Eigenvalue bounds for the Gramian operator of the heat equation

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ConDuction

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The problem framework

We analyse the spectral properties of the Gramian operator associated to the heat equation

$$\begin{cases} y_t - \Delta y = v 1_\omega & \text{in} \quad \Omega \times (0,\infty), \\ y = 0 & \text{on} \quad \partial\Omega \times (0,\infty) \\ y(0) = y_0 & \text{in} \quad \Omega. \end{cases}$$

The (infinite-time) Gramian is given by

$$X_\omega = \int_0^\infty \exp(\Delta t) 1_\omega \exp(\Delta t) dt. \tag{1}$$

and is the solution of the Lyapunov equation

$$\Delta X_\omega + X_\omega \Delta = -1_\omega.$$

The research was motivated by

L. Grubisić and D. Kressner, *On the eigenvalue decay of solutions to operator Lyapunov equations*, Systems & Control Letters, 73 (2014) 42–47.

M. Opmeer, *Decay of singular values of the Gramians of infinite-dimensional systems*, in Proceedings 2015 European Control Conference (ECC), Linz, 1183–1188.

They considered a general setting

$$AX + XA^* = -BB^*.$$

Characterisation of the Gramian

$$< X_\omega p_0, p_0 >= \int_0^\infty \int_\omega p^2 dxdt$$

where p is the solution of the heat eqwuation in the infinite-time horizon

$$\begin{cases} p_t - \Delta p = 0 & \text{in} \quad \Omega \times (0, \infty) \\ p = 0 & \text{on} \quad \partial\Omega \times (0, \infty) \\ p(0) = p_0 \in L^2(\Omega) & \text{in} \quad \Omega. \end{cases}$$

Lemma

The Gramian operator X_ω is a compact, self-adjoint operator on $L^2(\Omega)$. In particular, it allows for the spectral decomposition. Its eigenvalues μ_k constitute a sequence of positive real numbers accumulating at zero and the corresponding eigenfunctions ψ_k form an orthonormal basis of $L^2(\Omega)$.

(μ_k, ψ_k) – eigenpairs of X_ω

Clasical observability estimates

$$\sum_{j\geq 1} \exp(-a\lambda_j)|\hat{p}_{0,j}|^2 \leq C \int_0^T \int_\omega p^2 dxdt, \tag{2}$$

(λ_j, ϕ_j) – eigenpairs of $-\Delta$,
$\hat{p}_{0,j} = \langle \, p_0 \mid \phi_j \, \rangle$ - Fourier coefficients of p_0.

Improved ones

$$\sum_{j\geq 1} \exp(-a\sqrt{\lambda_j})|\hat{p}_{0,j}|^2 \leq C \int_0^T \int_\omega p^2 dxdt, \tag{2}$$

E. Fernández-Cara and E. Zuazua, *The cost of approximate controllability for heat equations: The linear case*, Advances in Differential Equations, 5(4-6) (2000) 465—514.

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The energy estimate:

$$\int_0^\infty\int_\Omega p^2 dxdt = \frac{1}{2}||p_0||^2_{H^{-1}(\Omega)}.$$

By combining the last estimate with (2), we get the following, two-sided bounds on the observed energy in the infinite time horizon

$$\frac{1}{C}\sum_{j\geq 1} \exp(-a\sqrt{\lambda_j})|\hat{p}_{0,j}|^2 \leq \int_0^\infty\int_\omega p^2 dxdt \leq \sum_{j\geq 1}\frac{1}{2\lambda_j}|\hat{p}_{0,j}|^2$$

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$$\sum_{j\geq 1} \exp(-a\sqrt{\lambda_j})|\hat{p}_{0,j}|^2 \leq C\int_0^T\int_\omega p^2\,dxdt, \qquad (2)$$

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Preliminary bounds on the eigenvalue decay for the Gramian

Proposition

There exist positive constants a and $\tilde{C}_1$ such that

$$\tilde{C}_1 \exp(-a\sqrt{\lambda_k}) \leq \mu_k \leq \frac{1}{2\lambda_k},$$

μ_k being the eigenvalues of the Gramian X_ω.

