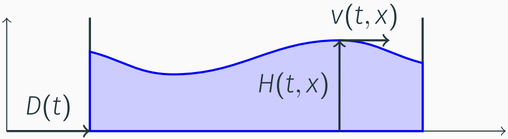

Linearized equation around $H = H_{eq}$, $v = 0$

$$\begin{cases} \partial_t h + H_{eq}\partial_x v = 0, & (t,x) \in (0,T) \times (0,L) \\ \partial_t v + g\partial_x h = -u(t), & (t,x) \in (0,T) \times (0,L) \\ v(t,0) = v(t,L) = 0 & t \in (0,T) \end{cases}$$

$h(t, L-x) = -h(t,x)$, $v(t, L-x) = v(t,x)$; not controllable. But moving the tank and such that the water is still at the start and end is possible if $T > T_* = L/\sqrt{gH_{eq}}$.

Theorem (Control using the return method, Coron 2002)

Local controllability in large time: there exists $T > 0$, $\eta > 0$ such that if

$$\begin{aligned} &\|H_0 - 1\|_{C^1} + \|v_0\|_{C^1} < \eta, \\ &\|H_1 - 1\|_{C^1} + \|v_1\|_{C^1} < \eta, \\ &\|D_1 - D_0\| < \eta \end{aligned}$$

then there exists a trajectory such that $H(t = 0) = H_0$, $H(t = T) = H_1$, $v(t = 0) = v_0$, $v(t = T) = v_1$, $D(0) = D_0$, $D(T) = D_1$, $\dot{D}(0) = \dot{D}(T) = 0$.

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Theorem (Lack of local controllability when the time is not large enough, Coron-K-Nguyen 2021)

For $T < 2T_*$, lack of local controllability with controls small in C^0: there exists $\eta > 0$ such that if $H(t=0) = H(t=T) = H_{\text{eq}}$, $v(t=0) = v(t=T) = 0$, $\dot{D}(0) = \dot{D}(T) = 0$, and if $\|u\|_{C^0} < \eta$, then $u = 0$.

Proof strategy: $(H, v) \approx$ linearized $+$ quadratic, and the quadratic term is $\geq c\|u\|^2_{H^{-1}}$.

Rescalling
$L = 1$, $H_{eq} = 1$, $g = 1$, $T_* = 1$.

Linearized equation

$$\partial_t h_1 + \partial_x v_1 = 0$$
$$\partial_t v_1 + \partial_x h_1 = -u(t)$$
$$v_1(t, 0) = v_1(t, 1) = 0$$

Rescalling
$L = 1$, $H_{\mathrm{eq}} = 1$, $g = 1$, $T_* = 1$.

Quadratic term

$$\partial_t h_2 + \partial_x v_2 = -\partial_x(h_1 v_1)$$
$$\partial_t v_2 + \partial_x h_2 = -\partial_x(v_1^2/2)$$
$$v_2(t,0) = v_2(t,1) = 0$$

Lemma

$$(h_2(T,\cdot),\phi) + (v_2(T,\cdot),\psi) = \int_{[0,T]^2} K_{T,\phi,\psi}(s_1,s_2)u(s_1)u(s_2)\,\mathrm{d}s_1\,\mathrm{d}s_2$$

for some explicitly computable kernel $K_{T,\phi,\psi}$.

Kernel for the Quadratic Approximation

Formula for the kernel (do not read)
With $\Phi(x) = (\phi(x) + \psi(x))/2$ for $0 < x < 1$ and $(\phi(-x) - \psi(-x))/2$ for $-1 < x < 0$,
$2K_{T,\phi,\psi}(s_1, s_2) =$

