Control of Waves on Time-Dependent Domains

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Boundary Control of Waves

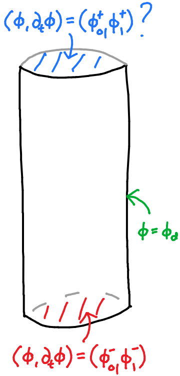
- Spatial domain: $\Omega \subseteq \mathbb{R}^n$ open, bounded.
- Control region: $\Gamma \subseteq (T_-, T_+) \times \partial\Omega$.
- Wave operators:

$$\Box := -\partial_t^2 + \Delta_x, \qquad \mathcal{L} := \Box + X(t,x) \cdot \nabla_{t,x} + V(t,x).$$

Problem (Dirichlet boundary controllability)

$\forall$ *initial/final data* $(\phi_0^\pm, \phi_1^\pm) \in L^2(\Omega) \times H^{-1}(\Omega)$:

- *Find boundary control* $\phi_d \in L^2(\Gamma)$*, such that solution of*
    - *wave equation:* $\mathcal{L}\phi|_{[T_-,T_+]\times\Omega} = 0$...
    - *initial data:* $(\phi, \partial_t\phi)|_{t=T_-} = (\phi_0^-, \phi_1^-)$...
    - *boundary data:* $\phi|_{(T_-,T_+)\times\partial\Omega} = \phi_d$...
- *satisfies* $(\phi, \partial_t\phi)|_{t=T_+} = (\phi_0^+, \phi_1^+)$.



Time-Dependent Domains

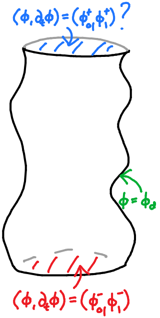
Consider now time-dependent domains:

$$\mathcal{U} = \bigcup_{T_- < \tau < T_+} (\{\tau\} \times \Omega_\tau).$$

- $\mathcal{U}$ has (timelike) moving boundary:

$$\mathcal{U}_b = \bigcup_{T_- < \tau < T_+} (\{\tau\} \times \partial\Omega_\tau).$$

- **Question.** Boundary control from $\Gamma \subseteq \mathcal{U}_b$?



By duality, controllability $\Leftrightarrow$ observability inequality:

$$\|(\psi, \partial_t\psi)(T_+)\|_{H^1 \times L^2} \lesssim \|\partial_\nu\psi\|_{L^2(\Gamma)},$$

$$\mathcal{L}_*\psi|_{\mathcal{U}} = 0, \qquad \psi|_{\mathcal{U}_b} = 0?$$

A Few Remarks

Remark. Some motivations:

1. Free boundary problems coupled to wave-type equations.
2. Geometric wave equations (non-stationary backgrounds), quasilinear waves.

Remark. No t-analyticity in $\mathcal{L}_*$ and in $\mathcal{U}$.

- Microlocal methods and GCC results do not apply.

Previous results for $\mathcal{L} = \Box$ and special $\mathcal{U}$:

- General n: (Bardos–Chen, Miranda)
- $n = 1$: (Cui–Jiang–Wang, Sun–Li–Lu, Wang–He–Li, ...)

Question. General $\mathcal{U}$, for all n, for arbitrary $\mathcal{L}$?

A Simplified Problem

For simplicity, restrict to $\Box\psi = 0$ ($\mathcal{L} := \Box$).

- Can apply mutiplier methods (and avoid Carleman estimates).

Recall standard results for static domains $(T_-, T_+) \times \Omega$:

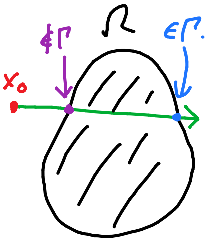
Theorem (Ho, Lions, ...)

Fix $x_0 \in \mathbb{R}^n$, and assume

$$T_+ - T_- > 2 \sup_{x \in \Omega} |x - x_0|.$$

Then, for $\mathcal{L} := \Box$, observability holds with:

$$\Gamma := (T_-, T_+) \times \{x \in \partial\Omega \mid (x - x_0) \cdot \nu > 0\}.$$



Classical Multiplier Argument

1. Integrate by parts, starting with

$$0 = \int_{[T_-, T_+] \times \Omega} \Box\psi \, \mathcal{S}^0_* \psi, \qquad \mathcal{S}^0_* \psi := (x - x_0) \cdot \nabla_x \psi + \frac{n-1}{2} \psi.$$

2. Combine with energy conservation $\Rightarrow$

$$(T_+ - T_-) \mathcal{E}(T_\pm) \leq 2R \cdot \mathcal{E}(T_\pm) + \frac{1}{2} \int_{(T_-, T_+) \times \partial\Omega} [(x - x_0) \cdot \nu] |\partial_\nu \psi|^2.$$

- $\mathcal{E}(t) = \frac{1}{2} \int_{\{t\} \times \Omega} |\nabla_{t,x} \psi|^2$.
- $R = \sup_{x \in \Omega} |x - x_0|$.

