IX Partial differential equations, optimal design and numerics
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Kantorovich-Rubinstein duality for the Hessian and convex order.

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Initial motivation: optimal design of a grillage (OGP)

The (OGP) problem consists in finding a measure $\sigma : \mathbb{R}^d \mapsto \mathcal{S}^{d\times d}$ (symmetric tensor valued) solving

$$\boxed{\min_{\mathcal{M}(\mathbb{R}^d, \mathcal{S}^{d\times d})} \left\{ \int \rho^0(\sigma) \ : \ \mathrm{spt}(\sigma) \subset \overline{\Omega} \ , \ \mathrm{div}^2 \sigma = f \right\}}$$

where:

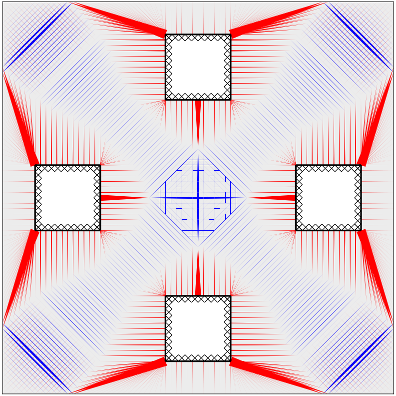
- $\Omega \subset \mathbb{R}^d$ convex design domain ($d = 2, 3$)
- $f \in \mathcal{M}(\overline{\Omega})$ a source term (distribution of order one)
- $\rho^0(S) := \sum_{i=1}^d |\lambda_i(S)|$ for all $S \in \mathcal{S}^{d\times d}$ (schatten norm)

If $\Omega = \mathbb{R}^d$ and f a signed measure, existence of optimal σ iff:

$$\int f = 0 \ , \ [f] := \int x \ f(dx) = 0.$$

Optimal grillage clamped on four collumns

σ^{opt} approximated by rank one measures (M. Gilbert, L He, Rozvany)



$\Omega =$ the large square , $\Sigma_0 =$ 4 small squares , $f = \mathcal{L}^2 \llcorner (\Omega \setminus \Sigma_0)$

Important issues

- dimension of the optimal measure σ and justification of its approximation by trusses of bars in bending regime ("grillage")

- how to relate the support of an optimal measure σ to the support of f when $\Omega = \mathbb{R}^d$?

$$\mathrm{spt}(\sigma) \subset \mathrm{co}(\mathrm{spt}(f)) \ ?$$

- relation with an OT problem ?

- how to handle loads of 1st order like $f = f_0 - \operatorname{div} F$ where $f_0 \in \mathcal{M}(\mathbb{R}^d)$ and $F \in \mathcal{M}(\mathbb{R}^d, \mathbb{R}^d)$ (s. t. $[f_0] + \int F = 0$) ?

The Hessian constraint problem

Assume that $f = \nu - \mu$, where $\mu, \nu \in \mathcal{P}_2(\overline{\Omega})$ share the same barycenter $[\mu] = [\nu]$. By standard convex analysis and duality:

$$\min(OGP) = \mathcal{I}(f, \Omega)$$

- $\mathcal{I}(f, \Omega)$ is the maximum of the Hessian constrained problem:

$$\boxed{\max\left\{< f, u > : \ u \in W^{2,\infty}(\Omega)\ ,\ \rho(\nabla^2(u)) \leq 1 \quad \text{in } \Omega\right\}.}$$

- $\rho(A) = \sup\{|\lambda_i(A)|\}$ is the *spectral* norm related to ρ^0 through

$$\rho^0(S) = \sup\{\langle A, S\rangle : \rho(A) \leq 1\}.$$

Case $\Omega = \mathbb{R}^d$ (or Ω convex)

$$\mathcal{I}(f,\Omega) = \max_{u \in C^{1,1}(\Omega)} \left\{ \int u d\nu - \int u d\mu \ : \ \mathrm{Lip}(\nabla u) \leq 1 \right\}$$

- Generalizations to higher order derivatives haven been considered in V.M. Zolotarev (Theory Prob. Appl. 1977) and recently used by M. Fathi (arxiv)
- Up to now no link with optimal transport was evidenced

Novelty: an unexpected link with convex order

Let μ, ν in $\mathcal{P}_2(\mathbb{R}^d)$ such that $[\mu] = [\nu]$ and set:

$$\mathcal{V}(\mu, \nu) = \min \{ var(\rho) \ : \ \rho \succeq_c \mu \ , \ \rho \succeq_c \nu \} .$$

where $\succeq_c$ denotes the convex order relation.

