Thematic session Benasque 2024

Shape optimization problems in analysis and in mechanics

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Optimal design versus optimal transport

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Shape optimization in mechanical engineering

In many applications, the optimal shape is characterized as the support of an optimal stress measure $\sigma : \mathbb{R}^d \mapsto \mathcal{S}^{d\times d}$ (d=2,3). An important case is when the criterium to be minimized is the compliance:

- Michell's truss problem ($\sigma : \Omega \subset \mathbb{R}^2 \mapsto \mathcal{S}^{2\times 2}$)
- Optimal grillage problem ($\sigma : \Omega \subset \mathbb{R}^2 \mapsto \mathcal{S}^{2\times 2}$)
- pre-stessed elastic membrane ($\sigma : \Omega \subset \mathbb{R}^2 \mapsto \mathcal{S}_+^{2\times 2}$)
    - Prager problem ($\sigma : \Omega \times (-h, h) \subset \mathbb{R}^3 \mapsto \mathcal{S}^{3\times 3}$)
    - optimal vault problem (parametrized surface $z = u(x, y)$)

A general framework for the optimal compliance problem

- $\Omega \subset \mathbb{R}^d$ design domain ($d = 2, 3$)
- $F \in \mathcal{M}(\overline{\Omega}; \mathbb{R}^n)$ a source or a load
- Σ_0 a compact subset of $\overline{\Omega}$ (Dirichlet zone)
- $A : \mathcal{D}(\mathbb{R}^d; \mathbb{R}^n) \to \mathcal{D}(\mathbb{R}^d; Y)$ a linear differential operator
- $\rho : Y \to \mathbb{R}_+$ a norm on the finite dimentional Euclidean space Y.

Given $m_0 > 0$ and $p > 1$, we look for a measure μ solving :

$$\min_{\mu \in \mathcal{M}_+(\overline{\Omega})} \left\{ \mathcal{C}(\mu) \ : \ \mu(\overline{\Omega}) \leq m_0 \right\} \qquad (MOP)$$

where the compliance $\mathcal{C}(\mu)$ is the convex, weak* l.s.c. functional:

$$\boxed{\mathcal{C}(\mu) = \sup_{u \in \mathcal{D}(\mathbb{R}^d; \mathbb{R}^n)} \left\{ < F, u > - \frac{1}{p} \int |\rho(Au)|^p \, d\mu \ : \ u = 0 \text{ in } \Sigma_0 \right\}}$$

From (MOP) to a linear program

Let $\mathcal{I} = \mathcal{I}(F, \Omega, \Sigma_0)$ given by the linear constraint program:

$$\boxed{\mathcal{I} := \sup_{u\lfloor_{\Sigma_0}=0} \left\{< F, u >: \ \rho(Au) \leq 1 \text{ in } \overline{\Omega}\right\}} \qquad (LCP)$$

Then by classical duality we get:

$$\min(MOP) = \frac{\mathcal{I}^{p'}}{p'} \frac{1}{m_0^{p'-1}}.$$

Proof: $$\inf_{\mu(\overline{\Omega})\leq m_0} \sup_{u\lfloor_{\Sigma_0}=0} = \sup_{u\lfloor_{\Sigma_0}=0} \inf_{\mu(\overline{\Omega})\leq m} \left\{< F, u > - \frac{1}{p} \int (\rho(Au))^p \, d\mu\right\}$$

$$= \sup_{u\lfloor_{\Sigma_0}=0} \left\{< F, u > - \frac{m_0}{p} (\sup_{\overline{\Omega}} \rho(Au))^p\right\}$$

Recovering optimal μ and stress measure σ

(I. Fragala-GB , ARMA 2007)

- By dualizing (LCP) from $C^0(\overline{\Omega}; Y)$ to $\mathcal{M}(\overline{\Omega}; Y)$, we arrive to the so called *Beckman's* formulation:

$$\inf_{\sigma \in \mathcal{M}(\overline{\Omega}; M_{d,p})} \left\{ \int \rho^0(\sigma) : \ A^*(\sigma) = F \text{ in } \mathcal{D}'(\mathbb{R}^d \setminus \Sigma_0) \right\}$$

being ρ^0 the dual norm:

$$\boxed{\rho^0(S) = \sup\{< S, M >: \rho(M) \leq 1\}}.$$

- The polar decomposition $\sigma = S\ \mu$ with the normalization $\rho^0(S) = k$ of any solution σ provides an optimal measure μ for (MOP); the constant k is tuned so that $\int \mu = m_0$.

