Optimal convex dominance – relations to mechanics and optimal transport

Centro de Ciencias de Benasque Pedro Pascual

Workshop: **X Partial differential equations, optimal design and numerics**
Benasque Aug 18 – Aug 30 2024

Thematic session: *Shape Optimization in Analysis and in Mechanics*

Karol Bołbotowski

Lagrange Mathematics and Computation Research Center, Paris
& Warsaw University of Technology, Faculty of Civil Engineering

In collaboration with:

Guy Bouchitté	Université du Sud Toulon-Var, IMATH
Guillaume Carlier	Université Paris-Dauphine, CEREMADE
Quentin Mérigot	Université Paris-Saclay, Institut de Mathématique d'Orsay
Filippo Santambrogio	Université Claude Bernard - Lyon 1, Institut Camille Jordan

Convex order of probability measures

Let $\mu, \nu \in \mathrm{P}(\mathbb{R}^d)$ be two probability measures. We say that ν is greater than μ in the **convex order**, i.e.

$$\mu \leq_c \nu \qquad \Leftrightarrow \qquad \int \varphi \, d\mu \leq \int \varphi \, d\nu \qquad \forall \, \varphi \text{ convex}$$

Necessary conditions

- $[\mu] = [\nu], \quad [\mu] = \int x \, \mu(dx)$

  $\Big(\text{we test with } \varphi(x) = \pm(a \cdot x + b), \;\; a \in \mathbb{R}^d, \; b \in \mathbb{R}\Big)$

- $\mathrm{var}(\mu) \leq \mathrm{var}(\nu), \quad \mathrm{var}(\mu) = \int |x|^2 \mu(dx)$

  $\Big(\text{we test with } \varphi(x) = |x|^2\Big)$

- $\mathrm{C}(\mu) \preceq \mathrm{C}(\nu), \quad \mathrm{C}(\mu) = \int x \otimes x \, \mu(dx)$

  $\Big(\text{we test with } \varphi(x) = (a \cdot x)^2, \;\; a \in \mathbb{R}^d\Big)$

Example: Gaussian case

- $d = 1, \; \mu_\sigma = \mathcal{N}(0, \sigma^2)$

  $$\mu_{\sigma_1} \leq_c \mu_{\sigma_2} \qquad \Leftrightarrow \qquad \sigma_1 \leq \sigma_2$$

- $d > 1, \; \mu_\Sigma = \mathcal{N}(0, \Sigma)$

  $$\mu_{\Sigma_1} \leq_c \mu_{\Sigma_2} \qquad \Leftrightarrow \qquad \Sigma_1 \preceq \Sigma_2$$

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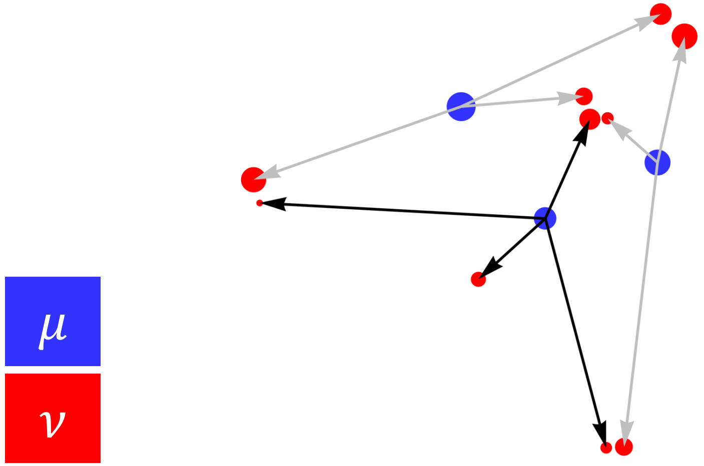
$$\mu \leq_c \nu \qquad \Leftrightarrow \qquad \int \varphi \, d\mu \leq \int \varphi \, d\nu \qquad \forall \, \varphi \text{ convex}$$

We say that $\gamma \in \mathrm{P}(\mathbb{R}^d \times \mathbb{R}^d)$ is a **martingale transport plan** between μ and ν, i.e.

$$\gamma \in \mathrm{MT}(\mu, \nu) \qquad \Leftrightarrow \qquad \begin{cases} \mu = \pi_x \# \gamma, \;\; \nu = \pi_y \# \gamma, \\ [\gamma^x] = x \;\; \mu\text{-a.e.} \quad \text{where } \; \gamma = \mu \otimes \gamma^x \end{cases}$$



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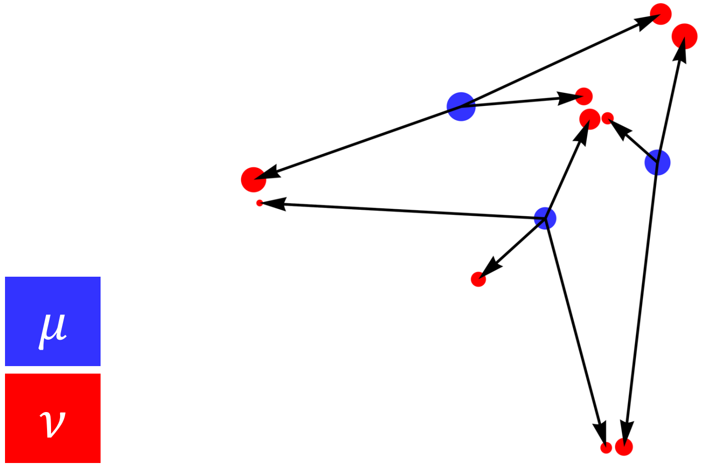
Theorem (Strassen 1965)

Let $\mu, \nu \in \mathrm{P}(\mathbb{R}^d)$ be two probability measures. The following conditions are equivalent:

(i) $\mu \leq_c \nu$,

(ii) there exists a matringale plan $\gamma \in \mathrm{MT}(\mu, \nu)$.



Optimal Convex Dominance

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$$\inf \left\{ \int f(z) \varrho(dz) \ : \ \varrho \in \mathrm{P}(\mathbb{R}^d), \quad \mu \leq_c \varrho, \ \ \nu \leq_c \varrho \right\} \qquad \textbf{(P)}$$

$$\sup \left\{ \int \varphi \, d\mu + \int \psi \, d\nu \ : \ \varphi, \psi \text{ convex}, \ \varphi(z) + \psi(z) \leq f(z) \right\} \qquad \textbf{(P*)}$$

Cost function:

$$f : \mathbb{R}^d \to [0, \infty), \ \text{convex}$$

Example:

$$f(z) = |z|^p, \qquad p \in [1, \infty)$$

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Optimal Transport

$$c(x,y) = |x-y|^p$$

$$\min \left\{ \iint c(x,y) \, \gamma(dxdy) \ : \ \gamma \in \mathrm{P}(\mathbb{R}^d \times \mathbb{R}^d), \ \ \pi_x \# \gamma = \mu, \ \pi_y \# \gamma = \nu \right\} \qquad \textbf{(OT)}$$

$$\sup \left\{ \int \varphi \, d\mu + \int \psi \, d\nu \ : \ \varphi, \psi \in C(\mathbb{R}^d), \ \varphi(x) + \psi(y) \leq c(x,y) \right\} \qquad \textbf{(OT*)}$$

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Proposition 1 (zero duality gap)

$$\sup(\mathcal{P}^*) = \inf(\mathcal{P}) < \infty$$

Proposition 2 (existence of minimizers)

Solution of $(\mathcal{P})$ exists if f is superlinear ($p > 1$). (no uniqueness)

Example: two Gaussians

$$f = |\cdot|^2, \ \mu = \mathcal{N}(0, M), \ \nu = \mathcal{N}(0, N)$$

$$R = M \vee N := M + (N - M)_+ = N + (M - N)_+$$

$$\rho = \mathcal{N}(0, R) \ \text{ solves } \ (\mathcal{P})$$

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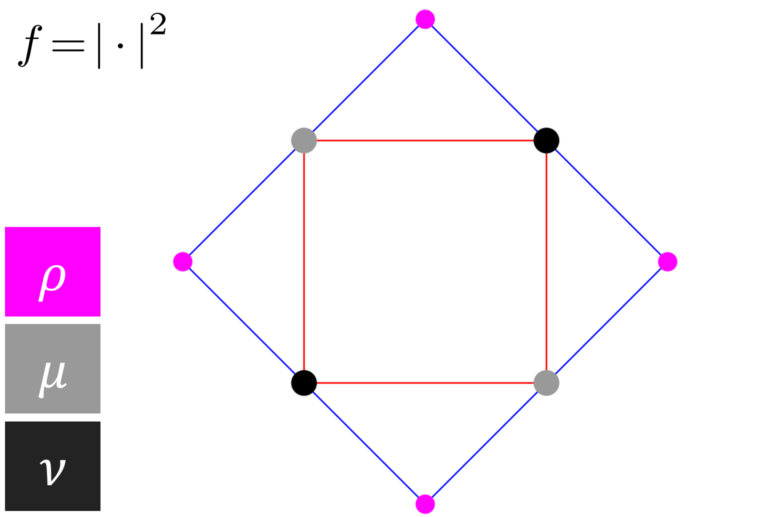
Example: two-point measures



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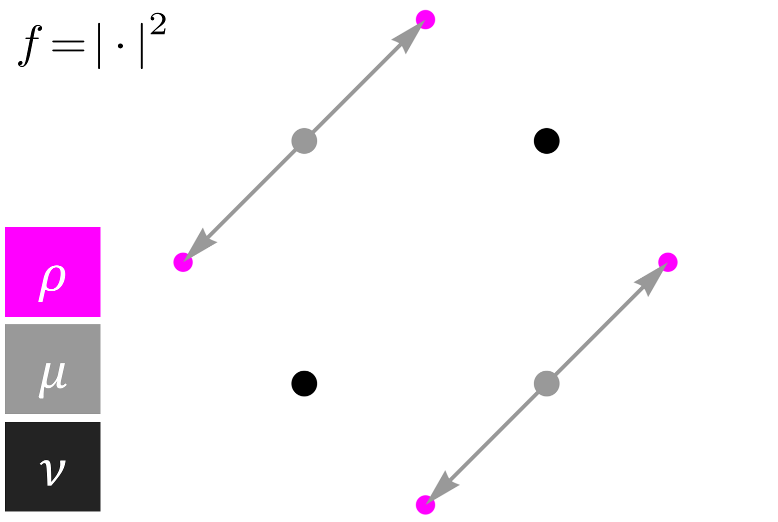

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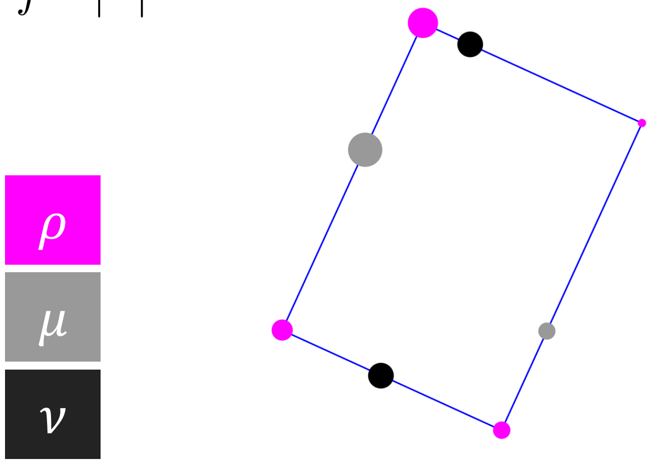
$f = |\cdot|^2$



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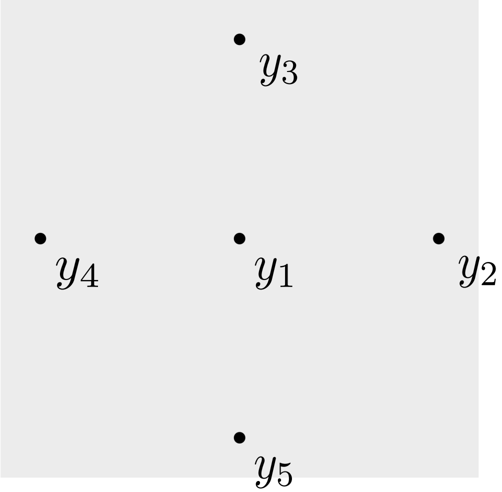
Example: Lebesgue vs 5 Dirac masses

$$f = |\cdot|^p, \quad \mu = \mathcal{L}^2 \llcorner Q, \quad \nu = \sum_{i=1}^{5} \frac{1}{5} \delta_{y_i}$$



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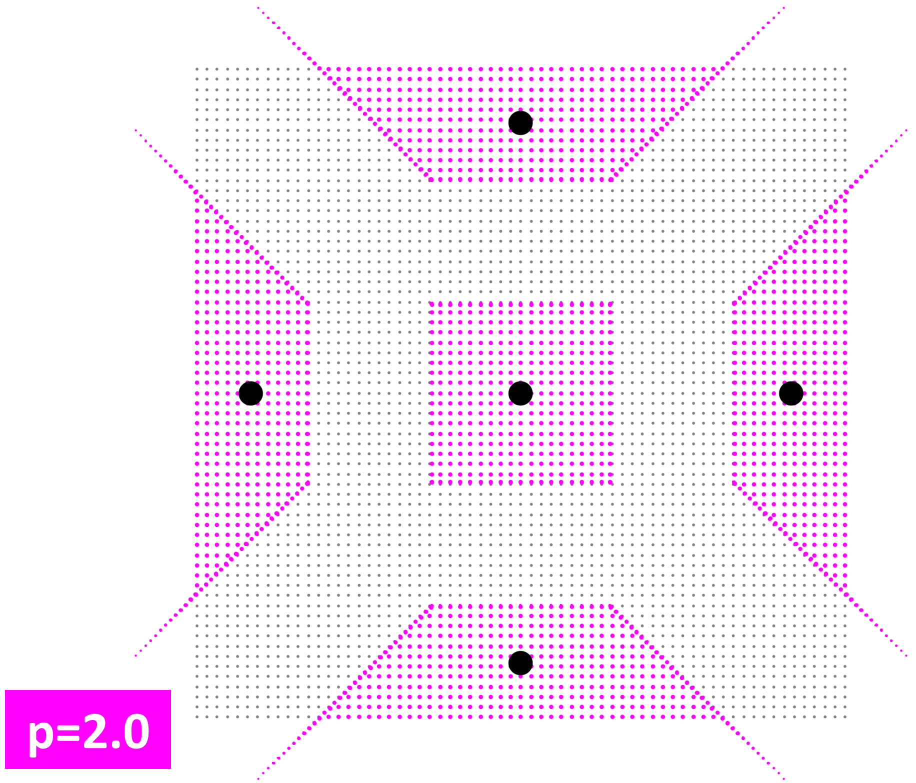

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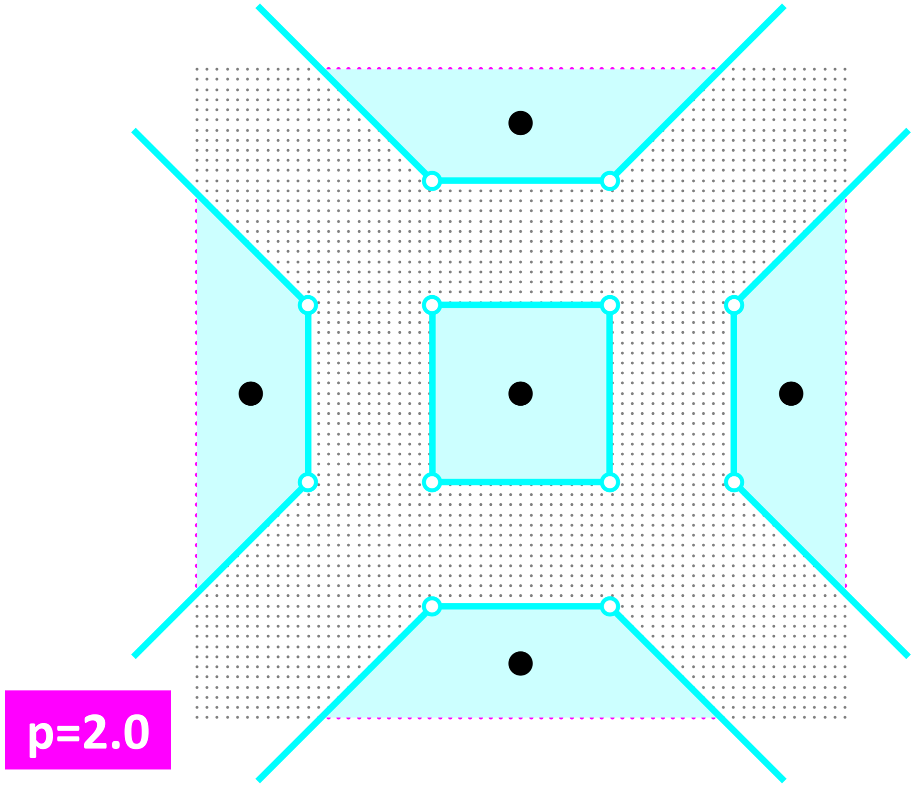

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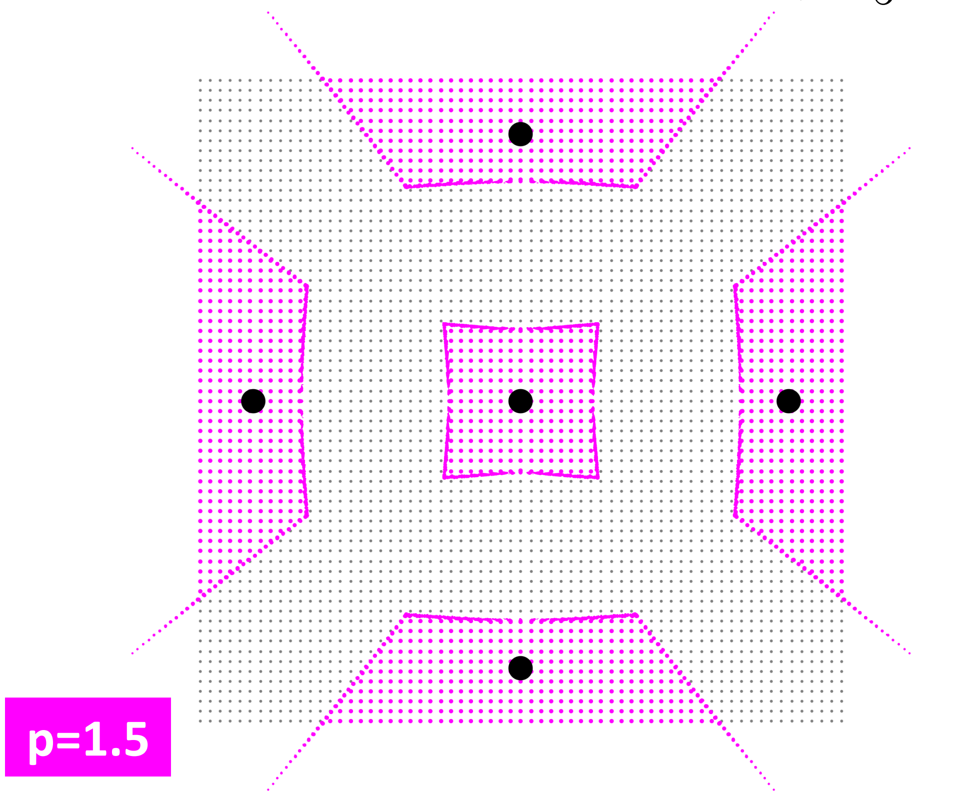