- This problem enters in the larger class of stochastic optimization problems under dominance constraints motivated by mathematical finance, statistical decision theory, economics, see for instance: D. Dentcheva and A. Ruszczynski, SIOPT (2003); A.Müller and M. Scarsini, SIOPT (2006); Johannes Wiesel and Erica Zhang (2023).
- Different objective functionals on $\mathcal{P}(\mathbb{R}^d)$ can be considered as for instance $F(\rho) = \int |x|^p \, d\rho$ for $p \neq 2$
work in progress: KB, G. Carlier, F . Santambrogio, Q. Merigot

Connection of (OG) with optimal transport

We will construct a 3 marginals OT problem whose admissible plans γ have marginals (μ, ν, ρ) with $\rho \succeq_c \mu$, $\rho \succeq_c \nu$.
As a result, for $\rho \in \operatorname{argmin} \mathcal{V}(\mu, \nu)$, we have the following equality:

$$\boxed{\mathcal{I}(f, \mathbb{R}^d) = var(\rho) - \frac{var(\mu) + var(\nu)}{2}}$$

Moreover:

- optimal stress measures σ for the (OG) problem can be recovered from optimal plans γ
- $\nu \succeq \mu \iff u(x) = \frac{|x|^2}{2}$ is optimal for $\mathcal{I}(f)$

  $\left(\nu \succeq \mu \;\Leftrightarrow\; \rho = \nu \text{ solves } \mathcal{V}(\mu, \nu) \;\Leftrightarrow\; \mathcal{I}(f) = \frac{1}{2}(var(\nu) - var(\mu))\right)$.

Plan of the talk

1. Back to the first order gradient case ;
2. OT formulation in the Hessian case ;
3. Relation between (OT3) and martingale OT;
4. Explicit examples.

1- Back to the first order gradient case.

Let $\Omega \subset \mathbb{R}^d$ a domain and $f \in \mathcal{M}(\Omega)$ compactly supported such that $\int f = 0$. We set

$$\mathcal{I}(f, \Omega) \; = \; \sup \left\{ \langle f, u \rangle \, : \, |\nabla u| \leq 1 \text{ in } \Omega \right\}.$$

- A classical duality scheme relates this to Beckmann problem

$$\mathcal{I}(f, \Omega) = \min_{\lambda \in \mathcal{M}(\overline{\Omega}; \mathbb{R}^d)} \left\{ \int |\lambda| \; : \; -\operatorname{div} \lambda = f \text{ in } \mathbb{R}^d \right\}$$

- If Ω is convex, the Euclidean distance $d_\Omega(x, y) = |x - y|$ is involved in the equivalence:

$$|\nabla u| \leq 1 \text{ a.e. in } \Omega \iff |u(x) - u(y)| \leq d_\Omega(x, y) \text{ in } \Omega^2$$

(otherwise d_Ω is the geodesic distance in Ω)

Connexion with Monge optimal transport

Def : Let $\mu, \nu \in \mathcal{M}_+(\overline{\Omega})$ such that $\mu(\overline{\Omega}) = \nu(\overline{\Omega})$; the Monge distance is given by

$$W_1(\mu,\nu) := \min \left\{ \int_{\overline{\Omega}\times\overline{\Omega}} \overbrace{|x-y|}^{\text{subadditive cost } c(x,y)} \gamma(dxdy) \ : \ \gamma \in \Pi(\mu,\nu) \right\}$$

(Kantorovich relaxation of $\inf\{\int_{\overline{\Omega}} |x - Tx|\, \mu(dx) : T^\sharp(\mu) = \nu\}$).
We set $W_1(\mu,\nu) = +\infty$ if $\mu(\overline{\Omega}) \neq \nu(\overline{\Omega})$

THM *(Kantorovich-Rubintein duality)* Let Ω be a convex domain, f balanced compactly supported in $\overline{\Omega}$. Then

$$\mathcal{I}(f,\Omega) = W_1(f_+, f_-)$$

Remarks and variants

- If Ω not convex, we need to define W_1 with, as cost $c(x,y)$, the geodesic distance in $\overline{\Omega}$;
- $\operatorname{spt}(f) \subset \overline{\Omega}$ is necessary to have $\mathcal{I}(f,\Omega) < +\infty$.
- The balance condition in $\mathcal{I}(f,\Omega)$ can be removed if we add a Dirichlet constraint $u = 0$ is prescribed on a non void compact subset Σ_0. If, for instance, $f \geq 0$, we get

$$\begin{cases} I(f,\Omega,\Sigma_0) = \min\{W_1(f,\nu)\ ;\ \operatorname{spt}(\nu) \subset \Sigma_0\} \\ u_{\rm opt} = d(x,\Sigma_0) \end{cases}$$