Two main questions related to the differentail operator A

- *Connection with Monge mass transport:* $\rightsquigarrow$ construction of optimal measures μ as superposition of mass transport along geodesic curves

- *Support of optimal measures:* for given F and Σ_0, we expect that if the design domain Ω is all $\mathbb{R}^d$ or an open ball B_{R_0} of large radius, any optimal σ will be compactly supported in B_{R_0}.
  - In the first order gradient casesc, it can be proved that $\text{spt}(\sigma) \subset \text{co}(\Sigma_0 \cup \text{spt}(F))$.
  - In all the other cases, this is a quite challenging issue to obtain an estimate of R_0 (even the existence of $R_0 < +\infty$) . The grail would be to determine a safety compact using a geometric construction starting from data F and Σ_0.

Important examples

1- **The heat equation:** $A = \nabla, \ \rho(z) = |z|$

2- **Michell's problem:** $u : \mathbb{R}^2 \to \mathbb{R}^2, \ Au = \frac{1}{2}(Du + Du^T)$ and $\rho(M) =$ spectral norm of $M \in S^{2\times 2}$

3- **The optimal pre-stressed membrane Pb:**
$(u, w) : \mathbb{R}^2 \to \mathbb{R} \times \mathbb{R}^2, A(u, w) = (\nabla u, e(w))$ and
$\rho = j_C(z, M) \quad , \quad C = \{(z, M) \ : \ \frac{1}{2} z \otimes z + M \leq I_d\}.$
($v = id - w$ is a maximal monotone map which induces a natural metric)

4- **The optimal grillage problem:** $u : \mathbb{R}^2 \to \mathbb{R}, \ Au = \nabla^2 u$ and ρ the spectral norm.

Some references

- **Old:**
  - G.Buttazzo, GB, P.Seppecher: Characterization of optimal shapes and masses through Monge-Kantorovich equation., *CRAS(1997), JEMS (2001)*
  - I. Fragala, GB, *ARMA (2007)*
  - W.Gangbo, P.Seppecher, GB, *M3AS (2008)*
  - I.Fragala, P.Seppecher, GB *ARMA (2011)*
  - I.Fragala, I.Lucardesi, P. Seppecher GB, *SIMA (2012)*
- **New:**
  - K. Bolbotwski, GB: Optimal design versus maximal Monge-Kantorovich metrics, *Arch. Ration. Mech. Anal. 243 (2022), no. 3, 1449-1524.*
  - Karol Bolbotowski: Optimal vault problem – form finding through 2D convex program, *Comp. and Math. with Appli., (2022)*
  - K. Bolbotwski, GB:Kantorovich-Rubinstein duality theory for the Hessian, *Forthcoming .*

Notes on Michell's problem

- **Originally:** 2d -problem of finding trusses of elastic bars (in compression or in tension) supporting a given load $F : \overline{\Omega} \to \mathbb{R}^2$ and with minimal volume (Michell -1920). It is a discrete problem where many explicit solutions are known (see Lewinsky book).

- **Nowadays:** Optimal compliance of elastic structure of infinitesimal volume Kohn-Allaire (1993), Gangbo Seppecher-GB (2008), Olberman, Babadjian-Rindler-Urlano (2020-2023), GB -vanishing mass conj (2001)

Classical duality

- **Strain problem** We assume that F is compactly supported and orthogonal to rigid motions $\int F = 0$ and $\int (x_1 F_2 - x_2 F_1) = 0$. Then $\exists u \in \cap_{p<\infty} W^{1,p}(\Omega)$ solving:

$$\mathcal{I}(F,\Omega) := \sup\{< F, u > : \ \rho(e(u) \leq 1 \quad \text{a.e.in } \Omega\} \qquad (LCP)$$

 where ρ is the spectral norm. The contraint on u can be recast by a two-point condition:

$$\boxed{|< u(x) - u(y), x - y >| \ \leq |x - y|^2 \quad \forall (x, y) \in \overline{\Omega}^2}$$

- **Stress problem** The unknown is a tensor measure $\sigma \in \mathcal{M}(\overline{\Omega}, S^{2\times 2})$ minimizing Beckman's problem:

$$\min \left\{ \int \rho^0(\sigma) \ : \ \text{spt}(\sigma) \subset \overline{\Omega}\ , \ -\operatorname{div} \sigma = F \right\} = \mathcal{I}(F, \Omega),$$

 where $\rho^0(S) = |\lambda_1(S)| + |\lambda_2(S)|$ (the Schatten norm).