$$\begin{cases} \int_{-2T+2s_2}^{0} \Phi(s + T - s_2)\,\mathrm{d}s + 2(T - s_2)\Phi(T - s_2) - 4(T - s_2)\Phi(T - s_1) \\ \qquad \text{if } 2T - 1 < s_1 + s_2 < 2T \\ \int_{s_2 - s_1}^{2-2T+s_2+s_1} \Phi(s - s_2 + T)\,\mathrm{d}s + (4T - 1 - 3s_2 - s_1)\Phi(T - s_2) - (1 + 2T - 3s_2 + s_1)\Phi(T - s_1) \\ \qquad \text{if } 2T - 2 < s_1 + s_2 < 2T - 1 \\ \int_{2-2T+2s_2}^{0} \Phi(s + T - s_2)\,\mathrm{d}s + (1 + 2T - 2s_2)\Phi(T - s_2) - (-1 + 4T - 4s_2)\Phi(T - s_1) \\ \qquad \text{if } 2T - 3 < s_1 + s_2 < 2T - 2 \\ \int_{s_2 - s_1}^{4-2T+s_2+s_1} \Phi(s + T - s_2)\,\mathrm{d}s + (-2 + 4T - 3s_2 - s_1)\Phi(T - s_2) - (2 + 2T - 3s_2 + s_1)\Phi(T - \\ \qquad \text{if } 2T - 4 < s_1 + s_2 < 2T - 3 \end{cases}$$

Lemma

$\Phi(x) = (\phi(x) + \psi(x))/2$ for $0 < x < 1$ and $(\phi(-x) - \psi(-x))/2$ for $-1 < x < 0$. If $1 < T < 2$ and if the control u steers the linearized equation from 0 to 0 (apart from maybe moving the tank),

$$(h_2(T,\cdot),\phi) + (v_2(T,\cdot),\psi) = \int_{[0,T-1]^2} K^{\mathrm{red}}_{T,\phi,\psi}(s_1,s_2)u(s_1)u(s_2)\,\mathrm{d}s_1\,\mathrm{d}s_2$$

with

$$K^{\mathrm{red}}_{T,\phi,\psi}(s_1,s_2) = \frac{3}{2}(1-|s_2-s_1|)\left(\Phi(T-s_1\vee s_2) - \Phi(T-s_1\wedge s_2)\right)$$

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Proposition

Φ 1-periodic, $\Phi(s) = s$ for $s \in [1,T]$. For $1 < T < 2$ and $U(s) = \int_0^s u(s')\,\mathrm{d}s'$

$$(h_2(T,\cdot),\phi) + (v_2(T,\cdot),\psi) \geq 3(2-T)\|U\|^2_{L^2(0,T-1)}$$

End of the proof.

$$(h,v) \approx \underbrace{(h_1,v_1)}_{\text{linear in } u} + \underbrace{(h_2,v_2)}_{\text{quadratic in } u} \qquad \square$$

Conclusion

Quadratic obstruction for small-time local controllability

- Finite-dimensional systems
- Schrödinger equation with bilinear controls
- Viscous Burgers equation
- Some nonlinear heat equations
- KdV

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Water-tank

- A trajectory which is natural for the water-tank is possible for the linearized equation but not for the nonlinear equation.
- Minimal time for the local-controllability to hold?

That’s all folks!

Bonus: Coercivity of an arbitrary scalar product for the water tank

Question
Coercivity of Q_Ψ:

$$Q_\Psi(u) = \int_{[a,b]^2} u(s_1)u(s_2)(1+\epsilon|s_2-s_1|)\big(\Psi(s_1 \wedge s_2) - \Psi(s_1 \vee s_2)\big)\,\mathrm{d}s_1\,\mathrm{d}s_2?$$

(with $\Psi = -\Phi(T-s)$, $Q_\Psi = \langle\Phi,$ order 2 for the water-tank$\rangle$.)

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(with $\Psi = -\Phi(T-s)$, $Q_\Psi = \langle\Phi$, order 2 for the water-tank$\rangle$.)

Lemma

$\Psi \in C^1$, $\Psi' \geq c > 0$. *Then,*

$$Q_\Psi(U') \geq \alpha \|U\|^2_{L^2} \text{ for every } U \in H^1_0(a,b)$$

iff

$$\int_a^b \Psi'(s)\,\mathrm{d}s \int_a^b \frac{1}{\Psi'(s)}\,\mathrm{d}s < (b-a+2\epsilon^{-1})^2$$

Proof.
Integrate by parts; consider the resulting formula as a quadratic form on $L^2(\Psi'(s)\,\mathrm{d}s)$; see that on a stable space with codimension 2, $Q_\Psi = \mathrm{Identity}$; compute explicitly the 2×2 matrix on the orthogonal and study its positivity. □

Control of the KdV Equation

KdV equation

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KdV equation linearized around 0

$$\begin{cases} \partial_t y_1 + \partial_x y_1 + \partial_x^3 y_1 = 0, & (t,x) \in (0,T) \times (0,L) \\ y_1(t,0) = y_1(t,L) = 0, \partial_x y_1(t,L) = u(t) & t \in (0,T) \end{cases}$$

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If $L \in \mathcal{N}$*, there is some finite dimensional unreachable space* $\mathcal{M}$*.*

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If $L \notin \mathcal{N}$, the nonlinear KdV equation is small-time locally controllable.