G.Buttazzo, P. Seppecher, GB (1997), G.Buttazzo, GB: JEMS (2001)

Recovering $(u_{\mathrm{opt}}, \sigma_{\mathrm{opt}})$ from γ_{opt}

- Let $\gamma_{\mathrm{opt}} \in \Pi(\mu, \nu)$ solving MK-problem. Then a solution λ_{opt} to Beckmann problem is given by the vector measure:

$$\lambda_{\mathrm{opt}} = \iint \lambda^{x,y} \, \gamma_{\mathrm{opt}}(dxdy) \ ,$$

where $\boxed{\lambda^{x,y} = \mathcal{H}^1 \llcorner [x,y] \, \frac{y-x}{|y-x|}}$ $\quad (-\operatorname{div} \lambda^{x,y} = \delta_y - \delta_x)$

- All solutions λ are associated to such an optimal γ (thanks to Smirnov decomposition Thm)

- Any optimal u_{opt} satisfies

$$|u_{\mathrm{opt}}(y) - u_{\mathrm{opt}}(x)| = |x - y| \quad \gamma_{\mathrm{opt}} \, a.e.$$

Hence u_{opt} is affine with slope 1 on every $[x, y]$ such that $(x, y) \in \mathrm{spt}(\gamma_{\mathrm{opt}})$.

Support of optimal measures λ

- As a consequence of the representation through optimal plan γ and whenever Ω is convex:

$$\mathrm{spt}(\lambda_{\mathrm{opt}}) \ \subset \ \mathrm{co}(\mathrm{spt}(f)).$$

- the same inclusion holds if f belongs to the completion $\mathcal{M}_{0,1}(\Omega)$ of balanced measures with respect to the MK norm ($f = f_0 - \mathrm{div}\, F$, where $F \in \mathcal{M}(\overline{\Omega}; \mathbb{R}^d)$ is *tangential*).
T. Champion, Jimenez, GB, *Revista Parma (2005)* and A. Arroyo-Rabasa, GB *in progress*

- If Ω is not convex, the same inclusion holds with $\mathrm{co}(\cdot)$ replaced by the geodesic envelope.

2- OT formulation in the Hessian case

Recall that for $f = \nu - \mu$:

$$\mathcal{I}(f,\Omega) = \max_{u \in C^{1,1}(\Omega)} \left\{ \int u d\nu - \int u d\mu \ : \ \mathrm{Lip}(\nabla u) \leq 1 \right\} .$$

The two-points cost is then: $c(x,y) = \mathcal{I}(\delta_x - \delta_y) = \delta_\Omega(x,y)$.

Difficulties are two fold:

- Need a counterpart for $\mathrm{Lip}(\nabla u) \leq 1$ of the two-point condition: $|\nabla u| \leq 1 \iff u(x) - u(y) \leq |x - y|$.
- the barycenter condition $[\mu] = [\nu]$ fails if $\mu = \delta_x$ and $\nu = \delta_y$.

$$\mathcal{I}(\delta_x - \delta_y) = +\infty \quad \text{whenever } x \neq y.$$

The three points alternative

Consider a third point $z \in \mathbb{R}^d$ and the cost $c(x, y, z) = \mathcal{I}(f^{x,y,z})$ attached to the first order distribution:

$$\boxed{f^{x,y,z} := \delta_x - \delta_y - \operatorname{div}\left((z - x)\delta_x - (z - y)\delta_y\right)}$$

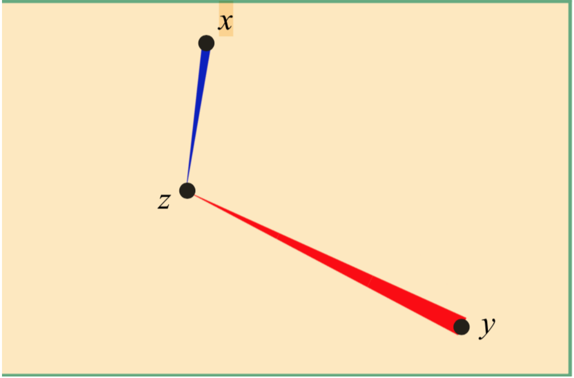
Note that $\operatorname{spt}(f^{x,y,z}) = \{x, y\}$ and that $\langle f^{x,y,z}, u\rangle = 0$ for all affine functions. Next we introduce $S^{d\times d}$- valued measures:

$$\sigma^{a,b} = \tau_{a,b} \otimes \tau_{a,b} \, \mathcal{H}^1 \llcorner [a, b] \ , \ \tau_{a,b} := \frac{b - a}{|b - a|}$$

$$\boxed{\sigma^{x,y,z}(d\xi) := |\xi - z| \left(\sigma^{z,y}(d\xi) - \sigma^{x,z}(d\xi)\right)}$$

(affine stress density, ≤ 0 on $[x, z]$ and ≥ 0 on $[z, y]$)

Density of $\sigma^{x,y,z}$ wrt. $\mathcal{H}^1 \llcorner ([x,z] \cup [z,y])$



Fundamental Lemma

- $\sigma^{x,y,z}$ satisfies $\mathrm{div}_2\, \sigma^{x,y,z} = f^{x,y,z}$.
- Assume that $z \in B(\frac{x+y}{2}, \frac{|x-y|}{2})$. Then $\sigma^{x,y,z}$ is optimal for $\mathcal{I}(f^{x,y,z})$ and

$$\mathcal{I}(f^{x,y,z}) = c(x,y,z) := \frac{1}{2}(|x-z|^2 + |y-z|^2).$$

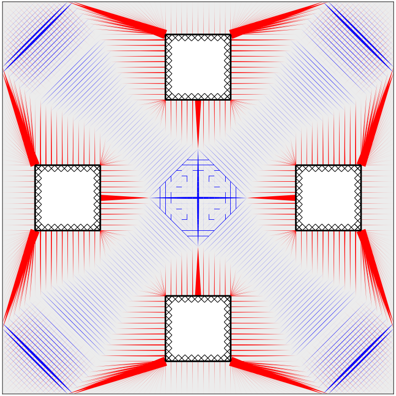
The idea is now to extend this result to any admissible measure f searching for optimal σ in the form

$$\sigma = \int \sigma^{x,y,z}\, \gamma(dxdydz)\ ,$$

being γ a three marginals plan satisfying

$$f = \int f^{x,y,z}\, \gamma(dxdydz).$$

Underlying mechanical intuition



The three points condition

We use the 3-points cost:

$$c(x,y,z) = \frac{1}{2}(|x-z|^2 + |y-z|^2)$$

Theorem((E. Le Gruyer (2009) Geom. Funct. Anal.)
For every $\varphi \in C^1(\mathbb{R}^d)$, the conditions (i)(ii)(iii) are equivalent:

(i) $\mathrm{Lip}(\nabla\varphi) \leq 1$

(ii) For all $(x,y,z) \in (\mathbb{R}^d)^3$, it holds:

$$\big(\varphi(x) + \langle \nabla\varphi(x), z-x\rangle\big) - \big(\varphi(y) + \langle \nabla\varphi(y), z-y\rangle\big) \leq c(x,y,z);$$

(iii) Forall $(x,y) \in (\mathbb{R}^d)^2$ and $z \in B(\frac{x+y}{2}, \frac{|x-y|}{2})$, it holds:

$$\big(\varphi(x) + \langle \Phi(x), z-x\rangle\big) - \big(\varphi(y) + \langle \Phi(y), z-y\rangle\big) \leq c(x,y,z)\ .$$

Moreover (iii) implies that $\Phi = \nabla\varphi$.

Extension à la "Kirschbaum"

Corollary 1: Let $E \subset \mathbb{R}^d$ and consider functions $\varphi : E \to \mathbb{R}$, $\Phi : E \to \mathbb{R}^d$ such that for any $(x, y) \in E \times E \times \mathbb{R}^d$:

$$\big(\varphi(x) + \langle \Phi(x), z - x\rangle\big) - \big(\varphi(y) + \langle \Phi(y), z - y\rangle\big) \leq c(x, y, z),$$

Then, it exists $u \in W^{2,\infty}_{\mathrm{loc}}(\mathbb{R}^d)$ such that

$$(u, \nabla u) = (\varphi, \Phi) \;\text{ in } E, \qquad \rho(\nabla^2 u) \leq 1 \quad \text{a.e. in } \mathbb{R}^d$$