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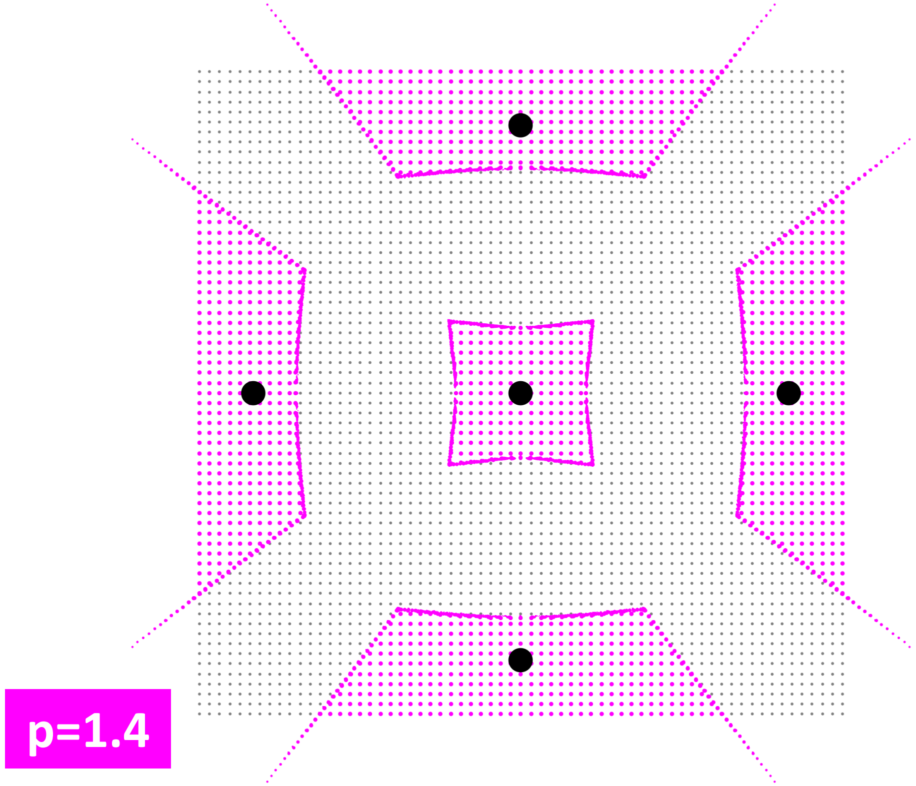

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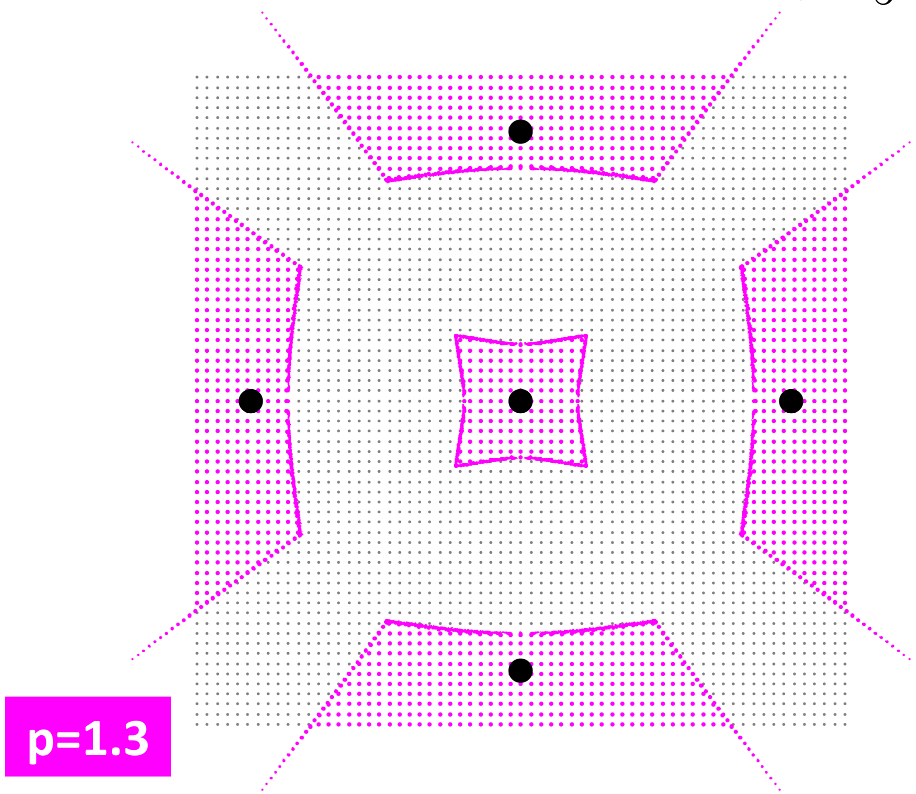

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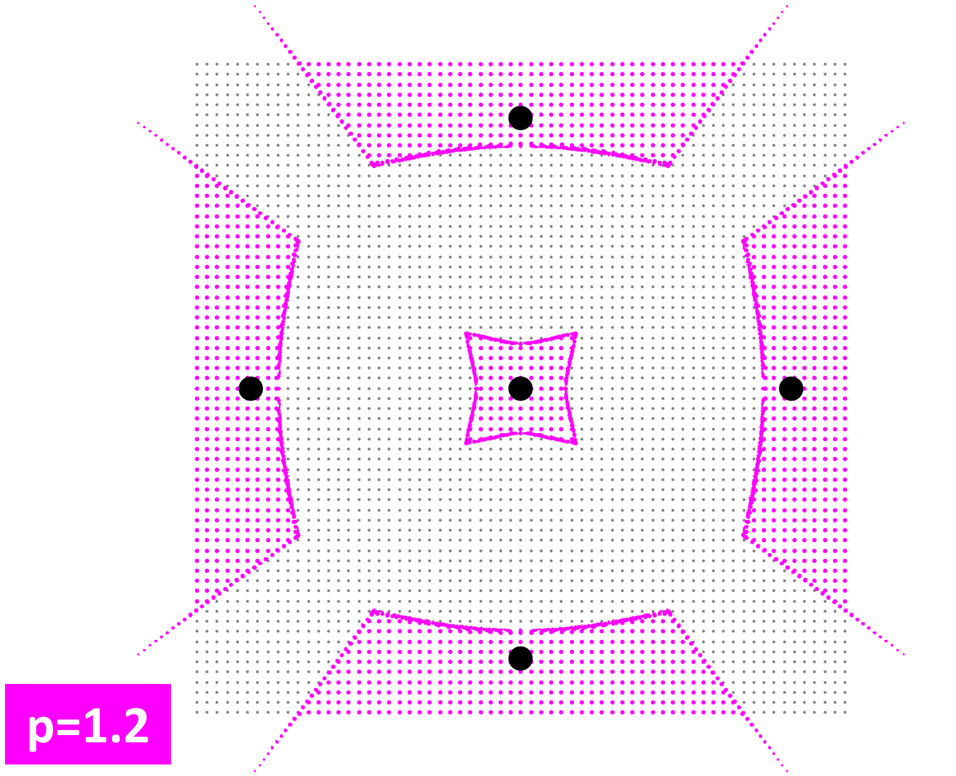

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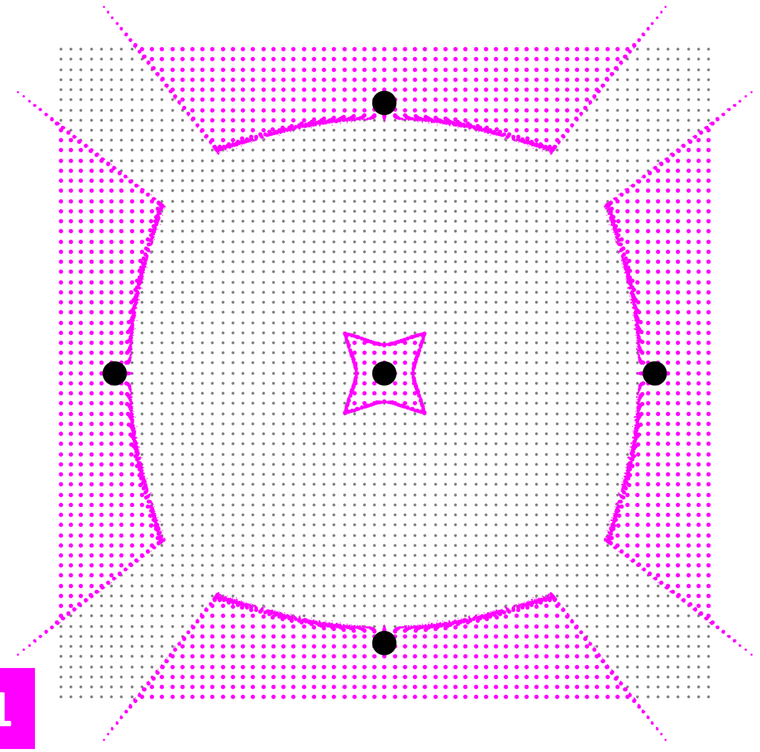

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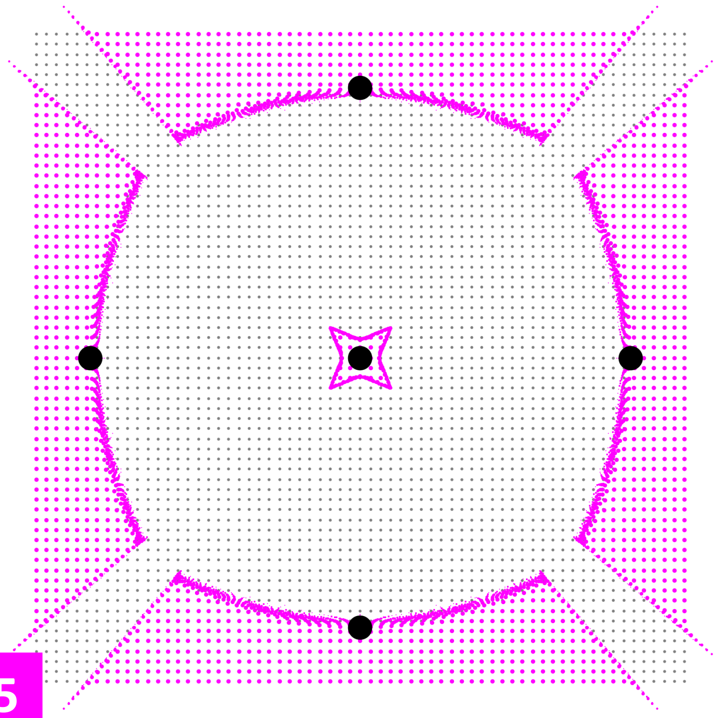

p=1.05

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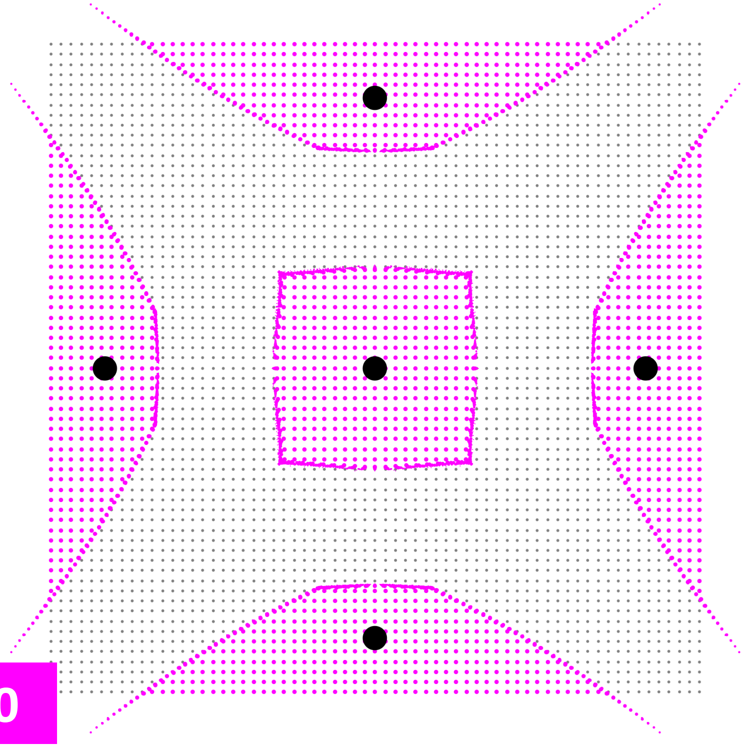


p=3.0

Optimal convex dominance

Let $\mu, \nu \in \mathrm{P}(\mathbb{R}^d)$ share the barycenter, i.e. $[\mu] = [\nu] = 0$. Assume that $\int f d\mu < \infty$, $\int f d\nu < \infty$.

$$\inf \left\{ \int f(z) \varrho(dz) \ : \ \varrho \in \mathrm{P}(\mathbb{R}^d), \quad \mu \leq_c \varrho, \quad \nu \leq_c \varrho \right\} \qquad \textbf{(P)}$$

$$\sup \left\{ \int \varphi d\mu + \int \psi d\nu : \varphi, \psi \text{ cvx}, \ \varphi(z) + \psi(z) \leq f(z) \right\} \qquad \textbf{(P*)}$$

Proposition 1 (zero duality gap)

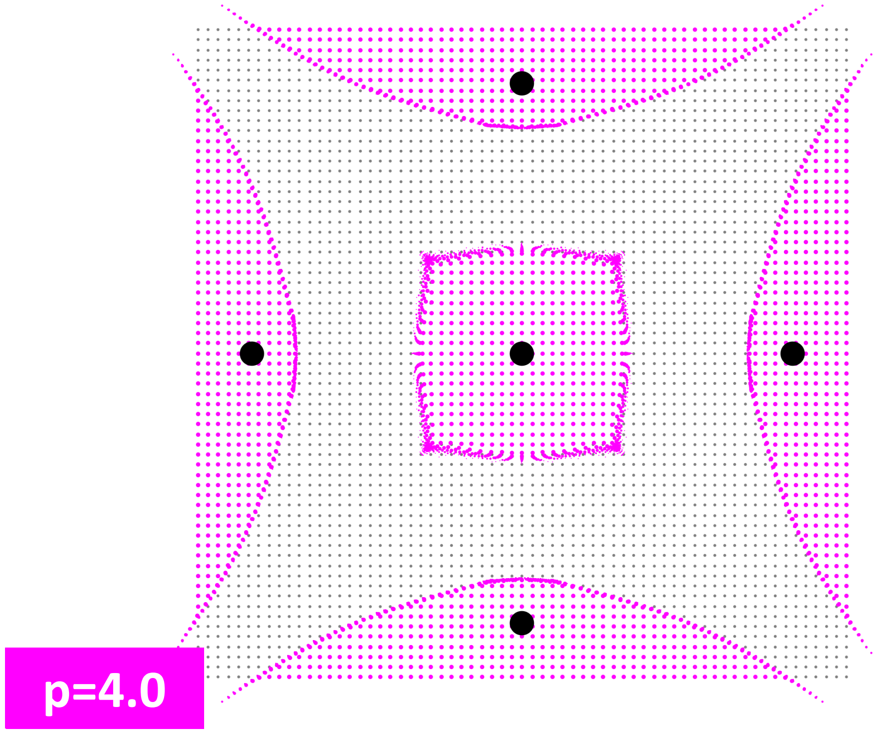
$$\sup(\mathcal{P}^*) = \inf(\mathcal{P}) < \infty$$

Proposition 2 (existence of minimizers)

Solution of $(\mathcal{P})$ exists if f is superlinear $(p > 1)$. (no uniqueness)

Example: Lebesgue vs 5 Dirac masses

$$f = |\cdot|^p, \quad \mu = \mathcal{L}^2 \llcorner Q, \quad \nu = \sum_{i=1}^{5} \frac{1}{5} \delta_{y_i}$$



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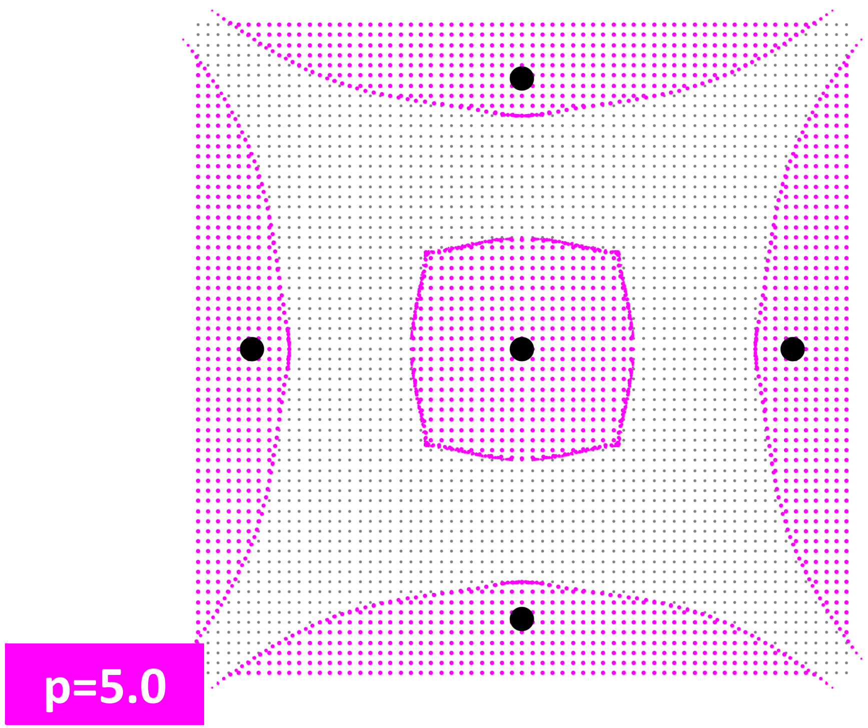


Optimal Grillage Problem

The motivation – Optimal Grillage Problem

Let $\mu, \nu \in \mathrm{P}(\mathbb{R}^d)$ share the barycenter, i.e. $[\mu] = [\nu]$. Assume that $\int |x|^2 \mu(dx) < \infty$, $\int |y|^2 \nu(dy) < \infty$.

$$\min\left\{ \int \rho^0(\sigma) \,:\, \sigma \in \mathcal{M}^b\big(\mathbb{R}^d; \mathbb{R}^{d\times d}_{\mathrm{sym}}\big),\ \ \mathrm{div}^2\sigma = \mu - \nu \ \text{ in } \mathcal{D}'(\mathbb{R}^d) \right\} \qquad \textbf{(OG)}$$

$$\max\left\{ \int u \, d(\mu - \nu) \,:\, u \in C^{1,1}(\mathbb{R}^d),\ \ \rho(\nabla^2 u) \leq 1 \text{ a.e.} \right\} \qquad \textbf{(OG*)}$$

designboom.com

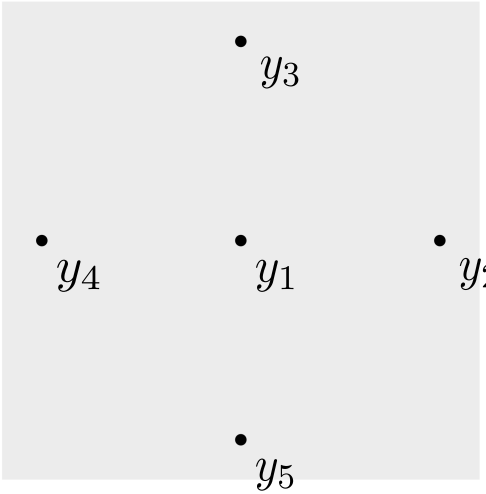
Gatti Wool Factory (Rome, 1951) by Pier Luigi Nerv

$$\rho^0(\eta) = \sum_{i=1}^{d} |\lambda_i(\eta)| \qquad \forall\, \eta \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$$

$$\rho(\xi) = \max_{i\in\{1,\dots,d\}} |\lambda_i(\xi)| \qquad \forall\, \xi \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$$

Example: Lebesgue vs 5 Diracs

$$\mu = \mathcal{L}^2 \llcorner Q, \qquad \nu = \sum_{i=1}^{5} \frac{1}{5}\delta_{y_i}$$