- Green $\Rightarrow$ $T_+ - T_- > 2R$.
- Orange $\Rightarrow$ $\Gamma := \{(x - x_0) \cdot \nu > 0\}$.

Remark. Does not work well for time-dependent domains.

Minkowski Geometry

Study $\Box$ via Minkowski (Lorentzian) geometry.

- Setting of special (general) relativity.

Euclidean geometry on $\mathbb{R}^n$:

$$d_E^2(x, \bar{x}) = \sum_{k=1}^{n} (x^k - \bar{x}^k)^2, \qquad g_E = (dx^1)^2 + \cdots + (dx^n)^2.$$

Minkowski geometry on $\mathbb{R}^{1+n}$:

$$d_M^2((t, x), (\bar{t}, \bar{x})) = -(t - \bar{t})^2 + \sum_{k=1}^{n} (x^k - \bar{x}^k)^2, \qquad g_M = -(dt)^2 + (dx^1)^2 + \cdots + (dx^n)^2.$$

Euclidean (Riemannian) concept $\Rightarrow$ Minkowski (Lorentzian) concept:

- Normal vectors, divergence theorem, IBP. (Curvature, exponential map.)

New Multiplier Argument I

1. Replace reference point x_0 by reference event (t_0, x_0).

2. Consider "square distance" $f := \frac{1}{4}[-(t - t_0)^2 + |x - x_0|^2]$.

3. Replace classical multiplier $\mathcal{S}_*^0\psi$ by

$$\mathcal{S}_*\psi := \underbrace{[(x - x_0) \cdot \nabla_x \psi + (t - t_0)\, \partial_t \psi]}_{2\cdot(\text{Mink. gradient of } f \text{ applied to } \psi)} + \tfrac{n-1}{2}\psi .$$

Apply (Minkowski) integration by parts in both t and x:

- Main difference: "flip sign" for all t-components.

New Multiplier Argument II

$$0 = \int_{\mathcal{U}} \Box\psi S_* \psi = \int_{\mathcal{U}} \Box\psi \underbrace{(\nabla^\alpha f \nabla_\alpha \psi)}_{\frac{1}{2}(\partial_t f \partial_t \psi + \nabla_x f \cdot \nabla_x \psi)} + \tfrac{n-1}{2} \int_{\mathcal{U}} \Box\psi \, \psi = I_1 + I_2 .$$

$$I_2 = -\tfrac{n-1}{4} \underbrace{\int_{\Omega_\tau}}_{\mathcal{U} \cap \{t = \tau\}} \partial_t \psi \, \psi \Big|_{\tau = T_-}^{\tau = T_+} - \tfrac{n-1}{4} \int_{\mathcal{U}} \underbrace{\nabla^\alpha \psi \nabla_\alpha \psi}_{-(\partial_t \psi)^2 + |\nabla_x \psi|^2} .$$

$$\begin{aligned} I_1 &= \int_{\Omega_\tau} [-\partial_t \psi (\nabla^\alpha f \nabla_\alpha \psi) + \partial_t f (\nabla^\alpha \psi \nabla_\alpha \psi)] \Big|_{\tau = T_-}^{\tau = T_+} + \tfrac{1}{2} \int_{\mathcal{U}_b} \underbrace{\mathcal{N} f (\mathcal{N} \psi)^2}_{\mathcal{N} := \text{Mink. normal to } \mathcal{U}_b} \\ &\quad + \int_{\mathcal{U}} [-\nabla^\alpha \psi \underbrace{\nabla_{\alpha\beta} f}_{\frac{1}{2}(g_M)_{\alpha\beta}} \nabla^\beta \psi + \tfrac{1}{2} \underbrace{\Box f}_{\frac{n+1}{2}} (\nabla^\alpha \psi \nabla_\alpha \psi)] \\ &= \int_{\Omega_\tau} [-\partial_t \psi (\nabla^\alpha f \nabla_\alpha \psi) + \partial_t f (\nabla^\alpha \psi \nabla_\alpha \psi)] \Big|_{\tau = T_-}^{\tau = T_+} + \tfrac{1}{2} \int_{\mathcal{U}_b} \mathcal{N} f (\mathcal{N} \psi)^2 + \tfrac{n-1}{4} \int_{\mathcal{U}} \nabla^\alpha \psi \nabla_\alpha \psi . \end{aligned}$$