Theorem (Coron and Crépeau 2004)

If L can be written in a unique way as $L = 2\pi\sqrt{\frac{k^2+kl+l^2}{3}}$ and that $k = l$, the nonlinear KdV equation is small-time locally controllable.

Theorem (Cerpa 2007, Crépeau and Cerpa 2009)

If $L \in \mathcal{N}$, there exists $T > 0$ such that the nonlinear KdV equation is locally controllable in time T.

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If $L \in \mathcal{N}$, there exists $T > 0$ such that the nonlinear KdV equation is locally controllable in time T.

Theorem (Coron K Nguyen 2020)

If $k \neq l \in \mathbb{N}^*$, $L = 2\pi\sqrt{\frac{k^2+kl+l^2}{3}}$ and $2k + l \notin 3\mathbb{N}$, lack of small-time local controllable of the nonlinear KdV equation for H^3 initial conditions with controls small in $H^1(0, T)$.

Order 2

$$\begin{cases} \partial_t y_1 + \partial_x y_1 + \partial_x^3 y_1 = 0, & (t,x) \in (0,T) \times (0,L) \\ y_1(t,0) = y_1(t,L) = 0,\ \partial_x y_1(t,L) = u(t) & t \in (0,T) \end{cases}$$

Order 2

$$\begin{cases} \partial_t y_2 + \partial_x y_2 + \partial_x^3 y_2 = -y_1 \partial_x y_1, & (t,x) \in (0,T) \times (0,L) \\ y_2(t,0) = y_2(t,L) = \partial_x y_2(t,L) = 0 & t \in (0,T) \end{cases}$$

Lemma

If $\dim(\mathcal{M}) = 2$*, we identify* $\mathcal{M} \approx \mathbb{C}$*, and then for some explicit* $p \in \mathbb{R}$ *and function* ϕ*.*

$$y_{2|\mathcal{M}}(t) = \int_0^L \int_0^t y_1(s,x)^2 e^{ip(t-s)} \phi(x) \, \mathrm{d}x \, \mathrm{d}s.$$

Theorem

If $L = 2\pi\sqrt{\frac{k^2+kl+l^2}{3}}$ with $2k + l \notin 3\mathbb{N}$, if T is small and if u steers y_1 from 0 to 0,

$$y_{2|\mathcal{M}} = \int_0^L \int_0^T y_1(s,x)^2 e^{ip(T-s)}\phi(x)\,\mathrm{d}x\,\mathrm{d}s = EN(u)^2(1 + O(T^{1/4}))$$

where $E \in \mathbb{C} \setminus \{0\}$ and $N(u) \sim \|u\|_{H^{-2/3}}$.

Proof.

- Take Fourier transform in t. For some explicitly computable function $\Lambda(x,z)$,
$$\hat{y}(z,x) = \hat{u}(z)\Lambda(z,x)$$
- Paley-Wiener: if, u steers the linearized equation from 0 to 0 then $\hat{u}$ and $\Lambda(\cdot,x)\hat{u}(\cdot)$ are entire and $|\hat{u}(z)| + |\hat{u}(z)\partial_x\Lambda(z,0)| \le Ce^{T|\Im(z)|}$.
- Computations $y_{2|\mathcal{M}} = \int \hat{u}(s)\overline{\hat{u}(s-p)}B(s)\,\mathrm{d}s, \quad B(s) \underset{s\to\pm\infty}{\sim} E|s|^{-4/3}$
- In the integral above, the part for $|s| \le m$ is $\le CmT^{1/2}\|u\|^2_{H^{-2/3}}$ (we use the Paley-Wiener property here). $\square$