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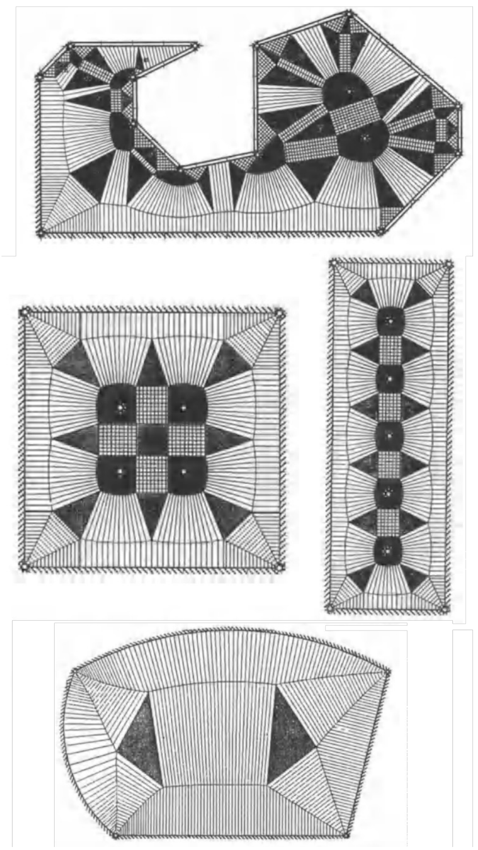
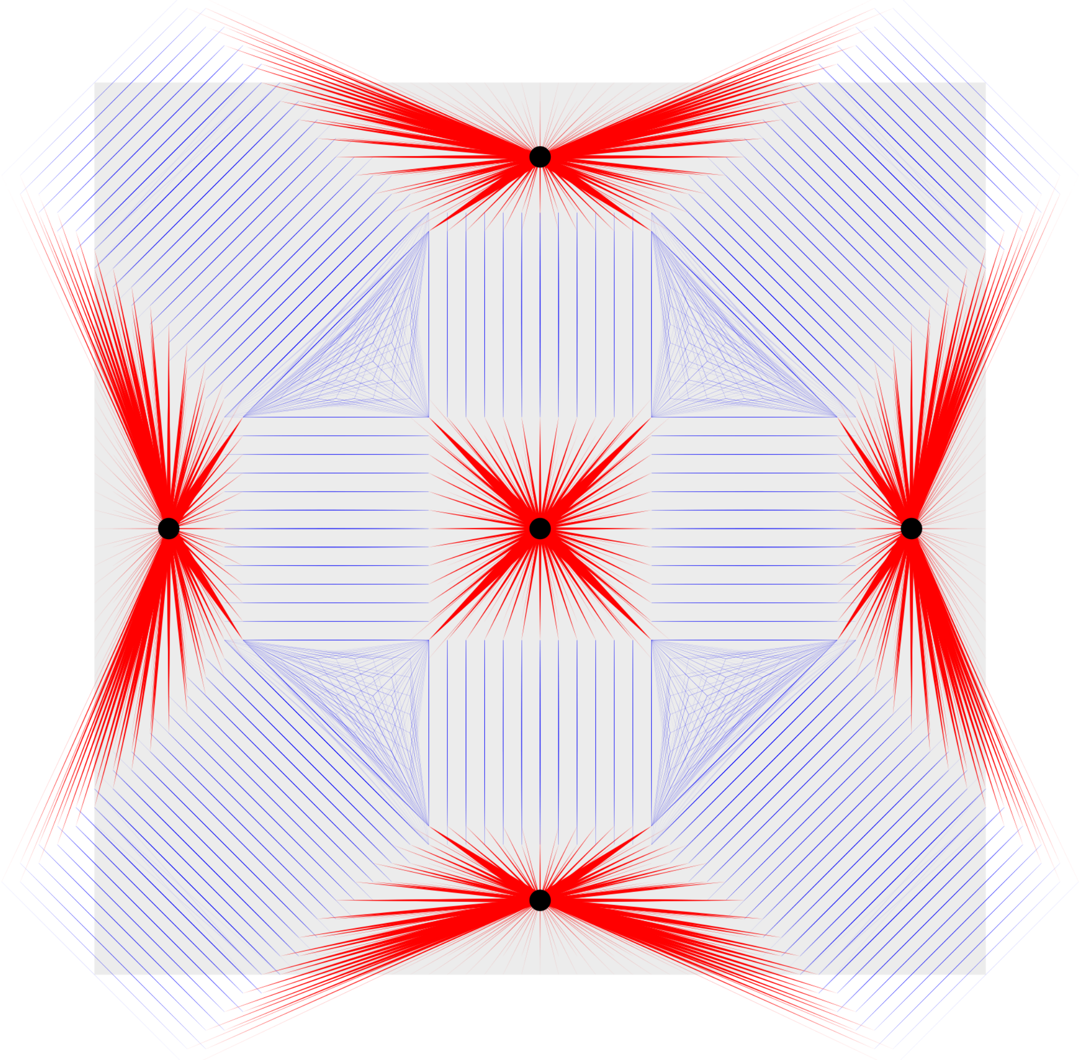


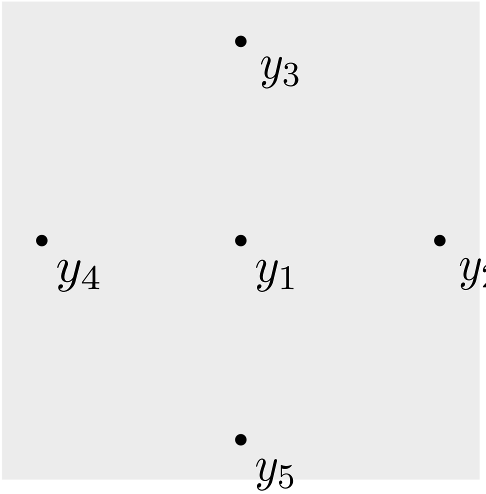
Rozvany (1972-1997)



Example: Lebesgue vs 5 Diracs

$$\mu = \mathcal{L}^2 \llcorner Q, \quad \nu = \sum_{i=1}^{5} \frac{1}{5} \delta_{y_i}$$

- $\sigma_h \ll \mathcal{H}^1 \llcorner \left(\bigcup_{(x,y) \in X \times X} [x, y] \right)$
- σ_h is rank-one
- positive and negative part



The case p = 2 versus the Optimal Grillage Problem

Let $\mu, \nu \in \mathrm{P}(\mathbb{R}^d)$ share the barycenter, i.e $[\mu] = [\nu]$. Assume that $\int |x|^2 \mu(dx) < \infty$, $\int |y|^2 \nu(dy) < \infty$.

$$\mathcal{I}(\mu,\nu) = \min\left\{ \int \rho^0(\sigma) \ : \ \sigma \in \mathcal{M}^b\big(\mathbb{R}^d; \mathbb{R}^{d\times d}_{\mathrm{sym}}\big), \ \ \mathrm{div}^2 \sigma = \mu - \nu \right\} \qquad \textbf{(OG)}$$

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Theorem

For $f = |\cdot|^2$ the equality holds true:

$$\mathcal{I}(\mu,\nu) = \mathcal{J}(\mu,\nu) - \frac{\mathrm{var}(\mu)+\mathrm{var}(\nu)}{2}$$

Corollary

For any solution ρ to the problem $(\mathcal{P})$ take any

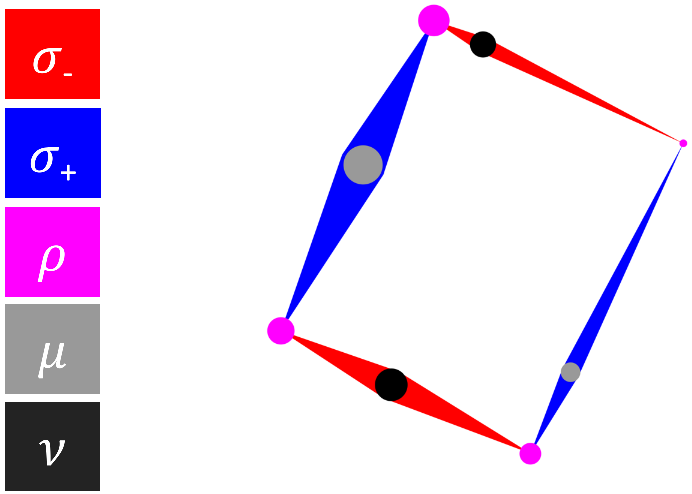
$$\gamma_+ \in \mathrm{MT}(\mu,\rho), \qquad \gamma_- \in \mathrm{MT}(\nu,\rho)$$

and define $\sigma_+, \sigma_- \in \mathcal{M}^b\big(\mathbb{R}^d; \mathbb{R}^{d\times d}_{\mathrm{sym},+}\big)$

$$\langle \sigma_\pm, \Theta\rangle = \iint \left(\int_0^1 t \Big\langle (x-z)\otimes(x-z), \Theta\big(z + t(x-z)\big) \Big\rangle \, dt \right) \gamma_\pm(dxdz)$$

Then $\sigma = \sigma_+ - \sigma_-$ solve the optimal grillage problem (OG).

Example: two-point measures



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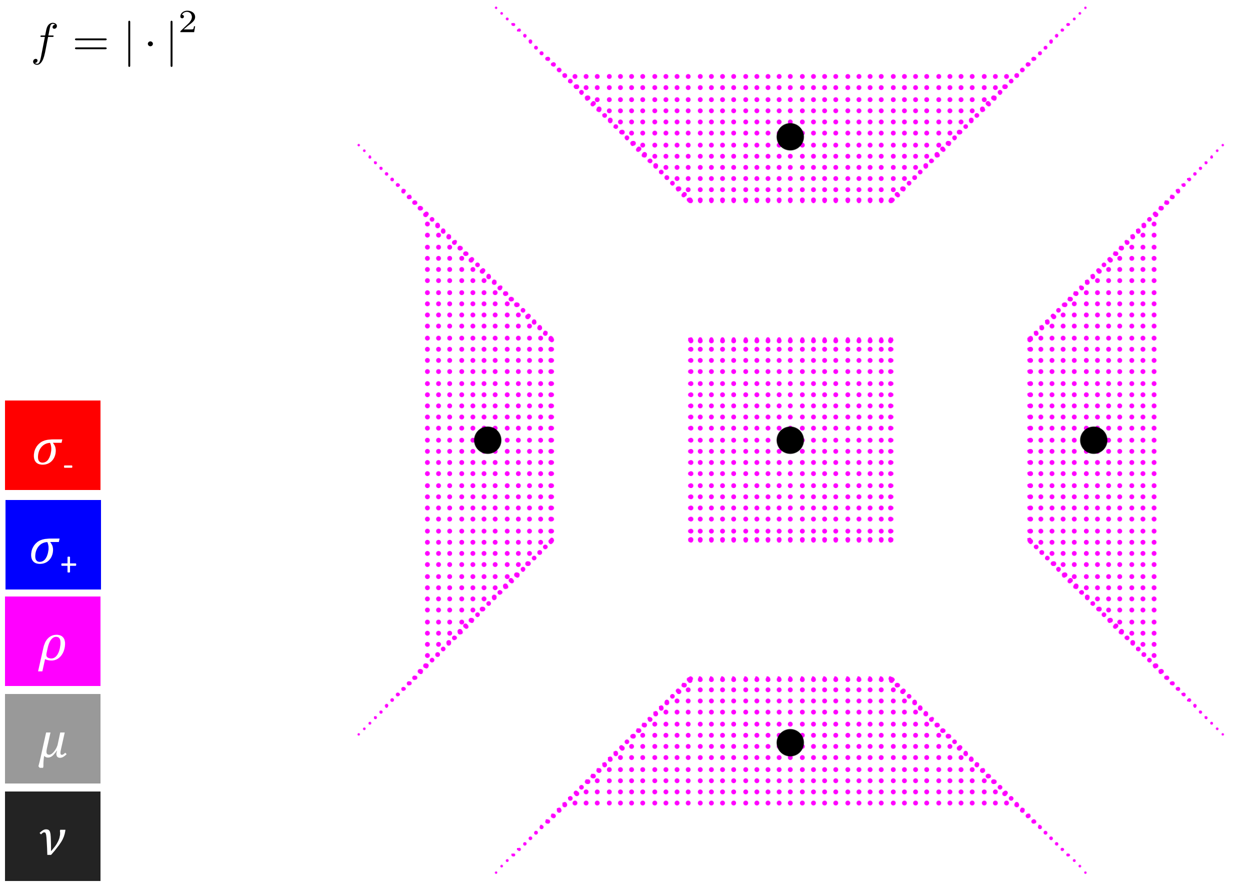
Example: Lebesgue vs 5 Dirac masses

$f = |\cdot|^2$



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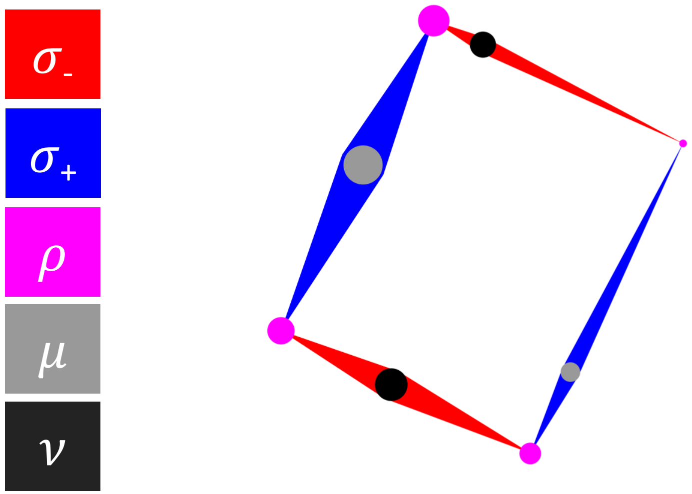

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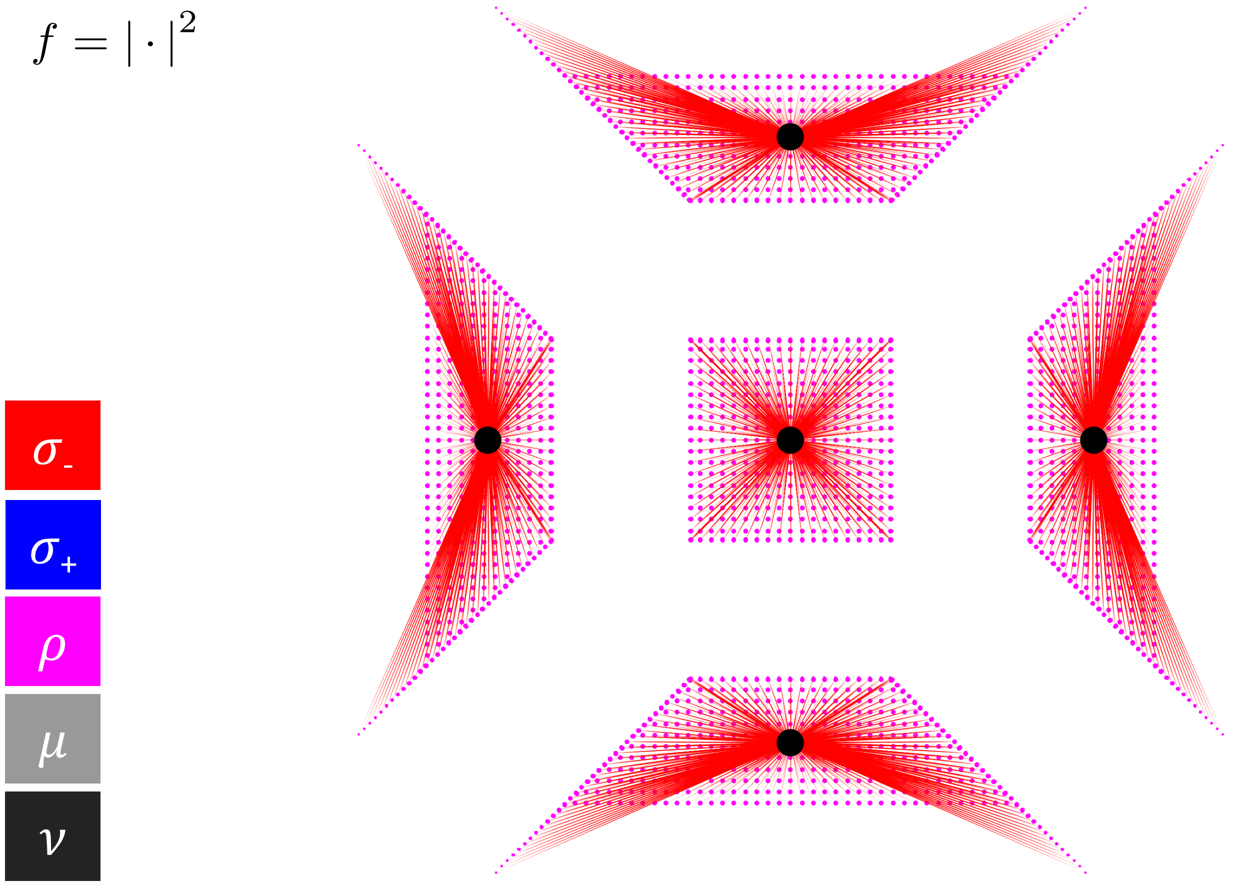

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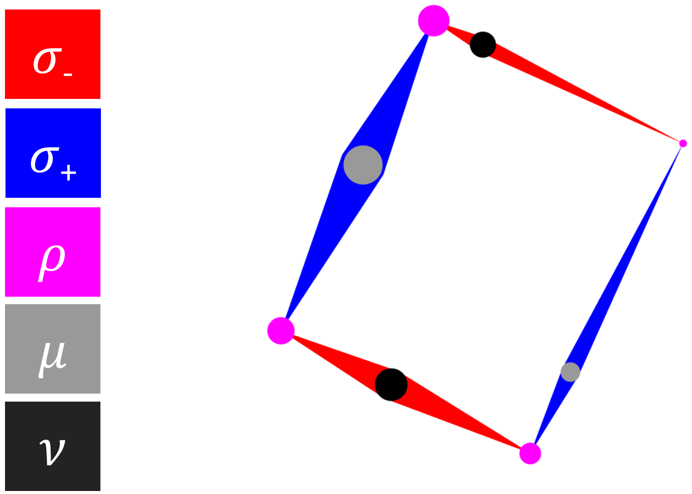

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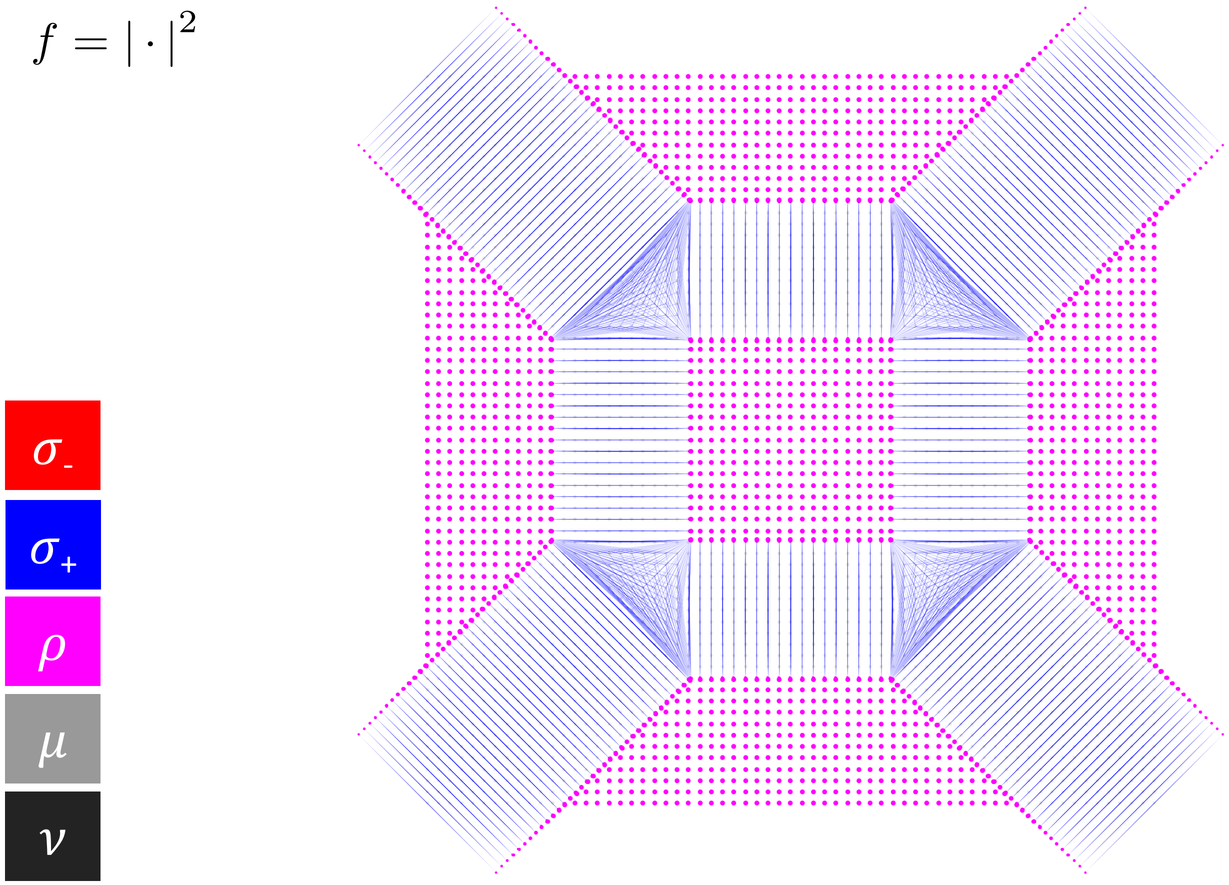

