How much does it cost to control near a shock?

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Plan of the talk

1. Shock profiles for Burgers' equation.
2. The control problem.
3. Main result and sketch of proof

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Shocks for inviscid Burgers' equation

Consider the 1D inviscid Burgers' equation

$$\begin{cases} \partial_t u + u \partial_x u = 0, & t \in \mathbb{R}^+, x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}. \end{cases} \tag{1}$$

By method of characteristics, jump discontinuities arise in finite time unless u_0 is non-decreasing.

Rankine-Hugoniot condition : A jump from u^- to u^+ $(u^- > u^+)$ is displaced at velocity

$$\text{shock speed} = \frac{u^- + u^+}{2}.$$

Family of stationary shocks

We restrict the study of (1) to a finite interval $[-L, L]$ and add Dirichlet conditions :

$$\begin{cases} \partial_t u + u\partial_x u = 0, & t \in \mathbb{R}^+, x \in (-L, L), \\ u(t, -L) = 1, & t \in \mathbb{R}^+, \\ u(t, L) = -1, & t \in \mathbb{R}^+, \end{cases} \tag{2}$$

There is an infinite family of stationary shocks given by

$$U_{x_0}(x) = \begin{cases} 1 & \text{if } x < x_0 \\ -1 & \text{if } x > x_0 \end{cases}, \quad x_0 \in (-L, L).$$

Any solution to (2) with initial datum of bounded variation converges to some U_{x_0}.

Stationary shock profiles for viscous Burgers.

We add a diffusive term $\varepsilon\partial_{xx}$ to the Burgers' equation for a regularizing effect.

$$\begin{cases} \partial_t u + u\partial_x u = \varepsilon\partial_{xx}u, & t \in \mathbb{R}^+, x \in (-L, L), \\ u(t, -L) = 1, & t \in \mathbb{R}^+ \\ u(t, L) = -1, & t \in \mathbb{R}^+. \end{cases} \tag{3}$$

A unique stationary solution exists, given by

$$U^\varepsilon(x) = -c \tanh\left(\frac{cx}{2\varepsilon}\right),$$

where $c > 0$ is in order to verify the boundary conditions.

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Uniform controllability problem.

We consider the problem of boundary null-controllability of Burgers equation linearized around the stationary shock.

$$\begin{cases} \partial_t u + \partial_x (U^\varepsilon u) = \varepsilon \partial_{xx} u, & t \in [0,T], x \in (-L,L), \\ u(t,-L) = h(t), & t \in [0,T], \\ u(t,L) = 0, & t \in [0,T], \\ u(0,x) = u_0(x), & x \in [-L,L], \end{cases} \tag{4}$$

with $U^\varepsilon(x) = -\tanh\left(\frac{x}{2\varepsilon}\right)$.

Uniform controllability

We want to find T_{unif} such that, for any $T > T_{\mathrm{unif}}$,

$$\exists \mathcal{C}, \forall \varepsilon > 0, \forall u_0 \in H^2 \cap H_0^1, \exists h \in L^2(0,T) :$$
$$u(T) \equiv 0 \text{ and } \|h\|_{L^2(0,T)} \leq \mathcal{C} \|u_0\|_{L^2(-L,L)}$$

Dual observability problem.

By standard duality argument, this is equivalent to the uniform observability of the adjoint system

$$\begin{cases} \partial_t v - U^\varepsilon \partial_x v = \varepsilon \partial_{xx} v, & t \in [0,T], x \in (-L,L) \\ v(t,-L) = v(t,L) = 0, & t \in [0,T] \\ v(0,x) = v_0(x), \quad x \in [-L,L] \end{cases} \tag{5}$$

Namely we want to investigate the minimal time T_{unif} such that, for any $T > T_{\mathrm{unif}}$,

$$\exists \mathcal{C} = \mathcal{C}(T) > 0, \forall \varepsilon > 0, \forall v_0 \in H^2 \cap H_0^1,$$
$$\mathcal{C}^2 \int_0^T |\varepsilon \partial_x v(t,-L)|^2 \,\mathrm{d}t \geq \|v(T,\cdot)\|^2_{L^2(-L,L)}$$

Related problems

Coron-Guerrero conjecture ('05) :

The uniform controllability of the transport equation

$$\partial_t y + M\partial_x y = \varepsilon\partial_{xx} y$$

in the vanishing viscosity limit is possible whenever $T > \frac{L}{M}$ if $M > 0$, or $T > \frac{2L}{|M|}$ if $M < 0$.