An extension function is given by $u = g^{**}_{\varphi,\Phi}(x) - \frac{|x|^2}{2}$ where:

$$g_{\varphi,\Phi}(x) := \inf_{\xi \in E} \left\{ \varphi(\xi) + \langle \Phi(\xi), x - \xi\rangle + \frac{1}{2}|x - \xi|^2 \right\} + \frac{|x|^2}{2}$$

Reducing $\mathcal{I}(f)$ to a three-points constrained problem

To any subset $E \subset \mathbb{R}^d$, we associate its *ball-extension* defined by:

$$\mathcal{B}(E) := \bigcup_{(x,y)\in E^2} B\Big(\frac{x+y}{2}, \frac{x-y}{2}\Big).$$

Collorary 2: Assume that $f = f_0 - \operatorname{div} F$ is balanced. Then:

$$\mathcal{I}(f) = \sup\Big\{ \int \varphi \, df + \int \Phi \, dF,$$

where the supremum is taken over all pairs $(\varphi, \Phi) \in (C^0(\mathbb{R}^d))^{1+d}$ s. t.

$$\big(\varphi(x) + \langle \Phi(x), z - x\rangle\big) - \big(\varphi(y) + \langle \Phi(y), z - y\rangle\big) \leq c(x, y, z),$$

for all $(x, y, z) \in \operatorname{spt} f \times \operatorname{spt} f \times \mathcal{B}\big(\operatorname{spt} f\big)$

- Same statement for $\mathcal{I}(f, \Omega)$ provided $\mathcal{B}(\operatorname{spt} f) \subset \overline{\Omega}$.
- From Corollary 1, can show that (OGP) admits a solution $\overline{\sigma} \in \mathcal{M}(\mathbb{R}^d, S^{d\times d})$ supported in $\mathcal{B}\big(\operatorname{spt} f\big)$ ($\operatorname{spt} \overline{\sigma} \subset \operatorname{co}(\operatorname{spt} f)$ is not true in general)

The 3-points OT problem

Given $\gamma \in \mathcal{M}_+(\overline{\Omega}^3)$ with bounded first moment, we denote:

$$\begin{cases} \gamma_1 := \Pi^1_\sharp(\gamma), & M_1(\gamma) := \Pi^1_\sharp\Big((z-x)\,\gamma\Big) \\ \gamma_2 := \Pi^2_\sharp(\gamma), & M_2(\gamma) := \Pi^2_\sharp\Big((z-y)\,\gamma\Big) \end{cases}$$

Let $\mu, \nu \in \mathcal{M}_+(\overline{\Omega})$ such that $\int d\mu = \int d\nu$, $\int x\, d\mu = \int y\, d\nu$. Then we may associate the admissible family:

$$\boxed{\Sigma(\mu,\nu) := \Big\{\gamma \in \mathcal{M}_+(\overline{\Omega}^3) \ : \ \gamma_1 = \mu,\ \gamma_2 = \nu,\ M_1(\gamma) = M_2(\gamma) = 0\Big\}}$$

It is convex *non empty* and weakly* compact.

Second order version of Kantorovich-Rubinstein

Theorem Let $f \in \mathcal{M}(\overline{\Omega}; \mathbb{R})$ be an admissible measure such that $\mathcal{B}(\mathrm{spt}\, f) \subset \overline{\Omega}$. Then:

(i) We have the no-gap equality $\mathcal{I}(f, \Omega) = \inf(OT3)$ where:

$$(OT3) \qquad \min \left\{ \int c(x, y, z)\, \gamma(dxdydz) \ : \ \gamma \in \Sigma(f_+, f_-) \right\}$$

(ii) There exists an optimal plan $\overline{\gamma}$ for (OT3) supported in $(\mathrm{spt}(f)^2 \times \mathcal{B}(\mathrm{spt}(f))$, thus a solution to (OGP)

$$\overline{\sigma} = \int \sigma^{x,y,z}\, \overline{\gamma}(dxdydz)$$

satisfying $\mathrm{spt}(\overline{\sigma}) \subset \mathcal{B}(\mathrm{spt}(f))$.

Proof: the inequality $\mathcal{I}(f) \leq \inf(OT3)$ is easy.