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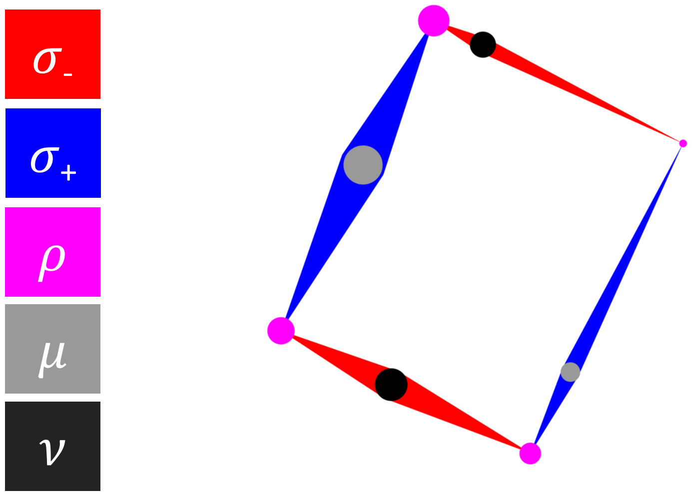

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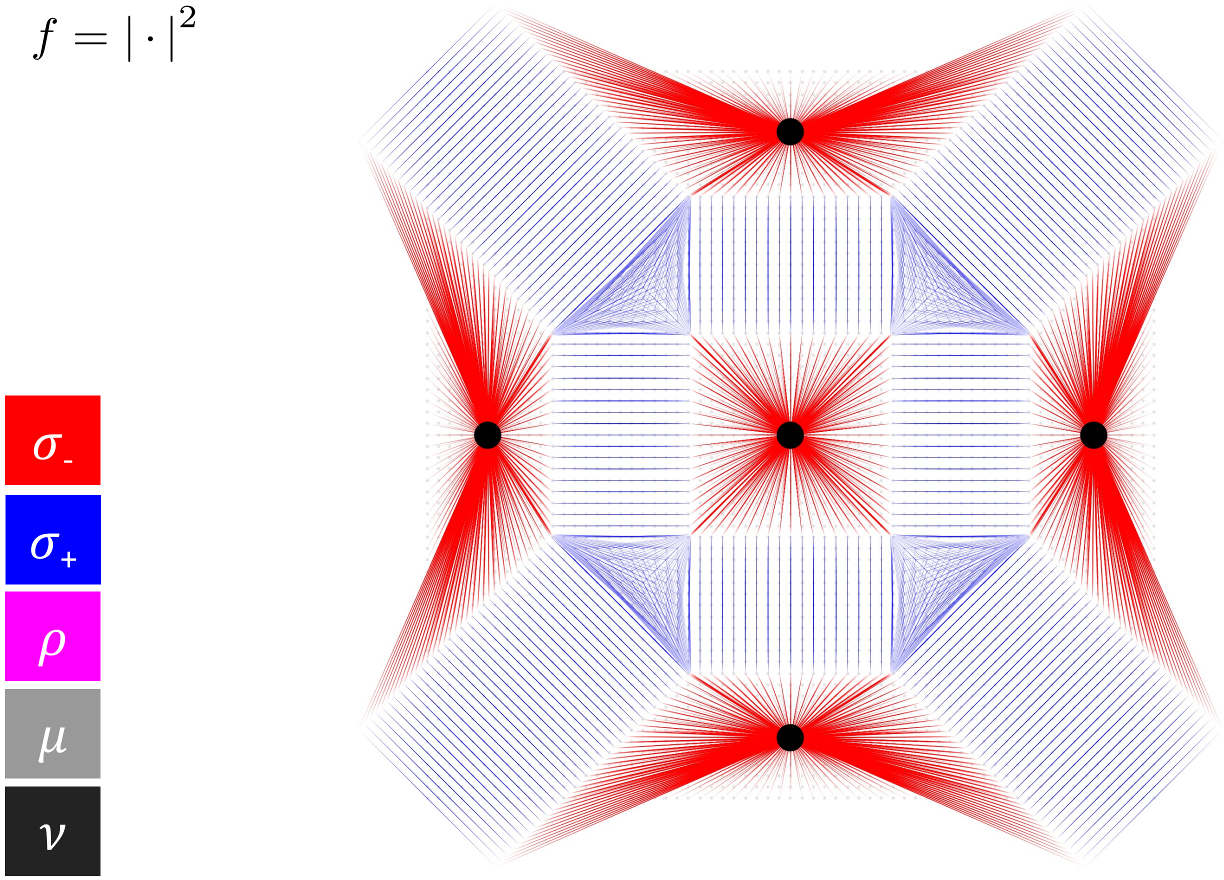

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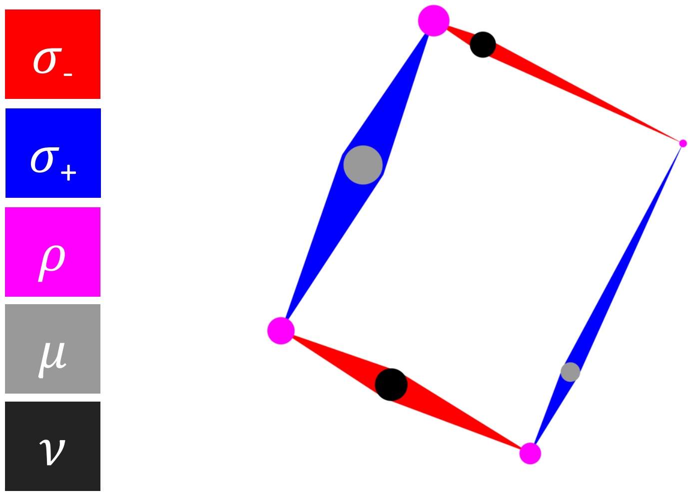

Numerical Strategies

Numerical strategy #1: discretizing the cone of convex f.

To discretize the dual problem

$$\sup\left\{\int \varphi\, d\mu + \int \psi\, d\nu \ :\ \varphi, \psi \text{ convex},\ \varphi(z) + \psi(z) \leq f(z)\right\} \quad \textbf{(P*)}$$

and then to dualize back.

Tackling numerically the convexity constraints in the literature:

- discretization by a point cloud:
  - Carlier, Lachand-Robert, Maury (2001),
  - Ekeland, Moreno-Bromberg (2010),
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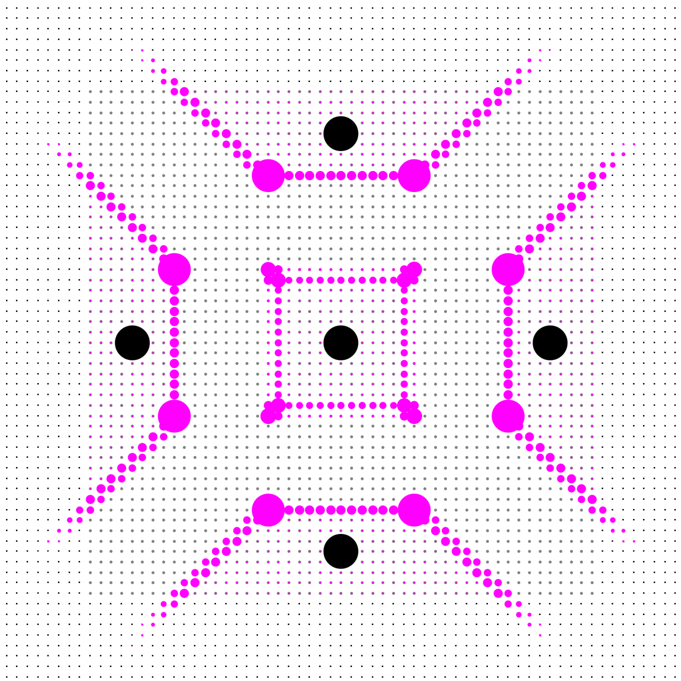
Ekeland, Moreno-Bromberg



- $n = 65^2 = 4225$ points,
- $n^2 \approx 18$mln constr,
- adaptivity $\rightarrow$ comp. cost $\approx n \log n$
- CPU time: 18s

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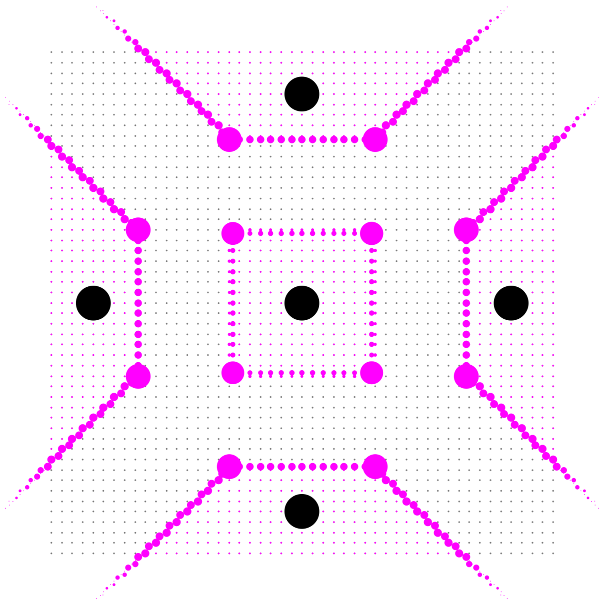

- $n = 193^2$ points,
- $n^2 \approx 1.4$bln constr,
- CPU time: 6min

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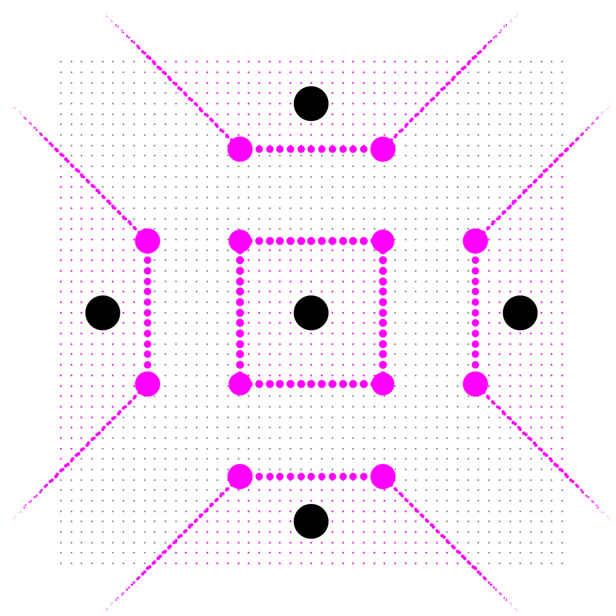

- $n = 385^2$ points,
- $n^2 \approx 22$bln constr,
- CPU time: 1.2h

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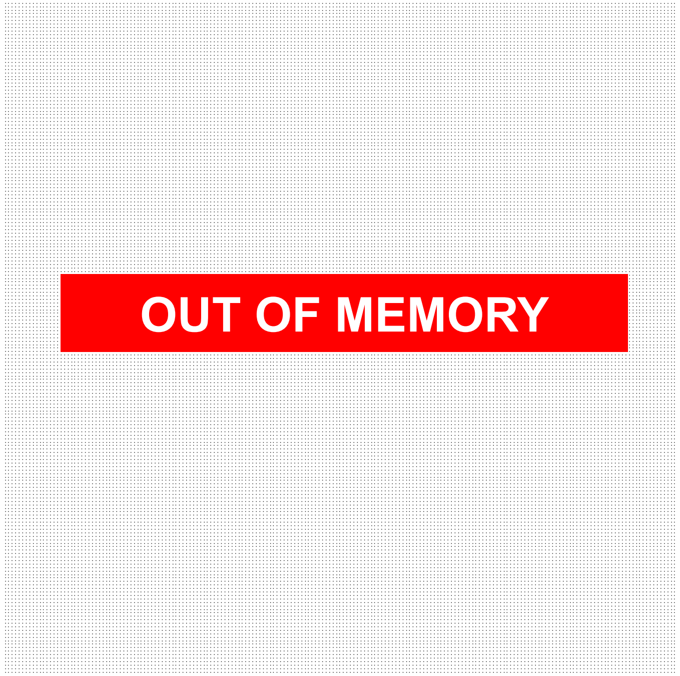

- $n = 577^2$ points,
- $n^2 \approx 111$bln constr,
- CPU time: -

Reformulation as a variant of Optimal Transport

Assume that f is strictly convex and differentiable (e.g. $f = |\cdot|^p$, $p > 1$).

$$\mathcal{J}(\mu,\nu) = \sup \left\{ \int \varphi \, d\mu + \int \psi \, d\nu \; : \; \varphi, \psi \text{ convex}, \; \varphi(z) + \psi(z) \leq f(z) \right\} \qquad \textbf{(P*)}$$

$$= \sup_{\substack{\varphi,\psi \in C(\mathbb{R}^d) \\ \Phi,\Psi \in C(\mathbb{R}^d;\mathbb{R}^d)}} \left\{ \int \varphi \, d\mu + \int \psi \, d\nu \; : \; \varphi(x) + \langle \Phi(x), z - x \rangle + \psi(y) + \langle \Psi(y), z - y \rangle \leq f(z) \quad \forall \, x, y, z \right\}$$

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(P̃)

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(P̃*)

Theorem

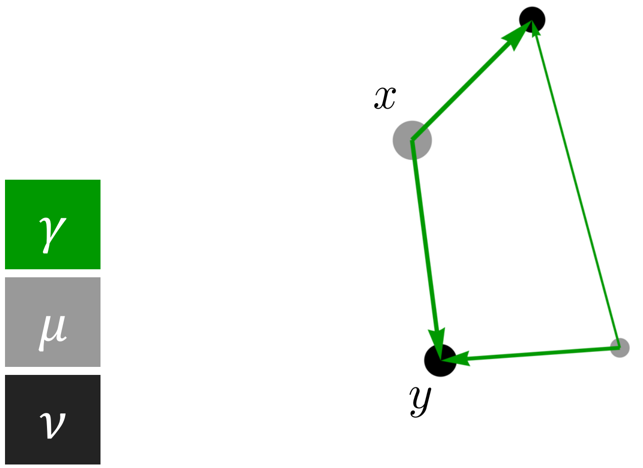
Take a solution (γ, q) of $(\widetilde{\mathcal{P}})$ and put

$$\zeta = \frac{dq}{d\gamma} : (x,y) \mapsto z \in \mathbb{R}^d$$

Then the following ρ solves $(\mathcal{P})$:

$$\rho = \zeta\#\gamma.$$

Example: two-point measures



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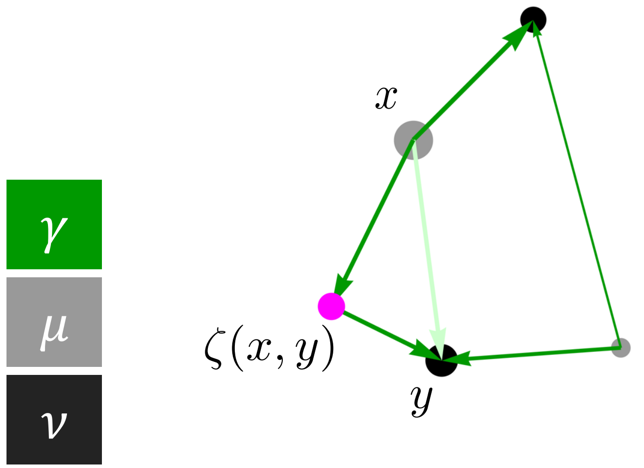

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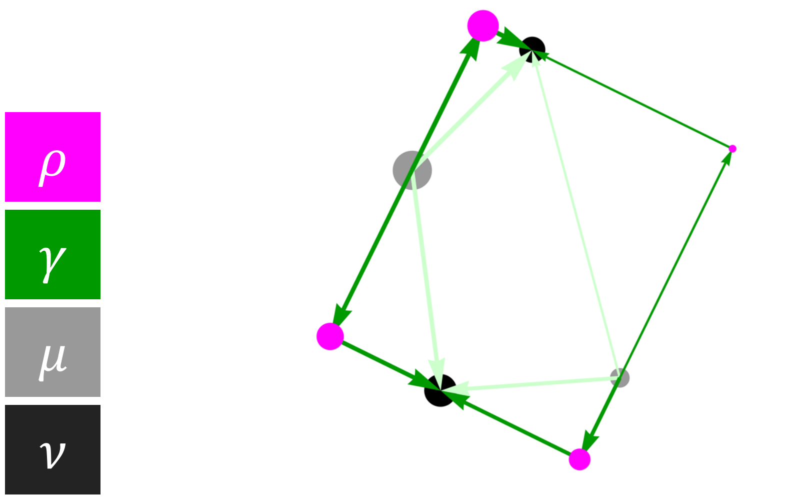
Example: two-point measures



Reformulation as a variant of Optimal Transport

$$\mathcal{J}(\mu,\nu) = \min\left\{ \iint f\left(\frac{dq}{d\gamma}\right) d\gamma \ : \ \begin{array}{lll} \gamma \in \mathrm{P}\big(\mathbb{R}^d \times \mathbb{R}^d\big), & \pi_x \# \gamma = \mu, & \pi_x \# \, x\, \gamma(dxdy) = \pi_x \# q, \\ q \in \mathcal{M}^b(\mathbb{R}^d \times \mathbb{R}^d; \mathbb{R}^d), & \pi_y \# \gamma = \nu, & \pi_y \# \, y\, \gamma(dxdy) = \pi_y \# q \end{array} \right\} \qquad (\tilde{\mathrm{P}})$$

$$\mathcal{J}(\mu,\nu) = \sup_{\substack{\varphi,\psi \in C(\mathbb{R}^d) \\ \Phi,\Psi \in C(\mathbb{R}^d;\mathbb{R}^d)}} \left\{ \int \varphi\, d\mu + \int \psi \, d\nu \ : \ \varphi(x) + \psi(y) \leq \langle \Phi(x), x\rangle + \langle \Psi(y), y \rangle - f^*\big(\Phi(x) + \Psi(y)\big) \quad \forall\, x,y \right\} \qquad (\tilde{\mathrm{P}}^*)$$

Theorem

Take a solution (γ, q) of $(\widetilde{\mathcal{P}})$ and put

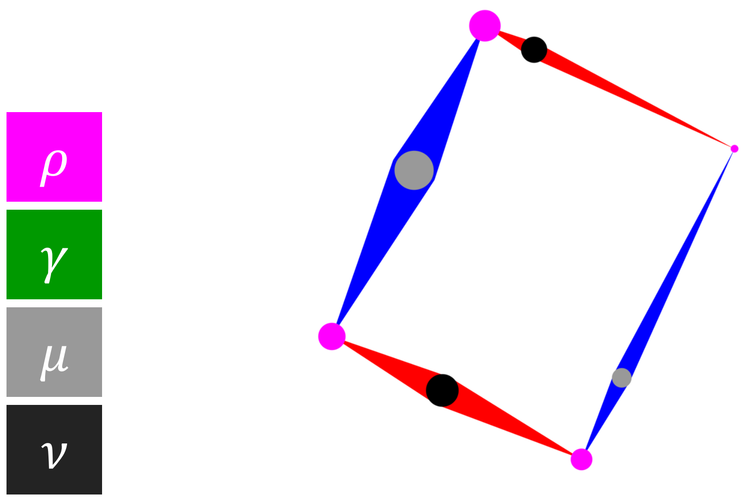
$$\zeta = \frac{dq}{d\gamma} : (x,y) \mapsto z \in \mathbb{R}^d$$

Then the following ρ solves $(\mathcal{P})$:

$$\rho = \zeta \# \gamma.$$

$$(\mathrm{id}, \zeta)\#\gamma \in \mathrm{MT}(\mu,\rho), \qquad (\zeta, \mathrm{id})\#\gamma \in \mathrm{MT}(\rho,\nu)$$

Example: two-point measures



Numerical strategy #2: conic programming

$$\mathcal{J}(\mu,\nu) = \min\left\{ \iint f\left(\frac{dq}{d\gamma}\right) d\gamma \ : \ \begin{array}{lll} \gamma \in \mathrm{P}\left(\mathbb{R}^d \times \mathbb{R}^d\right), & \pi_x \# \gamma = \mu, & \pi_x \# \, x\, \gamma(dxdy) = \pi_x \# q, \\ q \in \mathcal{M}^b(\mathbb{R}^d \times \mathbb{R}^d; \mathbb{R}^d), & \pi_y \# \gamma = \nu, & \pi_y \# \, y\, \gamma(dxdy) = \pi_y \# q \end{array} \right\}$$

($\tilde{\mathrm{P}}$)

$$\mathcal{J}(\mu,\nu) = \sup_{\substack{\varphi,\psi \in C(\mathbb{R}^d) \\ \Phi,\Psi \in C(\mathbb{R}^d;\mathbb{R}^d)}} \left\{ \int \varphi \, d\mu + \int \psi \, d\nu \ : \ \varphi(x) + \psi(y) \leq \langle \Phi(x), x\rangle + \langle \Psi(y), y \rangle - f^*\big(\Phi(x) + \Psi(y)\big) \quad \forall\, x, y \right\}$$

($\tilde{\mathrm{P}}$*)

If μ,ν are discrete and $f = |\cdot|^p$, then $(\widetilde{\mathcal{P}})$, $(\widetilde{\mathcal{P}}^*)$ is a finite dimensional conic programming problem.