New Multiplier Argument III

Combining the above, we get:

$$\int_{\Omega_\tau} \underbrace{\left[\partial_t \psi S_* \psi - \tfrac{1}{2} \partial_t f \nabla^\alpha \psi \nabla_\alpha \psi\right]}_{\frac{1}{4}(t-t_0)|\nabla_{t,x}\psi|^2 + \frac{1}{4}\partial_t\psi[\nabla^\alpha(|x-x_0|^2)\nabla_\alpha\psi - (n-1)\psi]} \Bigg|_{\tau=T_-}^{\tau=T_+} = \frac{1}{2} \int_{\underbrace{\mathcal{U}_b}_{\text{Or } \{\mathcal{N} f > 0\}}} \mathcal{N} f (\mathcal{N}\psi)^2 .$$

$$\left| \int_{\Omega_{T_\pm}} \tfrac{1}{4} \partial_t \psi [\nabla^\alpha (|x - x_0|^2) \nabla_\alpha \psi - (n-1)\psi] \right| \leq \frac{R_\pm}{4} \int_{\Omega_{T_\pm}} |\nabla_{t,x}\psi|^2 - \frac{n^2-1}{16 R_\pm} \int_{\Omega_{T_\pm}} \psi^2 .$$

$$R_\pm := \sup_{x \in \Omega_{T_\pm}} |x - x_0| .$$

$$\frac{T_+ - t_0}{2} \int_{\Omega_{T_+}} |\nabla_{t,x}\psi|^2 + \frac{t_0 - T_-}{2} \int_{\Omega_{T_-}} |\nabla_{t,x}\psi|^2 + \frac{n^2-1}{8} \left[\frac{1}{R_+} \int_{\Omega_{T_+}} \psi^2 + \frac{1}{R_-} \int_{\Omega_{T_-}} \psi^2 \right]$$
$$\leq \frac{R_+}{2} \int_{\Omega_{T_+}} |\nabla_{t,x}\psi|^2 + \frac{R_-}{2} \int_{\Omega_{T_-}} |\nabla_{t,x}\psi|^2 + \int_{\mathcal{U}_b \cap \{\mathcal{N} f > 0\}} \mathcal{N} f (\mathcal{N}\psi)^2 .$$

- Can absorb RHS into LHS if $T_+ - t_0 > R_+$ and $t_0 - T_- > R_-$.

The Multiplier Result

Theorem (S., 2019)

Consider $\mathcal{L} := \Box$ on moving domain $\mathcal{U}$. Assume

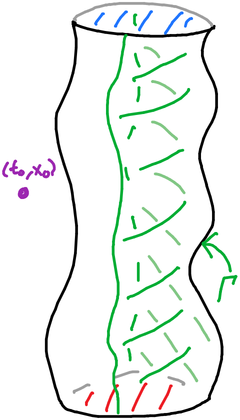
$$T_+ - T_- > R_+ + R_-, \qquad R_\pm := \sup_{x \in \Omega_{T_\pm}} |x - x_0|.$$

Then, we have observability:

$$\|(\psi, \partial_t \psi)(T_\pm)\|^2_{H^1 \times L^2} \lesssim \int_{\mathcal{U}_b \cap \{\mathcal{N} f > 0\}} |\mathcal{N} \psi|^2.$$

Remark. t_0 chosen s.t.

$$T_+ - t_0 > R_+, \qquad t_0 - T_- > R_-.$$



Some Remarks

Can recover all existing results for non-static domains.

- Attains GCC (optimal $T_+ - T_-$) when $n = 1$.
- Improves $T_+ - T_-$ when $n > 1$.
- $\mathcal{U} = (T_-, T_+) \times \Omega \Rightarrow$ classical multiplier result.

Remark. New term $(t - t_0)\partial_t \psi$ in multiplier $\Rightarrow$ energy estimate.

Control region $\{\mathcal{N} f > 0\}$ generalises $\{(x - x_0) \cdot \nu > 0\}$:

$\mathcal{U}$ time-independent	$\mathcal{N} f_0 > 0 \Leftrightarrow (x - x_0) \cdot \nu > 0$
$\mathcal{U}$ "expanding" from t_0	Need smaller Γ
$\mathcal{U}$ "contracting" from t_0	Need larger Γ

General Wave Operators

What about general $\mathcal{L} = \Box + X \cdot \nabla + V$?

- Similar results, using new geometric Carleman estimates.
- Carleman weight "given by $f^a e^{\lambda f^{1/2}}$".

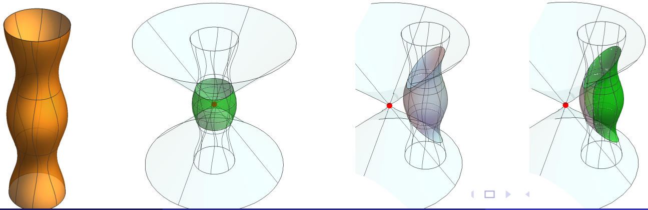
Theorem (S., 2019)

Multiplier observability estimate extends to general $\mathcal{L}$.

- *Can further improve observation region (almost) to*

$$\Gamma = \mathcal{U}_b \cap \{\mathcal{N} f > 0\} \cap \{f > 0\}.$$

- (t_0, x_0) *can be inside or outside of* $\mathcal{U}$.



Further Extensions

Additional results via similar methods:

- (Jena, 2021, 2022) Interior control for $\mathcal{L}$.
- (Fu–Liao, 2021; Fu–Liao–Lü, 2023) Some time-independent metrics.
- (S.–Jena, 2024) Some (time-dependent) Lorentzian manifolds.