Looking for optimal truss-like structures

If F is discrete, a natural approach consists in searching a minimum among finite trusses:

$$\sigma(\gamma) = \iint \sigma^{x,y}\,\gamma(dxdy)\ ,\ \sigma^{x,y} = \tau_{x,y} \otimes \tau_{x,y}\,\mathcal{H}^1 \llcorner [x,y]\ ,\ \tau_{x,y} := \frac{y-x}{|y-x|}$$

where $\gamma \in \mathcal{M}(\overline{\Omega}^2)$ is finitely supported. If the number N of bars is fixed, we obtain a linear program:

$$\min_{\sharp(\mathrm{spt}(\gamma)) \leq N} \left\{ \int_{\overline{\Omega}^2} |x-y|\,|\gamma|(dxdy)\ :\ \int_{\overline{\Omega}^2} (\delta_y - \delta_x)\tau_{x,y}\,d\gamma = F \right\}.$$

As $N \to \infty$, we expect a generalized optimal stress $\sigma(\gamma)$ to come out.

Existence of an optimal truss measure ?

Bad new ! A control of the cost $\int_{\overline{\Omega}^2} |x-y|\,|\gamma_N|$ along a minimizing sequence (γ_N) does not prevent $\iint |\gamma_N| \to +\infty$.
In fact curved bars may appear in the limit !

In the paper [Gangbo-Seppecher-GB (M3AS, 2008)], the class of bar stresses $\sigma^{x,y}$ is enlarged to $\{\sigma_C : C \in \mathcal{F}\}$ being $\mathcal{F}$ a class of Lipschitz curves with bounded curvature. This allowed to prove the optimality of some of these *generalized truss* measures γ supporting curves.

Remark In this representation, the curves in tension are associated with γ_+ and the ones in compression with γ_-. So we obtain two families of curves/bars which can only intersect orthogonally (in the principal directions of $e(u)$ for any u solving $\mathcal{I}(F,\Omega)$).

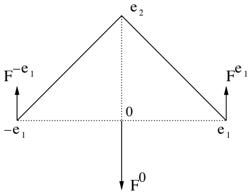
Open issues

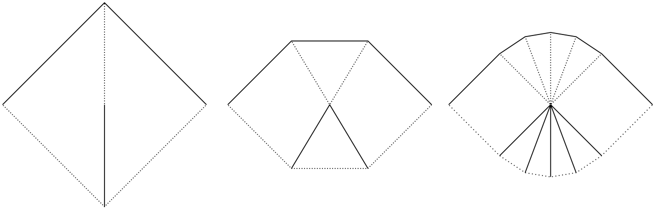
- Connection with Monge OT ? might be possible using OT for currents (see recent papers by B.Dacorogna and W.Gangbo) but, up to now...?
- Geometrical bounds on $\mathrm{spt}(\gamma)$? Clearly the inclusion $\mathrm{spt}(\gamma) \subset \mathrm{co}(\Sigma_0 \cup \mathrm{spt}(f))$ is false in the case of the bridge problem; A hope inspired from this example and from the forthcoming grillage problem would be that $\mathrm{spt}(\gamma) \subset \mathcal{B}(\Sigma_0 \cup \mathrm{spt}(f)$ where , for every $E \subset \mathbb{R}^2$:

$$\mathcal{B}(E) := \bigcup_{(x,y)\in E^2} B(\frac{x+y}{2}, \frac{x-y}{2}).$$

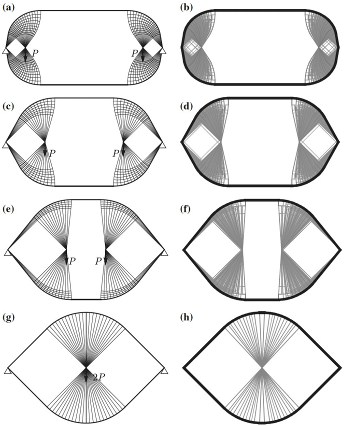
Unfortunately this enlarged set is still to small !

Bridge problem and minimizing sequence ($N = 5, 6, 10, 22, \ldots$)



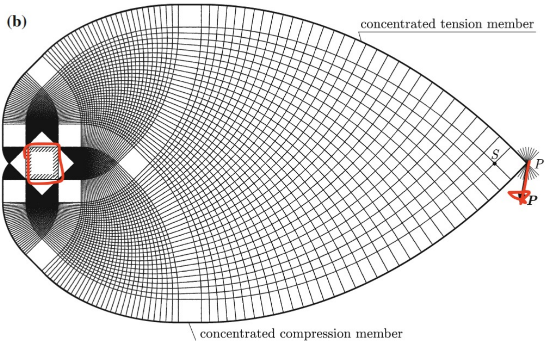


Four points variants, $\Omega = \mathbb{R}^2$ (T.Lewinski's book)



$\mathrm{spt}(\sigma) \subset \mathcal{B}(\Sigma_0 \cup \mathrm{spt}(f))$ is OK

An example with $\mathrm{spt}(\sigma)$ larger than $\mathcal{B}(\Sigma_0 \cup \mathrm{spt}(f))$



In red, the Dirichet zone and the load F