No need for discretizing the space!

The conic program can be tackled by an off-the-shelf software, e.g. MOSEK.

Numerical strategy #2: conic programming

$$\mathcal{J}(\mu,\nu) = \min \left\{ \iint f\left(\frac{dq}{d\gamma}\right) d\gamma \ : \ \begin{array}{l} \gamma \in \mathrm{P}(\mathbb{R}^d \\ q \in \mathcal{M}^b(\mathbb{R}^d \end{array} \right.$$

$$\mathcal{J}(\mu,\nu) = \sup_{\substack{\varphi,\psi \in C(\mathbb{R}^d) \\ \Phi,\Psi \in C(\mathbb{R}^d;\mathbb{R}^d)}} \left\{ \int \varphi \, d\mu + \int \psi \, d\nu \ : \ \varphi(x) \right.$$

If μ, ν are discrete and $f = |\cdot|^p$, then $(\widetilde{\mathcal{P}})$, $(\widetilde{\mathcal{P}}^*)$ is a finite dimensional conic programming problem.

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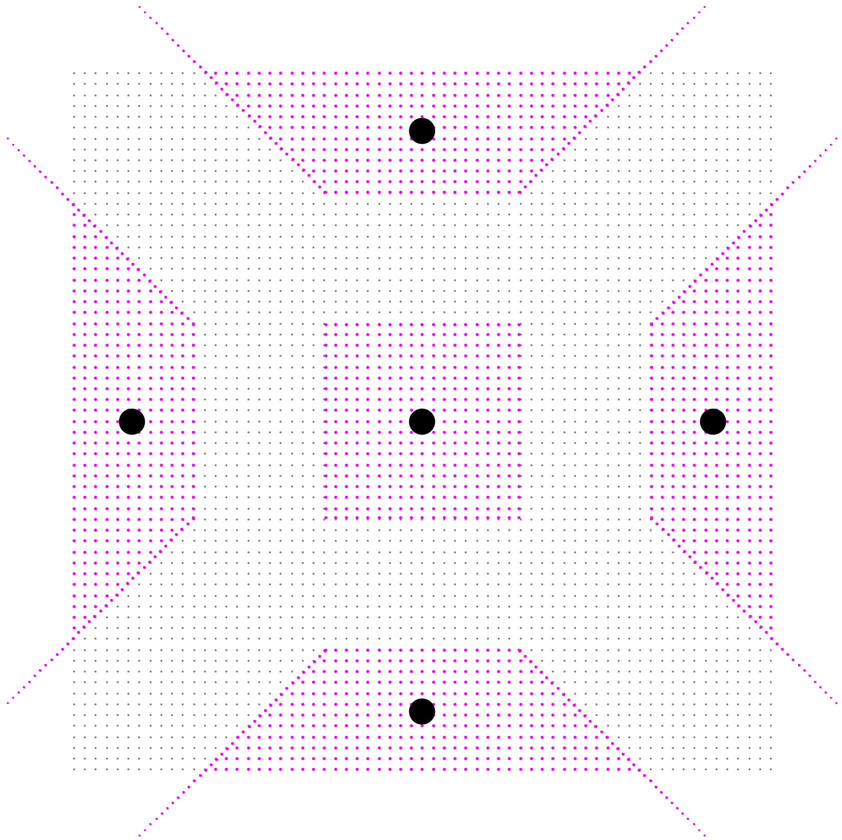
Example: Lebesgue vs 5 Dirac masses



- μ : 65 $\times$ 65 pts
- CPU time: 0.79s

Exact solution up to the accuracy of the solver

Numerical strategy #2: conic programming

$$\mathcal{J}(\mu,\nu) = \min \left\{ \iint f\left(\frac{dq}{d\gamma}\right) d\gamma \;:\; \begin{array}{l} \gamma \in \mathrm{P}(\mathbb{R}^d \times \\ q \in \mathcal{M}^b(\mathbb{R}^d \times \end{array} \right.$$

$$\mathcal{J}(\mu,\nu) = \sup_{\substack{\varphi,\psi \in C(\mathbb{R}^d) \\ \Phi,\Psi \in C(\mathbb{R}^d;\mathbb{R}^d)}} \left\{ \int \varphi \, d\mu + \int \psi \, d\nu \;:\; \varphi(x) + \right.$$

If μ, ν are discrete and $f = |\cdot|^p$, then $(\widetilde{\mathcal{P}})$, $(\widetilde{\mathcal{P}}^*)$ is a finite dimiensional conic programming problem.

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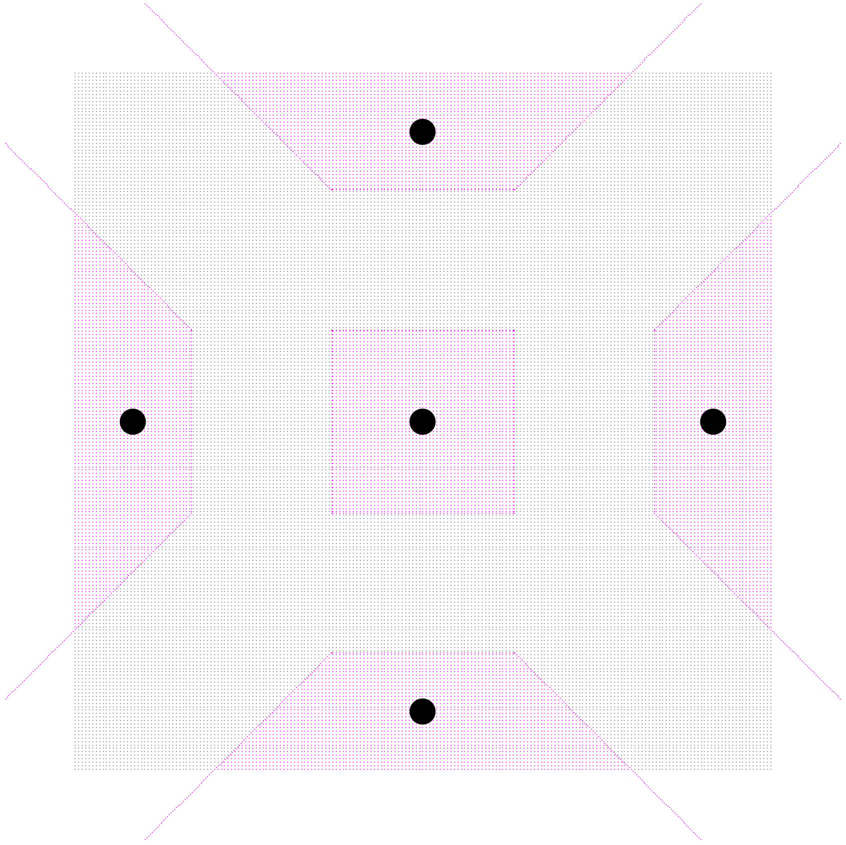
The conic program can be tackled by an off-the-shelf software, e.g. MOSEK.

Example: Lebesgue vs 5 Dirac masses



- μ : 200×200 pts
- CPU time: 10.3s

Exact solution up to the accuracy of the solver

Numerical strategy #2: conic programming

$$\mathcal{J}(\mu,\nu) = \min \left\{ \iint f\left(\frac{dq}{d\gamma}\right) d\gamma \ : \ \begin{array}{l} \gamma \in \mathrm{P}(\mathbb{R}^d \times \ldots \\ q \in \mathcal{M}^b(\mathbb{R}^d \times \ldots \end{array} \right.$$

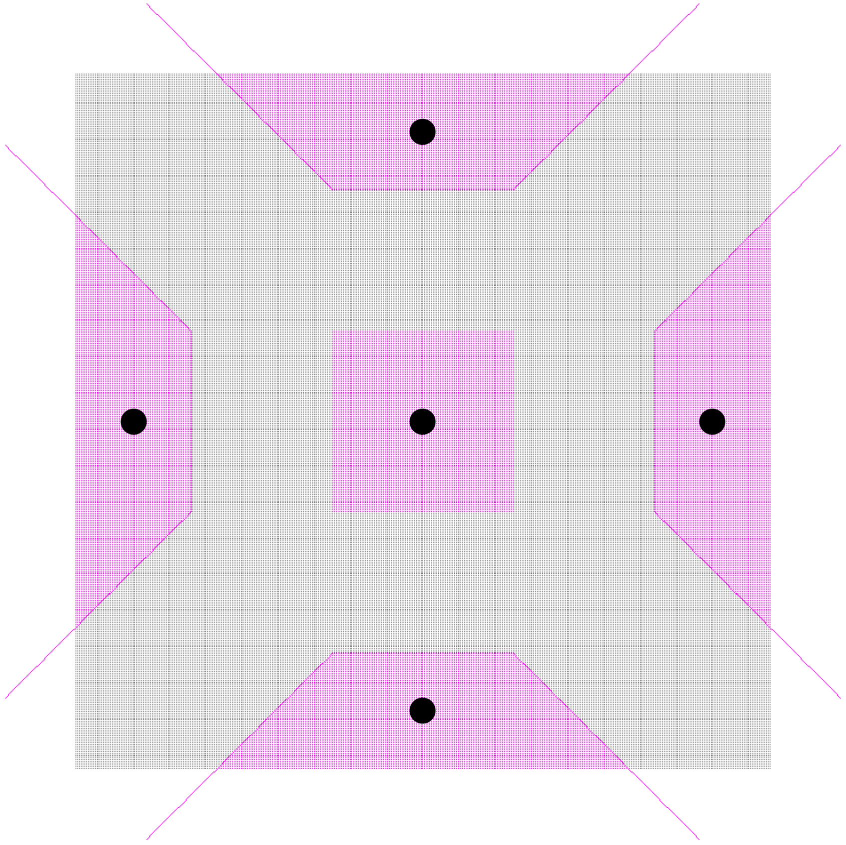
$$\mathcal{J}(\mu,\nu) = \sup_{\substack{\varphi,\psi\in C(\mathbb{R}^d) \\ \Phi,\Psi\in C(\mathbb{R}^d;\mathbb{R}^d)}} \left\{ \int \varphi\, d\mu + \int \psi\, d\nu \ : \ \varphi(x) + \ldots \right.$$

If μ,ν are discrete and $f = |\cdot|^p$, then $(\widetilde{\mathcal{P}})$, $(\widetilde{\mathcal{P}}^*)$ is a finite dimiensional conic programming problem.

No need for discretizing the space!

The conic program can be tackled by an off-the-shelf software, e.g. MOSEK.

Example: Lebesgue vs 5 Dirac masses



- μ : 400×400 pts
- CPU time: 46.5s

Exact solution up to the accuracy of the solver

Numerical strategy #2: conic programming

$$\mathcal{J}(\mu,\nu) = \min \left\{ \iint f\left(\frac{dq}{d\gamma}\right) d\gamma \ : \ \begin{array}{l} \gamma \in \mathrm{P}(\mathbb{R}^d \times \\ q \in \mathcal{M}^b(\mathbb{R}^d \times \end{array} \right.$$

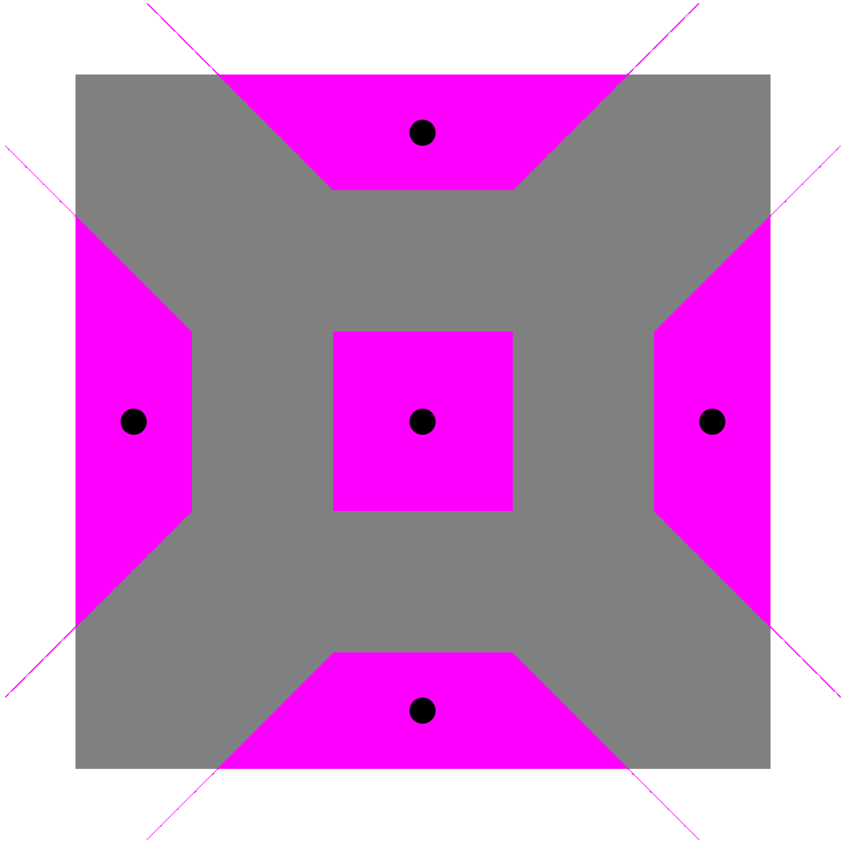
$$\mathcal{J}(\mu,\nu) = \sup_{\substack{\varphi,\psi \in C(\mathbb{R}^d) \\ \Phi,\Psi \in C(\mathbb{R}^d;\mathbb{R}^d)}} \left\{ \int \varphi \, d\mu + \int \psi \, d\nu \ : \ \varphi(x) + \right.$$

If μ, ν are discrete and $f = |\cdot|^p$, then $(\widetilde{\mathcal{P}})$, $(\widetilde{\mathcal{P}}^*)$ is a finite dimiensional conic programming problem.

No need for discretizing the space!

The conic program can be tackled by an off-the-shelf software, e.g. MOSEK.

Example: Lebesgue vs 5 Dirac masses



- μ : 800 $\times$ 800 pts
- CPU time: 4.6min

Exact solution up to the accuracy of the solver

Numerical strategy #3: entropic regularization

$$\mathcal{J}(\mu,\nu) = \min\left\{ \iint f\left(\frac{dq}{d\gamma}\right) d\gamma \;:\; \begin{array}{lll} \gamma \in \mathrm{P}\big(\mathbb{R}^d \times \mathbb{R}^d\big), & \pi_x\#\gamma = \mu, & \pi_x\#\, x\,\gamma(dxdy) = \pi_x\#q, \\ q \in \mathcal{M}^b(\mathbb{R}^d \times \mathbb{R}^d; \mathbb{R}^d), & \pi_y\#\gamma = \nu, & \pi_y\#\, y\,\gamma(dxdy) = \pi_y\#q \end{array} \right\} \qquad (\tilde{\mathrm{P}})$$

$$\mathcal{J}(\mu,\nu) = \sup_{\substack{\varphi,\psi \in C(\mathbb{R}^d) \\ \Phi,\Psi \in C(\mathbb{R}^d;\mathbb{R}^d)}} \left\{ \int \varphi\, d\mu + \int \psi\, d\nu \;:\; \varphi(x) + \psi(y) \leq \underbrace{\langle \Phi(x), x\rangle + \langle \Psi(y), y\rangle - f^*\big(\Phi(x) + \Psi(y)\big)}_{c_{\Phi,\Psi}(x,y)} \quad \forall\, x,y \right\} \qquad (\tilde{\mathrm{P}}^*)$$

Entropic regularization: $\big(h(r) = r\log(r) - r\big)$

$$\mathcal{J}_\varepsilon(\mu,\nu) = \sup_{\substack{\varphi,\psi \\ \Phi,\Psi}} \left\{ \int \varphi\, d\mu + \int \psi\, d\nu - \iint \exp\left(\frac{\varphi(x) + \psi(y) - \langle \Phi(x), x\rangle - \langle \Psi(y), y\rangle + f^*\big(\Phi(x) + \Psi(y)\big)}{\varepsilon} \right) \mu\otimes\nu(dxdy) \right\} \qquad (\tilde{\mathrm{P}}^*_\varepsilon)$$

$$\mathcal{J}_\varepsilon(\mu,\nu) = \min_{\gamma,q} \left\{ \iint f\left(\frac{dq}{d\gamma}\right) d\gamma + \iint \varepsilon\, h\left(\frac{d\gamma}{d\mu\otimes\nu}\right) d\mu\otimes d\nu \;:\; \begin{array}{ll} \pi_x\#\gamma = \mu, & \pi_x\#\, x\,\gamma(dxdy) = \pi_x\#q, \\ \pi_y\#\gamma = \nu, & \pi_y\#\, y\,\gamma(dxdy) = \pi_y\#q \end{array} \right\} \qquad (\tilde{\mathrm{P}}_\varepsilon)$$

$\gamma \ll \mu \otimes \nu$

Numerical strategy #3: entropic regularization

$$\mathcal{J}_\varepsilon(\mu,\nu) = \sup_{\substack{\varphi,\psi\\ \Phi,\Psi}} \left\{ \int \varphi\, d\mu + \int \psi\, d\nu - \iint \exp\left(\frac{\varphi(x)+\psi(y) - \langle \Phi(x), x\rangle - \langle \Psi(y), y\rangle + f^*\big(\Phi(x)+\Psi(y)\big)}{\varepsilon} \right) \mu\otimes\nu(dxdy) \right\}$$

$(\tilde{P}^*_\varepsilon)$

$$\mathcal{J}_\varepsilon(\mu,\nu) = \min_{\gamma,q} \left\{ \iint f\left(\frac{dq}{d\gamma}\right) d\gamma + \iint \varepsilon\, h\left(\frac{d\gamma}{d\mu\otimes\nu}\right) d\mu\otimes d\nu \ : \ \begin{array}{ll} \pi_x\#\gamma = \mu, & \pi_x\#\, x\,\gamma(dxdy) = \pi_x\#q, \\ \pi_y\#\gamma = \nu, & \pi_y\#\, y\,\gamma(dxdy) = \pi_y\#q \end{array} \right\}$$