Proof based on min-max (and max-min) characterization of the eigenvalues of symmetric operators:

$$\mu_k = \min_{\substack{\dim E = k-1 \\ E \subseteq L^2(\Omega)}} \left[\max \left[R_\omega(p_0) : p_0 \in E^\perp \setminus \{0\} \right] \right],$$

where R_ω stands for the Rayleigh quotient defined as the ratio of the observed energy and the initial one, i.e.

$$R_\omega(p_0) = \frac{< X_\omega p_0, p_0 >}{||p_0||^2_{L^2(\Omega)}} = \frac{\int_0^\infty \int_\omega p^2 dxdt}{\sum_{j\geq 1} |\hat{p}_{0,j}|^2}.$$

Numerical example

$\omega = [0, \pi/2]$, $\Omega = [0, \pi]$.
Finite dimensional approximation of the Gramian

$$X_N := P_{V_N} \circ (X_\omega)_{|V_N},$$

where $V_N = [\phi_1, \ldots, \phi_N] \subset L^2(\Omega)$ is the space spanned by first N eigenvectors of the Dirichlet Laplacian, P_{V_N} is the orthogonal projection from $L^2(\Omega)$ onto V_N, while $(X_\omega)_{|V_N}$ is the restriction of the Gramian to V_N.

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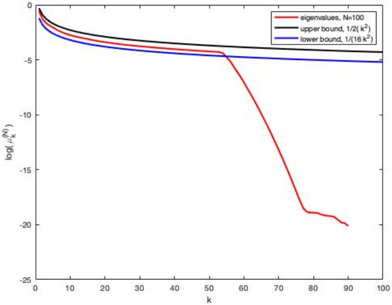
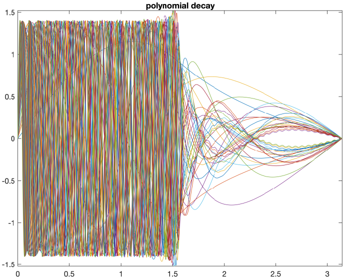
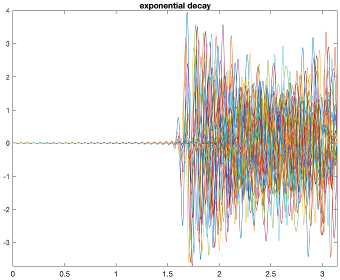


Figure: Eigenvalues decay for the finite dimensional approximation of the Gramian X_N for $N = 100$. Polynomial branch of the spectrum of X for $k \leq 55$, exponential branch for $55 < k < 80$ and irrelevant branch for $k \geq 80$.



(a)



(b)

Figure: Eigenvectors of the Gramian X_N ($N = 100$) corresponding to (a) polynomially and (b) exponentially decaying eigenvalues.

The largest eigenvalues try to maximize the Rayleigh quotient:

$$\mu_1^{(N)} = \max_{p_0 \in V_N} \frac{\int_0^\infty \int_\omega p^2 \, dx dt}{\|p_0\|^2_{L^2(\Omega)}}, \tag{3}$$

where p is the solution to the adjoint equation (3) with the initial datum p_0.

Lemma

(Improved observability estimates)
Let p be the solution to the heat equation (3) with initial datum $p_0 \in H^{-1}(\Omega)$ supported in a compact set $K \subset \omega$. Then for every $T \in \langle 0, \infty]$ there exists constants c_T, C_T independent of p_0, such that the following estimates hold:

1. $$\int_0^T \int_{\omega^c} p^2 dxdt \leq C_T ||p_0||^2_{H^{-2}(\Omega)},$$
2. $$\int_0^T \int_{\omega} p^2 dxdt \geq c_T ||p_0||^2_{H^{-1}(\Omega)}. \quad (4)$$

Proof:

1. – Multiplication by cut-off function supported on ω^c
   – Regularity and energy results for the solution to HE.
2. $$c_T ||p_0||^2_{H^{-1}(\Omega)} \leq \int_0^T \int_{\Omega} p^2 dxdt \leq \int_0^T \int_{\omega} p^2 dxdt + ||p_0||^2_{H^{-2}(\Omega)}.$$
   – Classical compactness-uniqueness arguments.