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Results in this sense :

	$M > 0$	$M < 0$
Coron,Guerrero ('05)	$T > \frac{4.3L}{M}$	$T > \frac{57.2L}{\lvert M\rvert}$
Glass ('09)	$T > \frac{4.2L}{M}$	$T > \frac{6.1L}{\lvert M\rvert}$
Lissy ('12)	$T > \frac{2.35L}{M}$	$T > \frac{4.35L}{\lvert M\rvert}$

Laurent, Léautaud ('23) : Similar results generalized to a wider class of problems of the form

$$\partial_t y + a\partial_x y + by = \varepsilon\partial_{xx} y,$$

under some regularity assumptions on a, b.

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Spectral analysis of $\mathcal{L}^\varepsilon = -U^\varepsilon \partial_x - \varepsilon \partial_{xx}$

Proposition (Mascia-Strani, 2012)

The spectrum of the operator $\mathcal{L}^\varepsilon$ consists of simple eigenvalues $\{\lambda_0 < \lambda_1 < \cdots\}$ such that :

$$\exists C > 0 : 0 < \lambda_0 < Ce^{-\frac{C}{\varepsilon}}, \quad \lambda_1 \geq \frac{C}{\varepsilon}$$

Two main issues compared with the constant transport case :

- A small eigenvalue, caracteristic of a metastability phenomenon.
- Lack of precise knowledge regarding the higher eigenvalues.

Result based on the conjugation of the operator :

$$\text{sech}\left(\frac{x}{2\varepsilon}\right) \mathcal{L}^\varepsilon \cosh\left(\frac{x}{2\varepsilon}\right) = \frac{1}{\varepsilon}\left(-\varepsilon^2 \partial_{xx} - \frac{1}{4} + \frac{1}{2}\tanh^2\left(\frac{x}{2\varepsilon}\right)\right) =: \frac{1}{\varepsilon} P^\varepsilon$$

Spectral analysis of P^ε

Schrödinger operator with a single-well symmetric potential $\implies$ good estimates for the bottom of the spectrum, loss of precision far from the well.

We are saved by the factorisation

$$P^\varepsilon = a^* a,$$

where $a = \varepsilon \partial_x + \frac{1}{2} \tanh\left(\frac{x}{2\varepsilon}\right)$.
We may compute

$$aa^* = -\varepsilon^2 \partial_{xx} + \frac{1}{4},$$

which is much simpler, and the eigenvalues of aa^* and a^*a are similar :

$$aa^*\varphi = \lambda\varphi \iff a^*a(a^*\varphi) = \lambda(a^*\varphi),$$
$$a^*a\psi = \mu\psi \iff aa^*(a\psi) = \mu(a\psi)$$

Spectral analysis of P^ε.

Solving the eigenvalue problem

$$\begin{cases} P^\varepsilon \varphi = \lambda \varphi, \\ \varphi(\pm L) = 0, \end{cases}$$

is equivalent to solving

$$\begin{cases} (-\varepsilon^2 \partial_{xx} + \frac{1}{4})\psi = \lambda \psi \\ a^* \psi(\pm L) = 0. \end{cases}$$

Proposition

The spectrum of $\mathcal{L}^\varepsilon$ consists of simple eigenvalues $\{\lambda_0 < \lambda_1 < \cdots\}$ that verify

$$\begin{aligned} &0 < \lambda_0 < Ce^{-\frac{C}{\varepsilon}}, \\ &\frac{1}{4\varepsilon} + \varepsilon \frac{k^2\pi^2}{4L^2} < \lambda_k < \frac{1}{4\varepsilon} + \varepsilon \frac{(k+1)^2\pi^2}{4L^2}, \quad k \geq 1 \\ &\sqrt{\lambda_{k+1} - \frac{1}{4\varepsilon}} - \sqrt{\lambda_k - \frac{1}{4\varepsilon}} > \sqrt{\varepsilon}\frac{\pi}{2L}, \quad k \geq 1 \end{aligned}$$

The main result

Theorem (L. 2024) [WIP].