$(\tilde{P}_\varepsilon)$

Theorem

Take a solution $\varphi, \Phi, \psi, \Psi$ of $(\widetilde{\mathcal{P}}^*_\varepsilon)$ and put

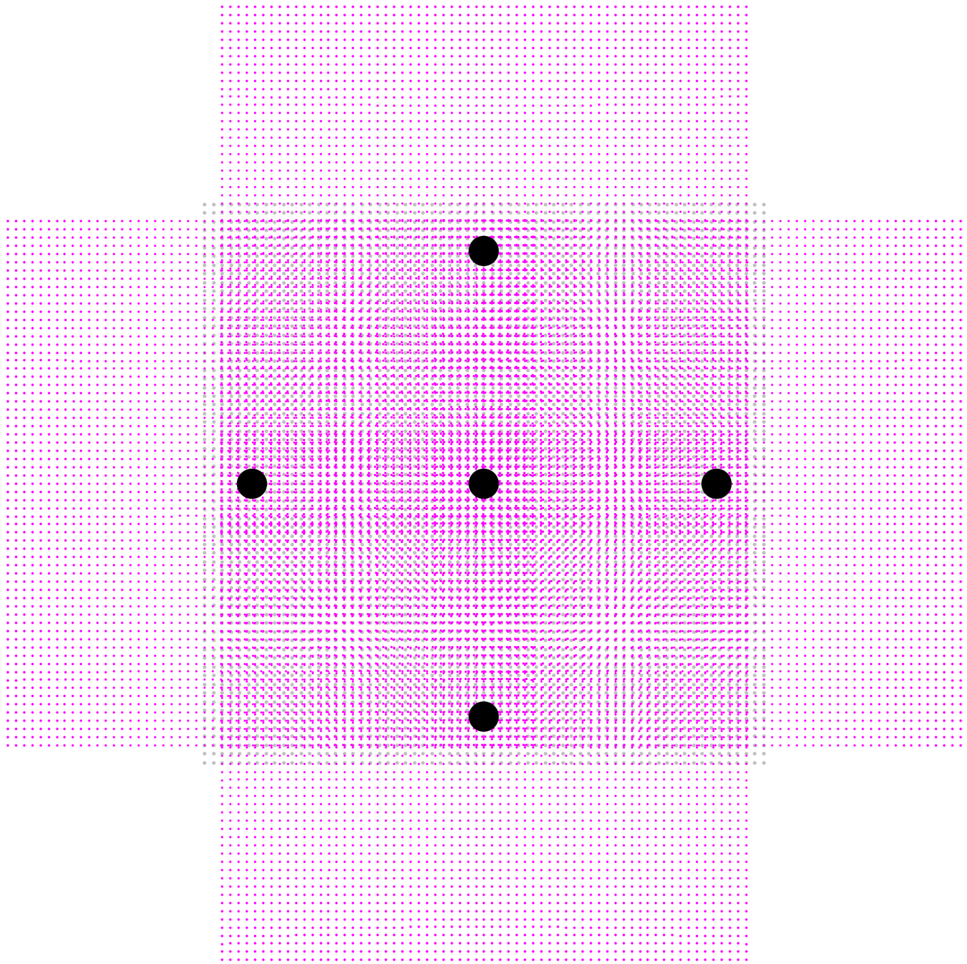
$$\gamma(dxdy) = \exp\left(\frac{\varphi(x)+\psi(y) - \langle \Phi(x), x\rangle - \langle \Psi(y), y\rangle + f^*(\Phi(x)+\Psi(y))}{\varepsilon} \right) \mu\otimes\nu(dxdy)$$

$$\zeta(x,y) = \nabla f^*\big(\Phi(x)+\Psi(y)\big)$$

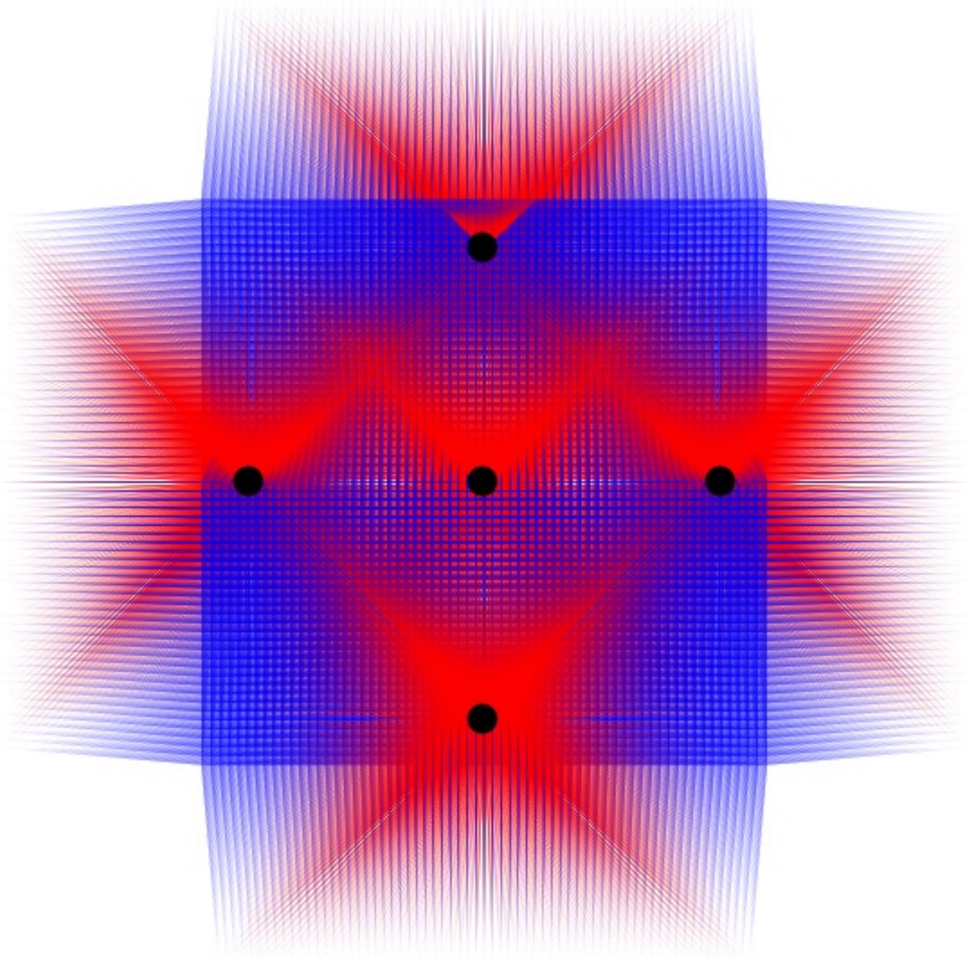
Then the following pair solves $(\widetilde{\mathcal{P}}_\varepsilon)$:

$$(\gamma, q) = (\gamma, \zeta\gamma)$$

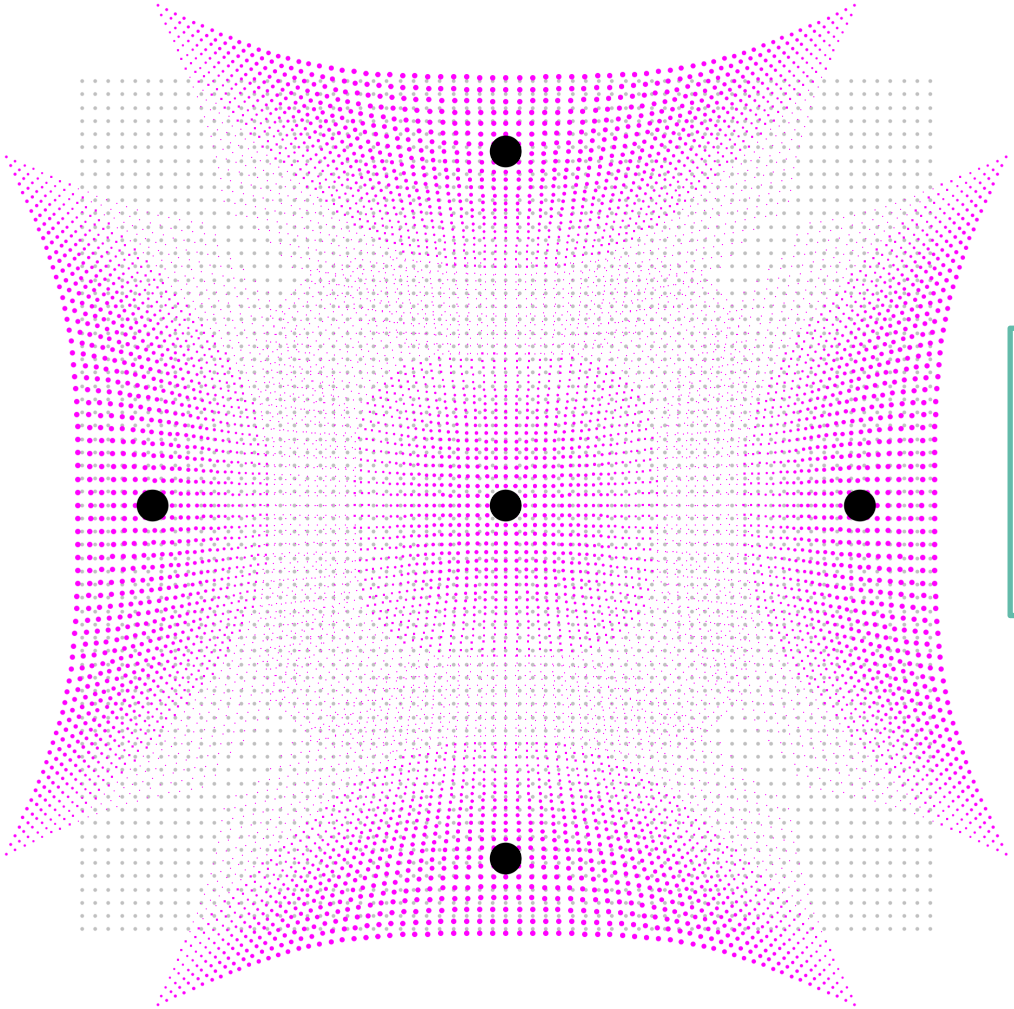
Solution by (modified) Sinkhorn algorithm



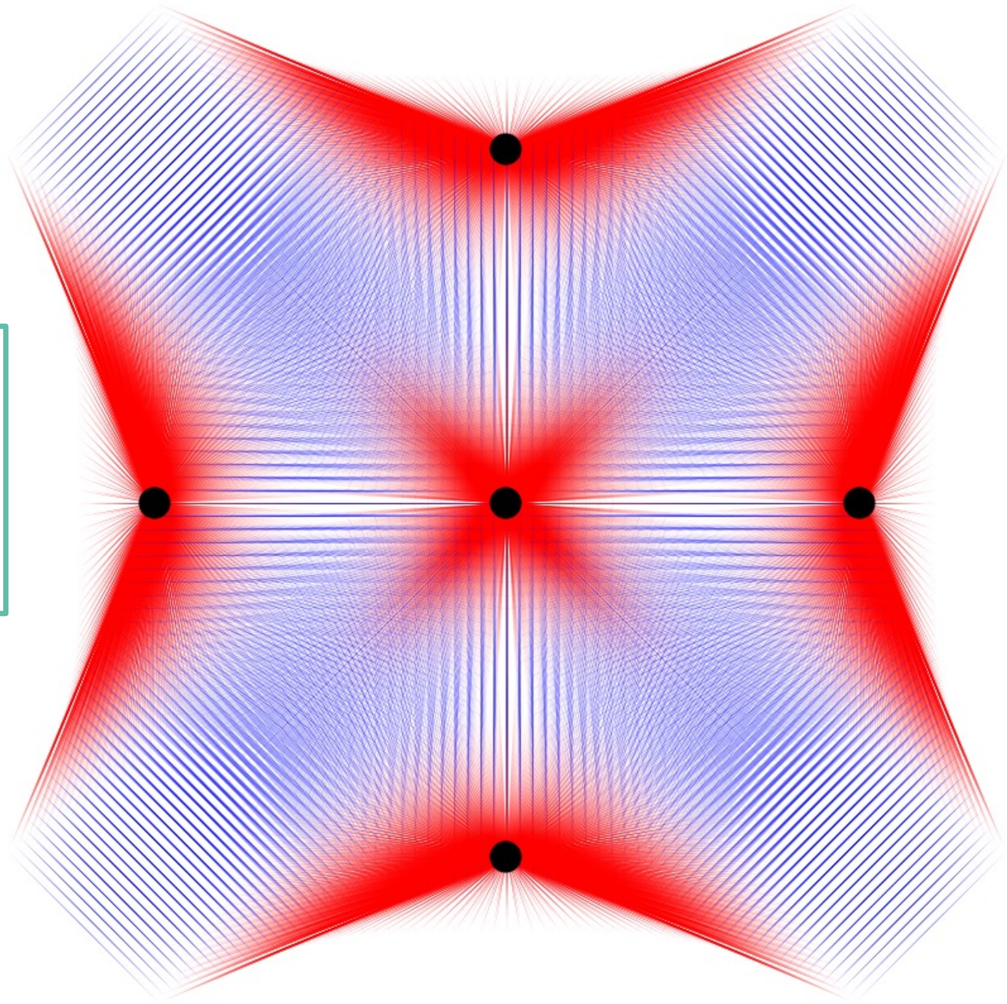
- $\varepsilon = 1.0$
- iter: 17
- CPU time: 0.07s



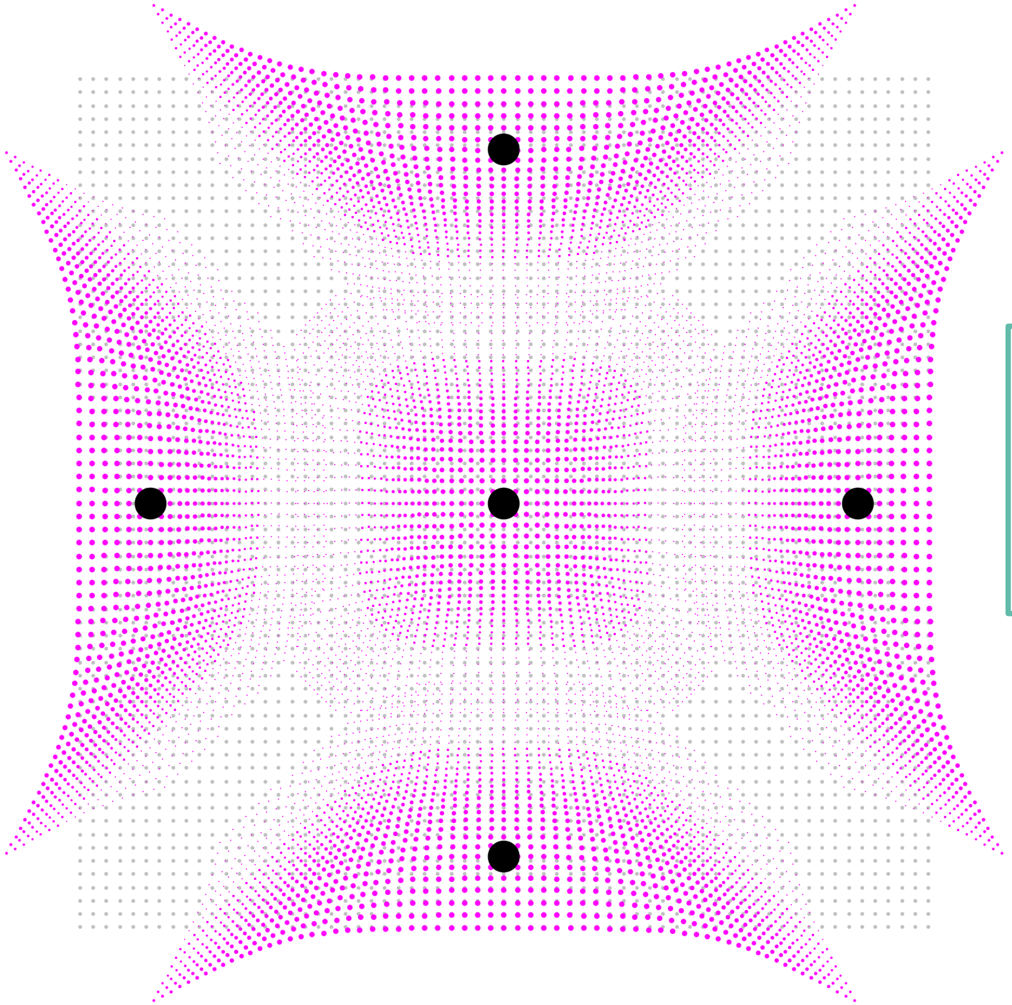
Solution by (modified) Sinkhorn algorithm



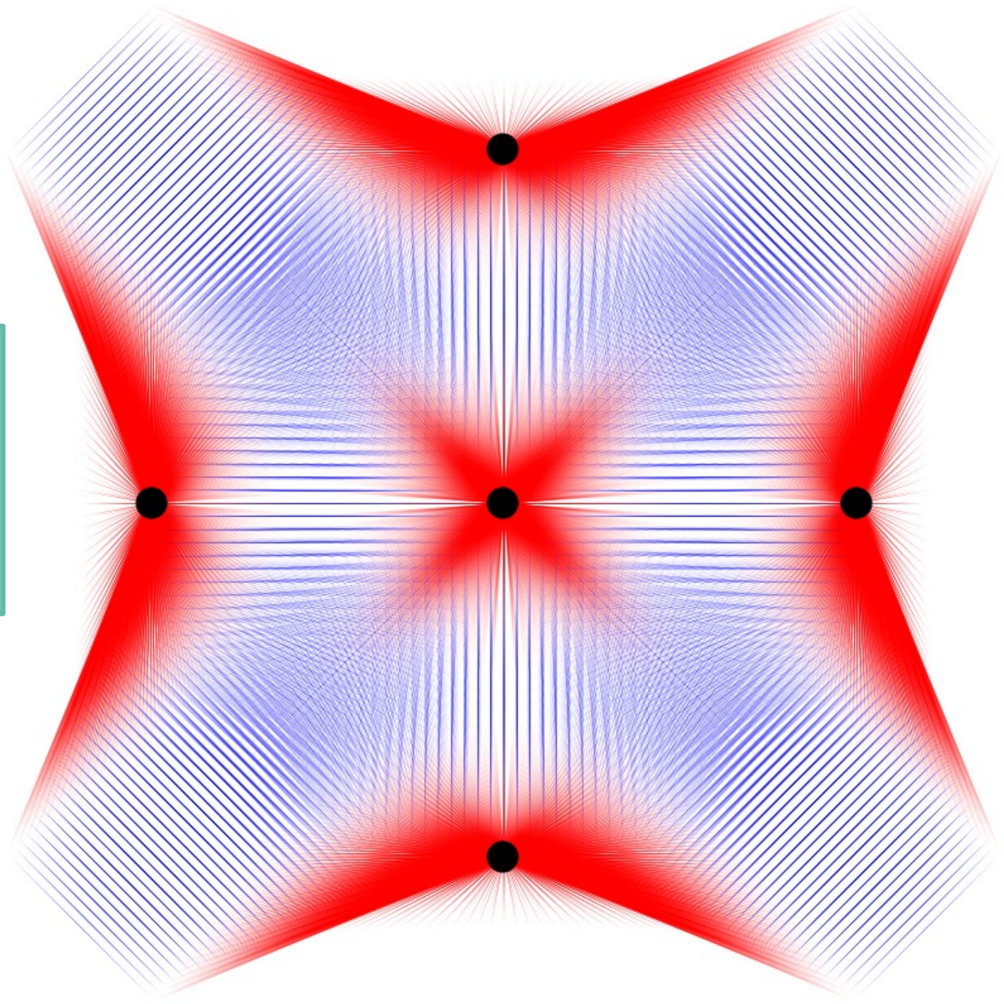
- $\varepsilon = 0.01$
- iter: 35
- CPU time: 0.13s