Lemma

The sequence of eigenvalues μ_k of the Gramian operator allow for a lower, polynomially decaying bound of the form

$$\frac{c_\infty}{\lambda_{\omega,k}} \le \mu_k,$$

where $\lambda_{\omega,k}$ are eigenvalues of the Dirichlet Laplacian on ω and c_∞ is the positive constant from (4) *(for $T = \infty$).*

We combine the last result

- with the previously obtained upper estimate
- with the Weyl's asymptotic law

$$\lambda_{\omega,k} \sim C(\omega) k^{2/d}.$$

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Theorem (**The main result**)

The sequence of eigenvalues μ_k of the Gramian operator X_ω given by (1) allow for a two-sided, polynomially decaying bounds of the form

$$C_1 k^{-2/d} \leq \mu_k \leq C_2 k^{-2/d},$$

where $C_{1,2}$ are positive constants that depend on ω and Ω only.

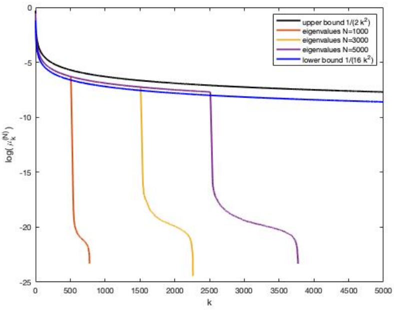

Figure: Eigenvalues decay rates for various approximation dimensions N.

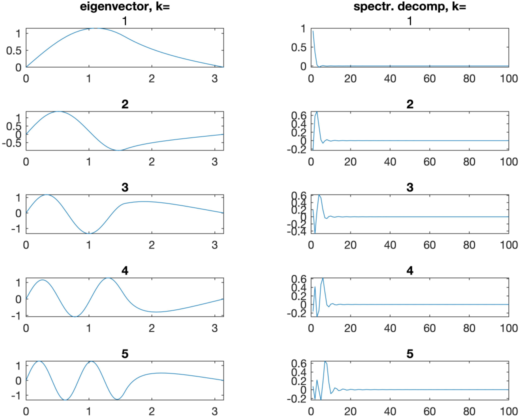

Figure: Eigenvectors $\psi_k^{(100)}$ of the Gramian X_N (left) and their spectral decomposition (right), for $k = 1..5$.

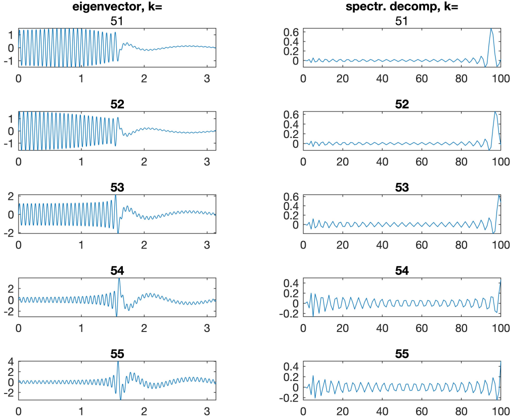

Figure: Eigenvectors $\psi_k^{(100)}$ of the Gramian X_N (left) and their spectral decomposition (right), for $k = 51..55$.

Precise estimates for rectangular domains

Theorem

Assume that the domain Ω is a rectangle in $\mathbf{R}^d$. Let $\omega \subseteq \Omega$ be a control region of a positive measure. Then there exists a positive integer $n \geq \sqrt[d]{|\Omega|/|\omega|}$ such that the eigenvalues of the Gramian X_ω satisfy

$$\frac{1}{2n^d \lambda_{kn}} \leq \mu_k \leq \frac{1}{2\lambda_k}. \tag{5}$$

The smallest value of the constant n for which the lower estimate in (5) holds depends in a non-increasing manner on the size of the control region $|\omega|$, and it equals to 1 when the latter coincides with the whole domain Ω.

Summary:

- lower bounds on the eigenvalues of the Gramian
- the obtained bounds are sharp
- the bounds are expressed in terms on eigenvalues of the Laplacian
- the results generalise to finite time Gramians
- the procedure does not apply in the context of boundary control or when the control acts in an arbitrary measurable subset of positive measure,

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Thanks for your attention!