There exists a time

$$T_{\mathrm{unif}} = 8.4L$$

such that, whenever $T > T_{\mathrm{unif},}$ the system

$$\begin{cases} \partial_t u + (\mathcal{L}^\varepsilon)^* u = 0 \\ \qquad u(t,-L) = h(t), \quad u(t,L) = 0, \\ \qquad u(0,x) = u_0(x) \end{cases}$$

is uniformly controllable **to the first mode** in the vanishing viscosity limit. Namely, for $T > T_{\mathrm{unif}}$, there exists a control h uniformly bounded with respect to ε such that

$$\left\langle \mathrm{sech}\left(\frac{x}{2\varepsilon}\right) u(T), \varphi_k \right\rangle = 0, \quad k \geq 1.$$

Moreover the controllability cost verifies

$$\mathcal{C}(T,\varepsilon) \leq c e^{-\frac{c}{\varepsilon}}.$$

Construction of a biorthogonal family

The first step is to construct a bi-orthogonal family to the $\left(e^{-\lambda_k t}\right)_{k\geq 1}$ with suitable estimates, namely a family $(q_j)_{j\geq 1} \subset L^2(0,T)$ such that

$$\int_0^T q_j(t) e^{-\lambda_k t}\,\mathrm{d}t = \delta_{j,k}$$

Estimates on λ_k allow us to build such a family with the bound

$$\|q_k\| \leq C \exp\left(f(L, \varepsilon, \lambda_k, T)\right)$$

where f is such that

$$-\lambda_k T + f(L, \varepsilon, \lambda_k, T) \leq -\alpha\lambda_k, \quad \alpha > 0 \tag{6}$$

for $T > 8.4L$.

Conclusion by method of moments.

Then, take any initial datum v_0 of the form

$$v_0(x) = \sum_{k\geq 1} c_k \psi_k,$$

where $\psi_k := \cosh\left(\frac{x}{2\varepsilon}\right)\varphi_k$ are the eigenvectors of $\mathcal{L}^\varepsilon$. The solution at time t is immediately given by

$$v(t,x) = \sum_{k\geq 1} c_k \psi_k e^{-\lambda_k t}$$

By taking the space derivative in $-L$ and integrating in time against q_j, we obtain

$$\int_0^T \varepsilon \partial_x v(t,-L) q_j(t)\,\mathrm{d}t = c_j \varepsilon \psi_j'(-L).$$

Conclusion by method of moments.

It follows, for $j \geq 1$,

$$|c_j| \leq \frac{\|\varepsilon\partial_x v(\cdot, -L)\| \|q_j\|}{\varepsilon\psi_k'(-L)}$$

Finally, we plug the latter estimate into $v(T)$:

$$\begin{aligned} \|v(T,\cdot)\| &= \left\| \sum_{k\geq 1} c_k \psi_k e^{\lambda_k T} \right\| \\ &\leq \|\varepsilon\partial_x v(\cdot, -L)\| \sum_{k\geq 1} e^{-\lambda_k T} \frac{\|q_k\| \cdot \|\psi_k\|}{\varepsilon\psi_k'(-L)} \end{aligned}$$

Using (6) finally gives

$$\|v(T,\cdot)\| \leq \|\varepsilon\partial_x v(\cdot, -L)\| \sum_{k\geq 1} e^{-\alpha\lambda_k} \cdot \frac{\|\psi_k\|}{\varepsilon\psi_k'(-L)},$$

and we conclude by having a precise description of the eigenfunctions ψ_k.

Work in progress and possible leads.

- Refine the constant 8.4 by improving the estimates on the bi-orthogonal family.
- Possibly obtain a time $T_{\min}$ for which we can prove the cost of controllability to the first mode explodes as $\varepsilon \to 0$ whenever $T < T_{\min}$.
- Prove non-uniform controllability for the full system for any finite time T.

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Thank you for your attention !