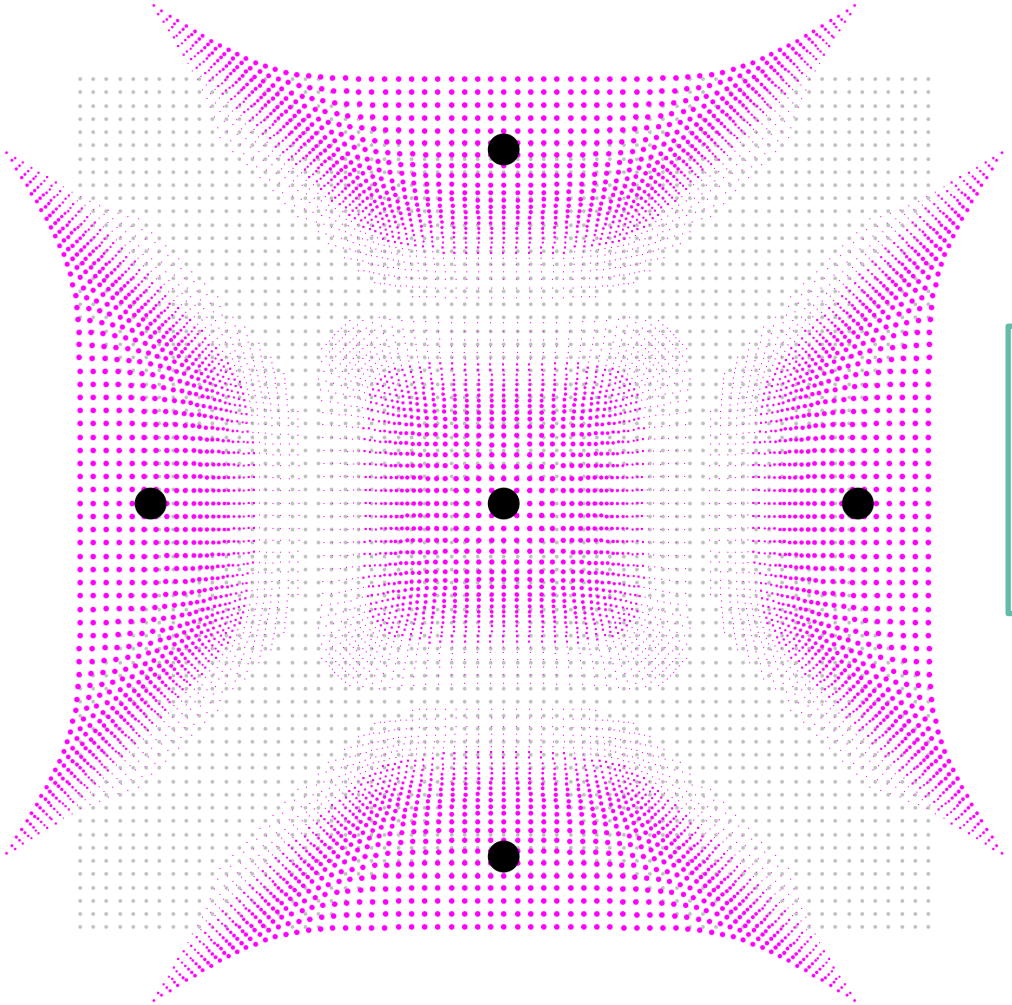
Solution by (modified) Sinkhorn algorithm



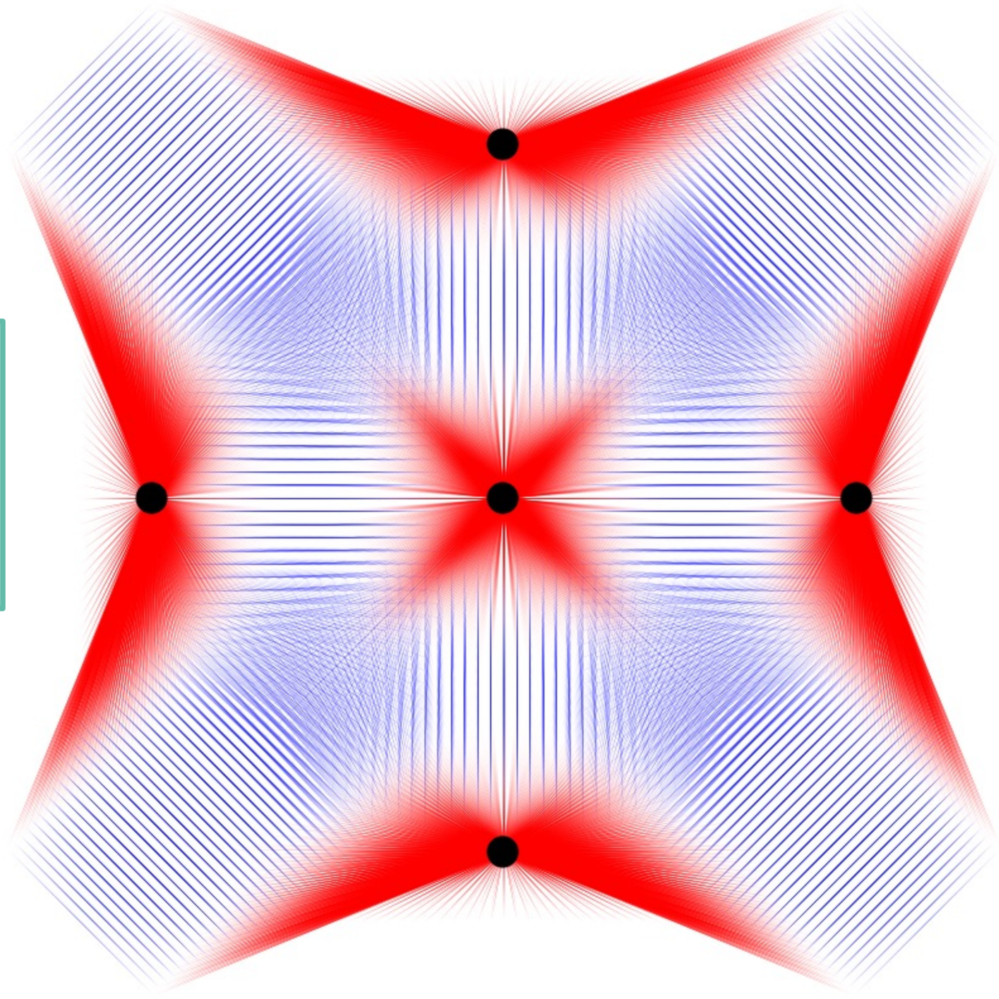
- $\varepsilon = 0.007$
- iter: 53
- CPU time: 0.19s



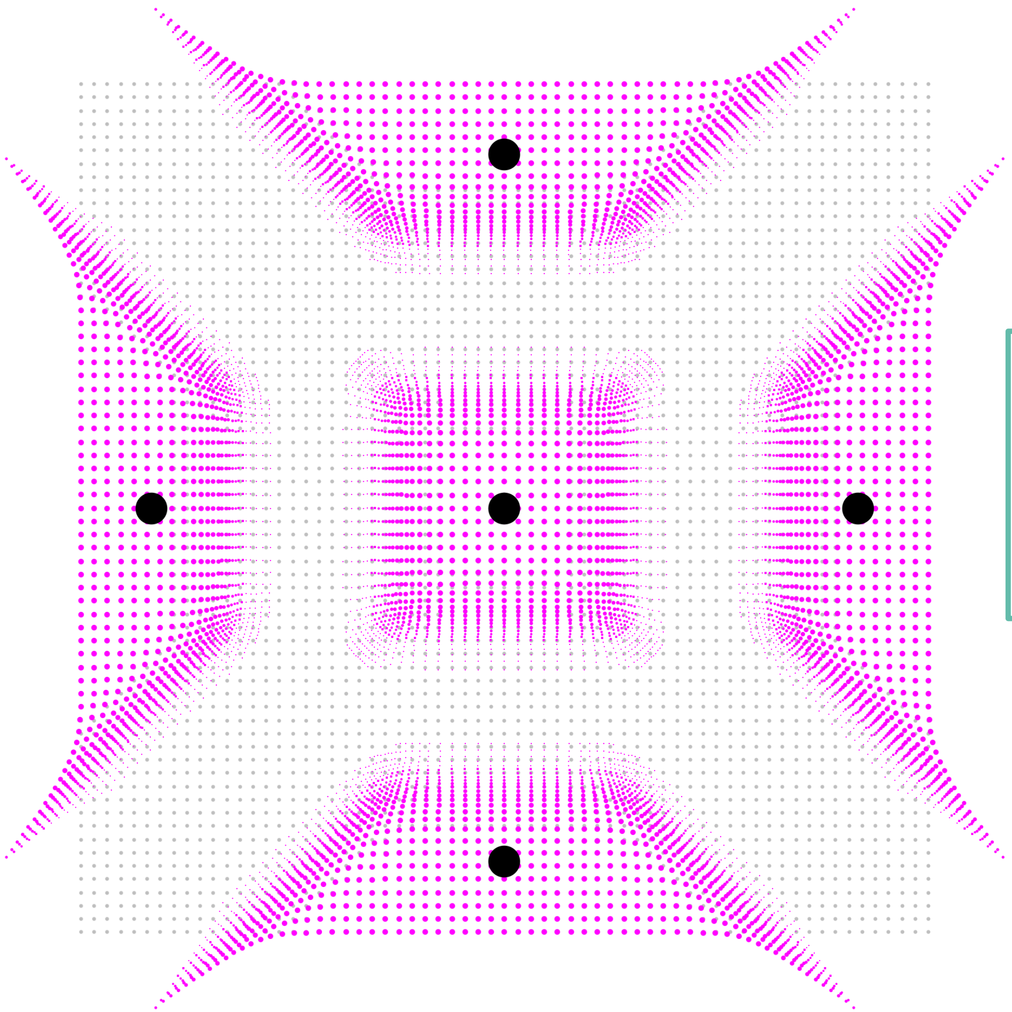
Solution by (modified) Sinkhorn algorithm



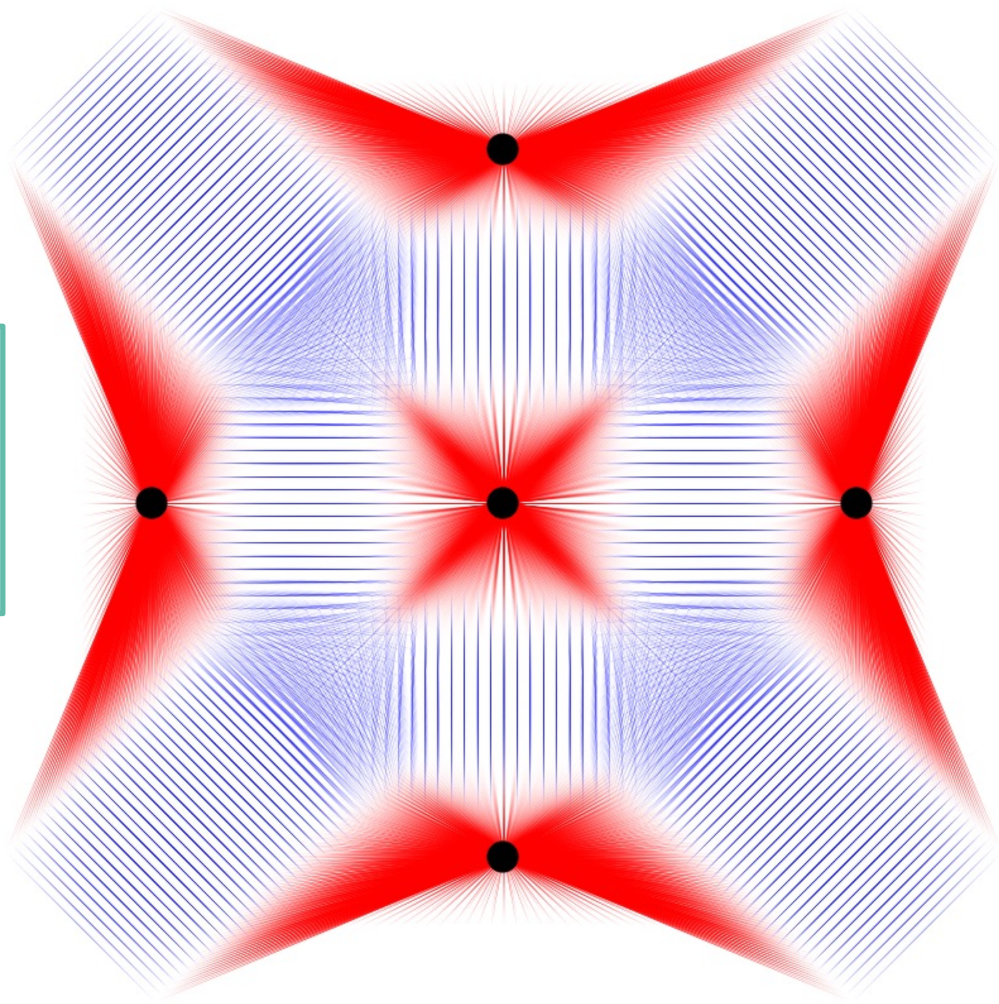
- $\varepsilon = 0.005$
- iter: 76
- CPU time: 0.27s



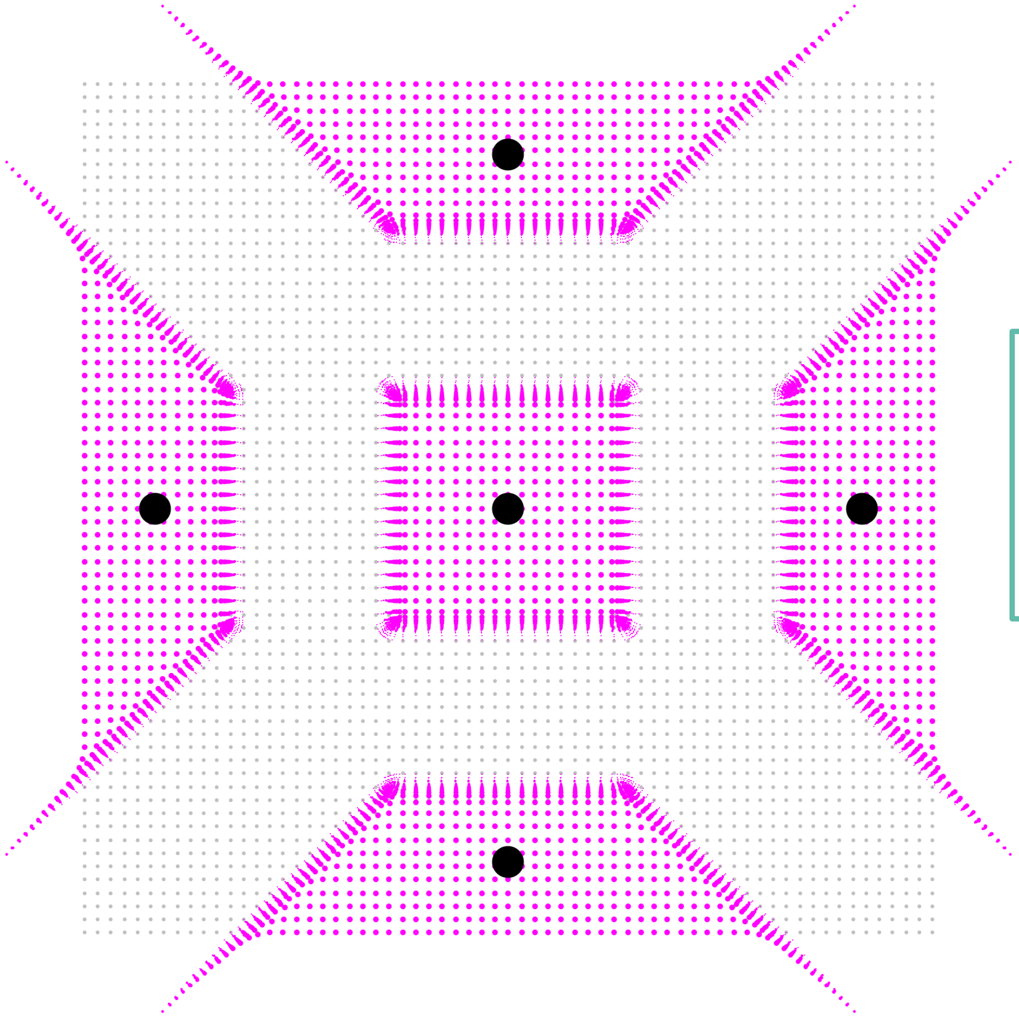
Solution by (modified) Sinkhorn algorithm



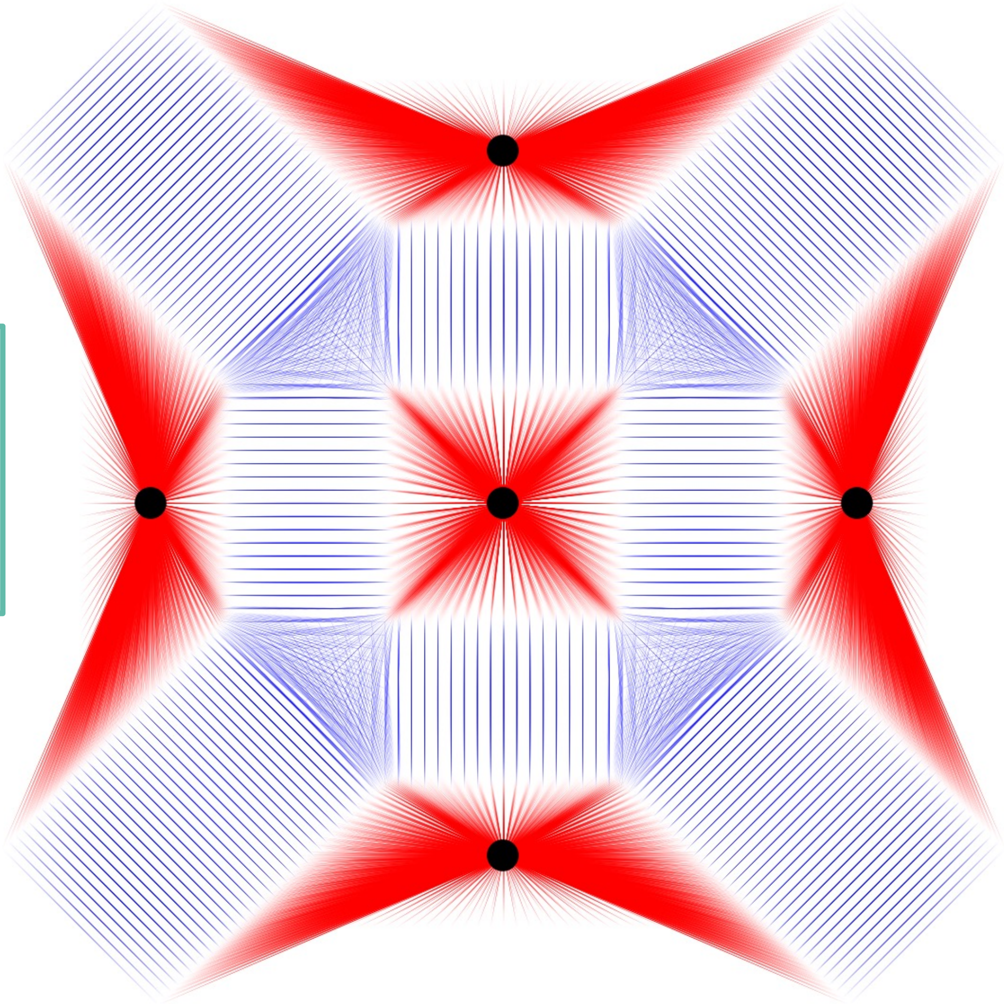
- $\varepsilon = 0.003$
- iter: 125
- CPU time: 0.41s



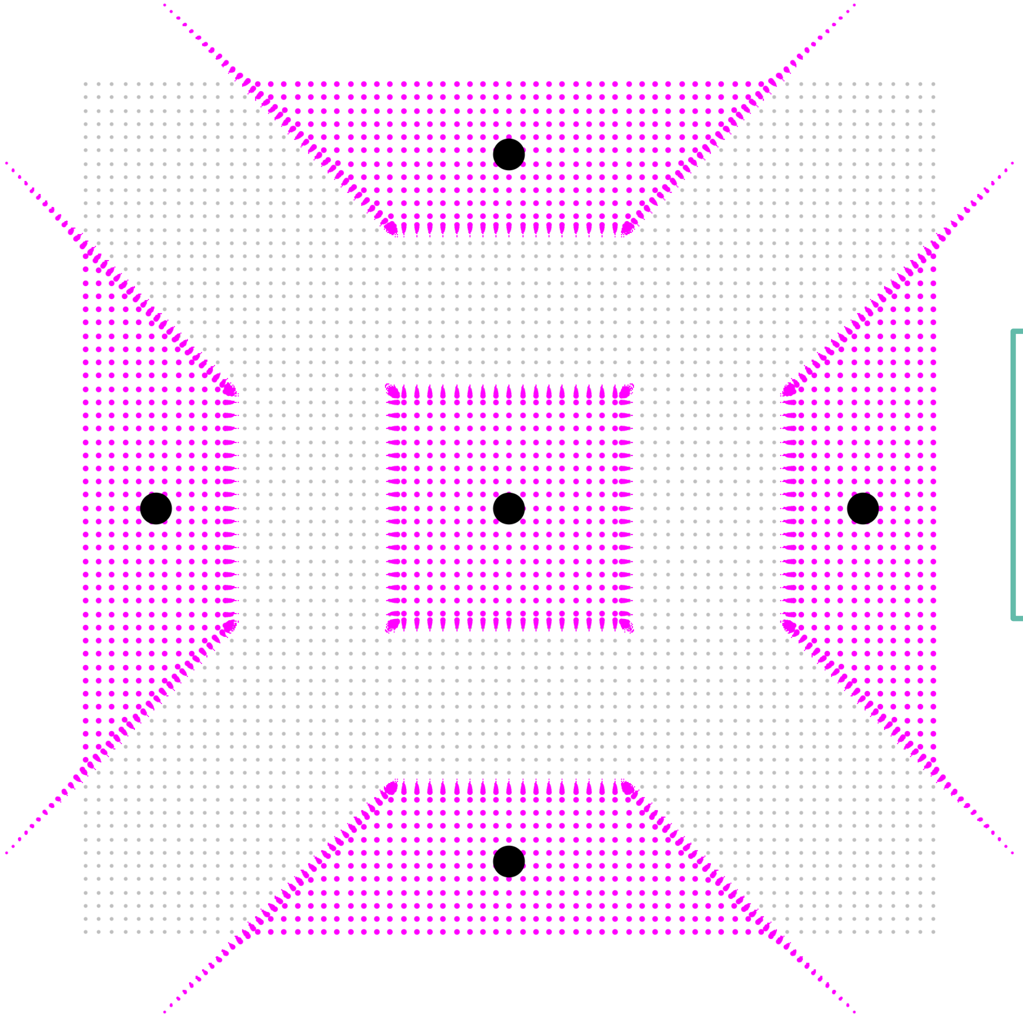
Solution by (modified) Sinkhorn algorithm



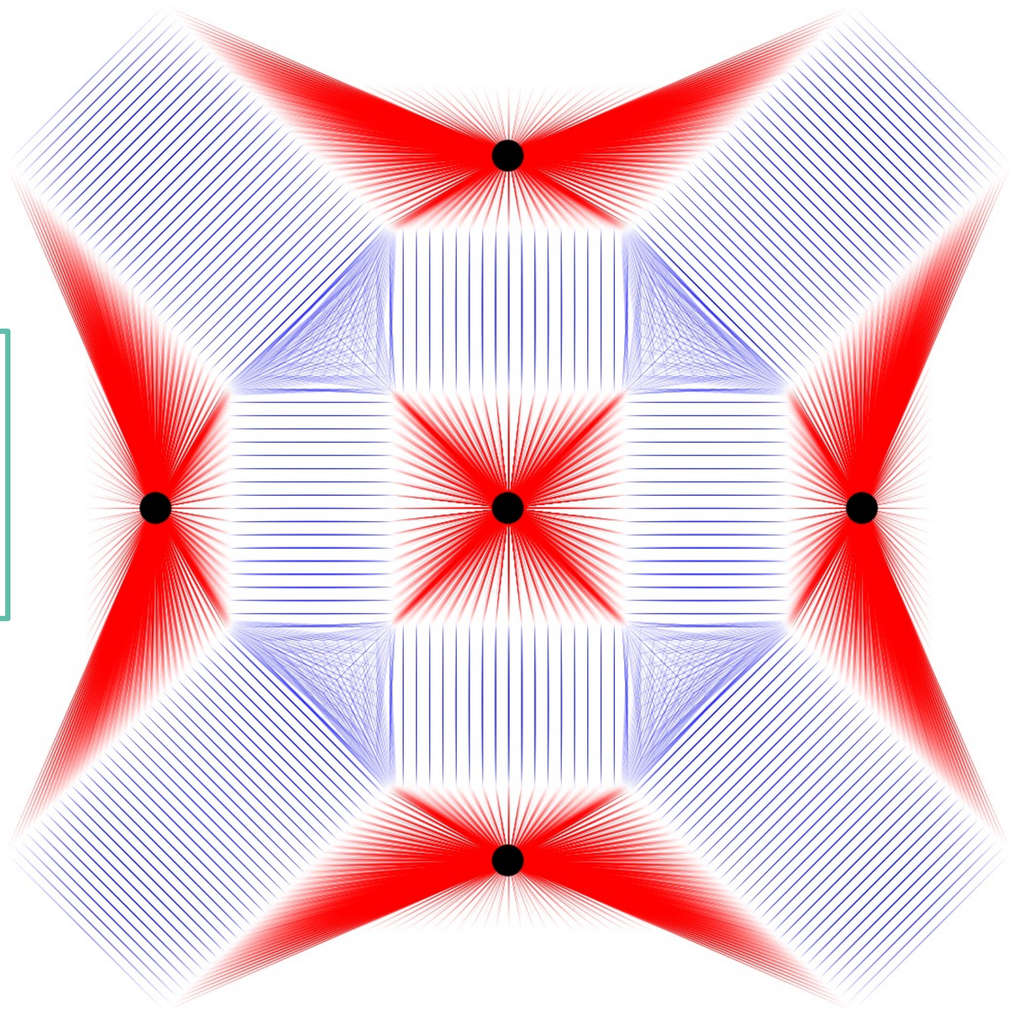
- $\varepsilon = 0.001$
- iter: 341
- CPU time: 1.1s



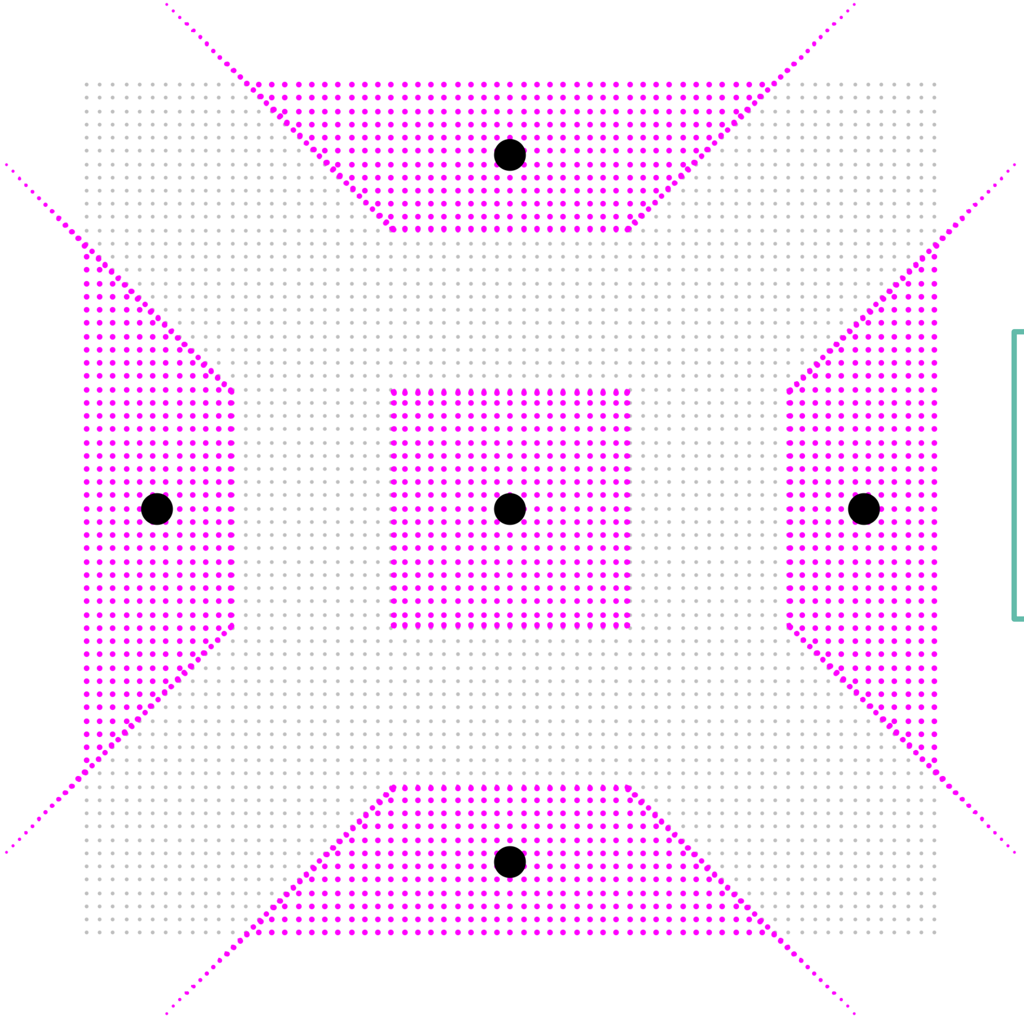
Solution by (modified) Sinkhorn algorithm



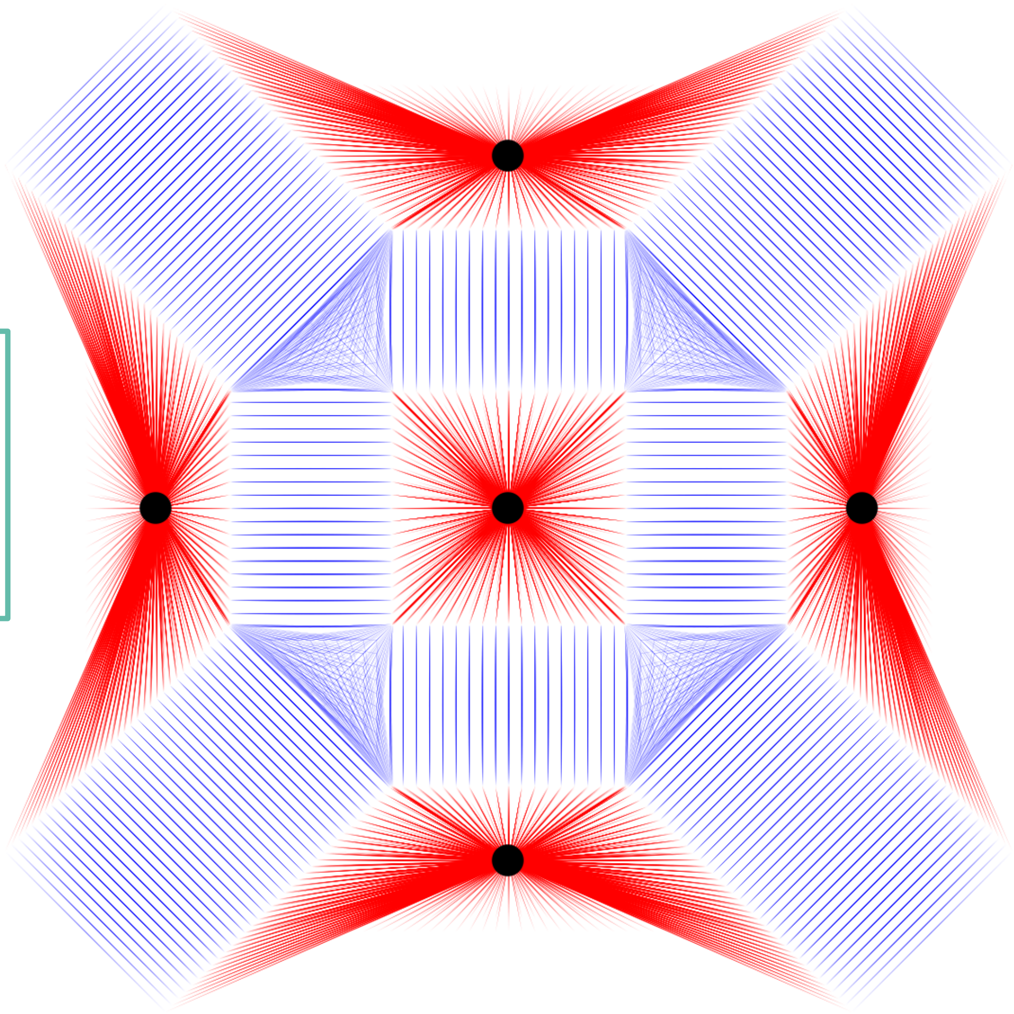
- $\varepsilon = 0.0005$
- iter: 646
- CPU time: 2.1s



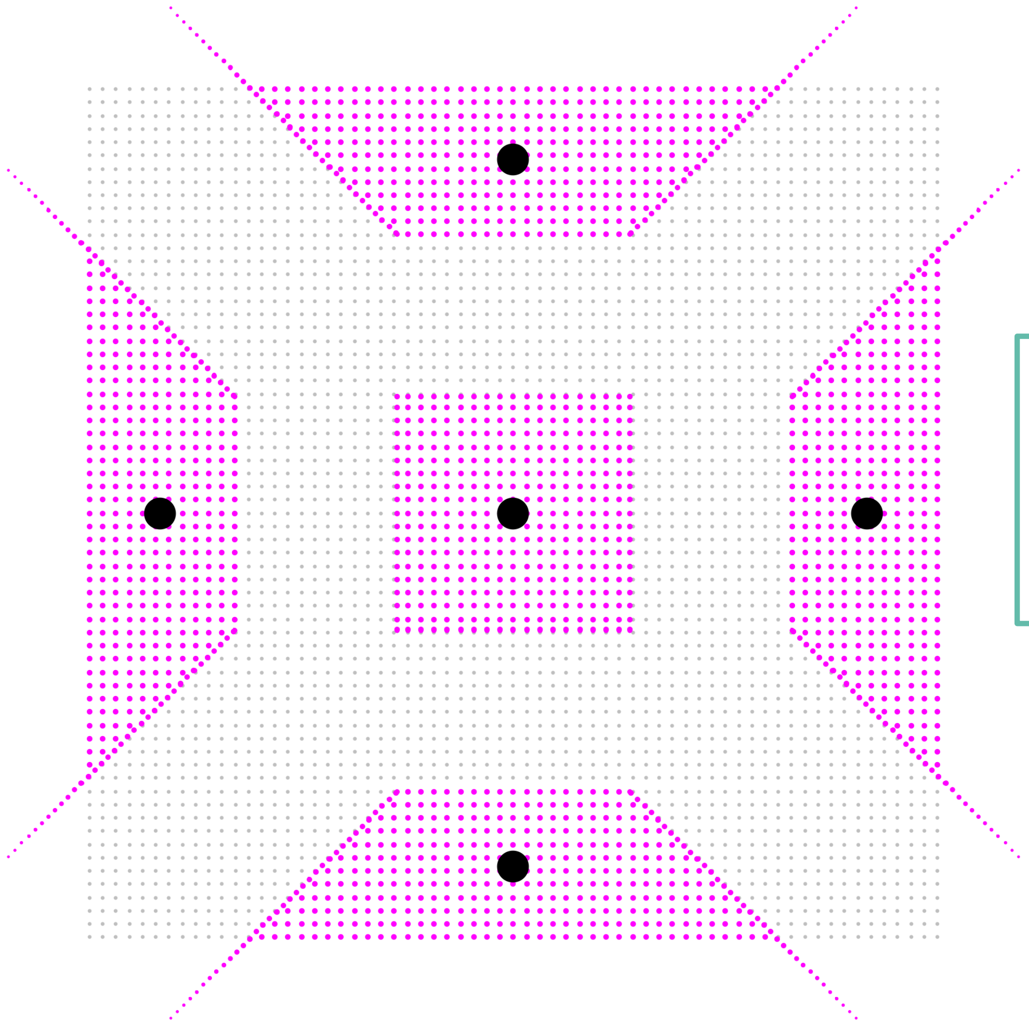
Solution by (modified) Sinkhorn algorithm



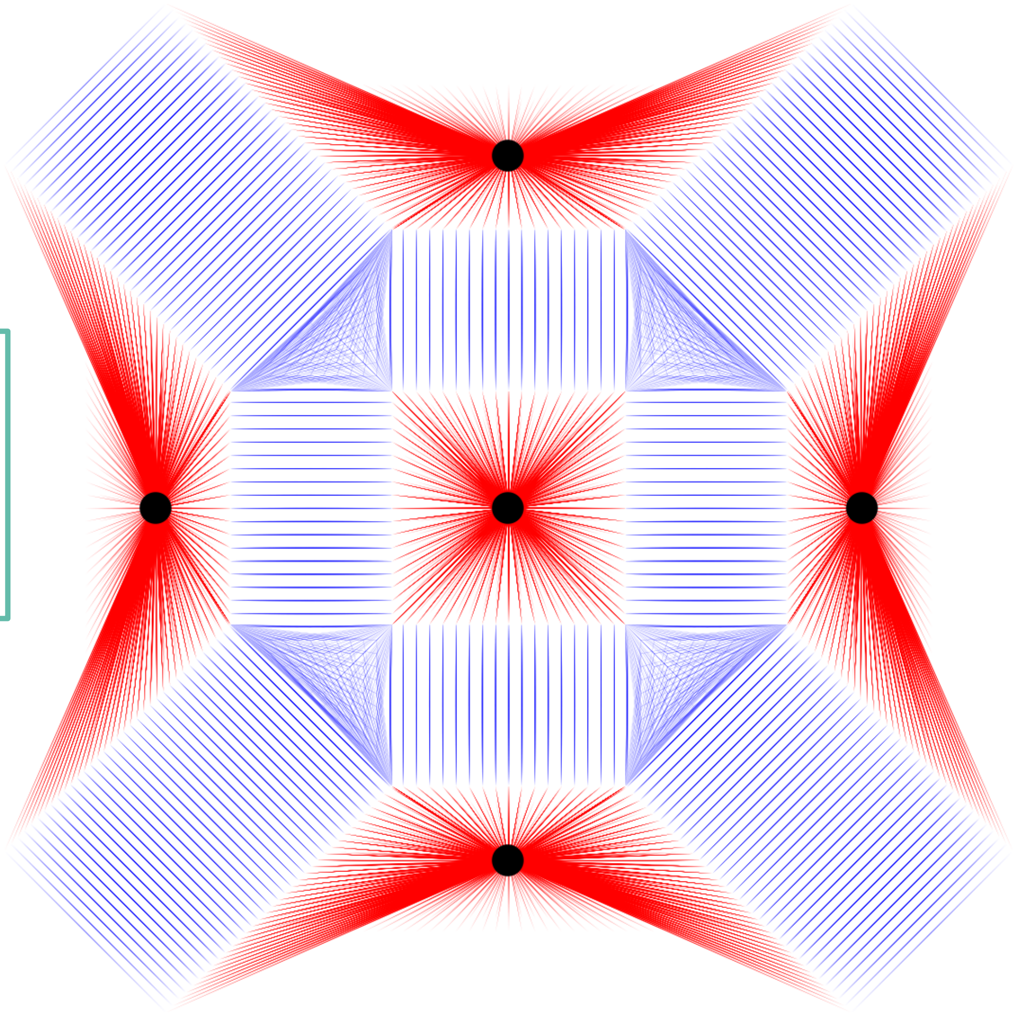
- $\varepsilon = 0.0001$
- iter: 2819
- CPU time: 7.5s



Solution by (modified) Sinkhorn algorithm



- $\varepsilon = 0.00001$
- iter: 23044
- CPU time: 60s



Thank you for your attention!