• From classical convex analysis and duality, we find that $\inf(OT3) = \sup(OT3)^\star$ where:

$$\sup\left\{ \int \varphi \, d\mu + \int \psi \, d\nu, \right. \qquad (OT3)^\star$$

where the supremum is taken over all pairs $(\varphi, \psi) \in (C^0(\mathbb{R}^d))^2$ such that, for suitable $\Phi, \Psi \in (C^0(\mathbb{R}^d))^d$, it holds :

$$\big(\varphi(x) + \langle \Phi(x), z - x \rangle\big) + \big(\psi(y) + \langle \Psi(y), z - y \rangle\big) \leq c(x, y, z)$$

• By restricting to pairs (φ, ψ) and (Φ, Ψ) such that $\psi = -\varphi$ and $\Psi = -\Phi$ and in virtue of Corollary 1 that we apply to $f = \nu - \mu$ and $F = 0$, we get:

$$\inf(OT3) \ = \ \sup(OT3)^\star \ \geq \ \mathcal{I}(f).$$

Proof of the no-gap equality $\mathcal{I}(f) = \inf(OT3)$

We need a second order version of the c-transform

c-transform: when $c(x,y) = |x-y|$, the c-transform of φ is $\varphi^c(y) := \inf_y \{|x-y| - \varphi(x)\}$. It satisfies $\operatorname{lip}\varphi^c \leq 1$ while $\varphi^c = -\varphi$ if φ is 1-Lipschitz.

⋄-tranform: Let $c(x,y,z) = \frac{1}{2}(|x-z|^2 + (y-z|^2)$.

$$\boxed{\varphi \in C^{0,1} \mapsto \varphi^\diamond(y) := g^{**}_{\varphi,\nabla\varphi}(y) - \frac{|y|^2}{2},}$$

$$g_{\varphi,\Phi}(y) := \inf_{\xi\in\mathbb{R}^d} \left\{\varphi(\xi) + \langle \Phi(\xi), y-\xi\rangle + \frac{1}{2}|y-\xi|^2\right\} + \frac{|y|^2}{2}$$

Lemma [Azagra, Legruyer, Mudarra (2018)]: $\varphi^\diamond$ is $C^{1,1}$ and $\operatorname{lip}(\nabla\varphi^\diamond) \leq 1$. Moreover:

$$\operatorname{lip}(\nabla\varphi) \leq 1 \implies \varphi^\diamond = -\varphi.$$

From admissible (φ, Φ) to $(\varphi^\diamond, \varphi^{\diamond\diamond})$

- The pairs (φ, Φ) and $(\varphi^\diamond, \nabla\varphi^\diamond)$ are also admissible while

$$\int \varphi d\mu + \int \varphi^\diamond d\nu \ \geq \ \int \varphi d\mu + \int \Phi d\nu.$$

- The pairs $(\varphi^{\diamond\diamond}, \nabla\varphi^{\diamond\diamond})$ and $(\varphi^\diamond, \nabla\varphi^\diamond)$ are also admissible while

$$\int \varphi^{\diamond\diamond} d\mu + \int \varphi^\diamond d\nu \geq \int \varphi d\mu + \int \varphi^\diamond d\nu.$$

Thus as $\varphi^{\diamond\diamond} = -\varphi^\diamond$ is admissible for $\mathcal{I}(f)$, we get

$$\mathcal{I}(f) \geq \int \varphi^\diamond d(\nu - \mu) \ = \int \varphi^{\diamond\diamond} d\mu + \int \varphi^\diamond d\nu \geq \ \int \varphi d\mu + \int \Phi d\nu.$$

□

Optimality conditions

An admissible pair (u, π) for $\mathcal{I}(f)$ and (OT3) respectively is optimal iff the equality below holds π-a.e. (x, y, z):

$$\boxed{u(x) + \langle \nabla u(x), z - x\rangle - (u(y) + \langle \nabla u(y), z - y\rangle = \frac{|x-z|^2 + |y-z|^2}{2}}$$

Remark: The equality above needs to be checked only for

$$\boxed{\overline{z}(x, y, u) = \frac{1}{2}(x + y + \nabla u(x) - \nabla u(y))}$$

Furhermore $\overline{\pi}$ solving (OT3) must be of the form

$$\langle \overline{\pi}, \varphi(x, y, z)\rangle = \iint \varphi(x, y, \overline{z}(x, y, u))\, \gamma(dxdy),$$

being u optimal for $\mathcal{I}(f)$ and $\gamma \in \Pi(\mu, \nu)$.

3- Connection with stochastic optimization under convex order constraint

Background: Given two probability measures $\mu, \nu \in \mathcal{P}_1(\mathbb{R}^d)$, we say that $\nu \succeq_c \mu$ if $\int f \, d\nu \geq \int f \, d\mu$ for every convex function $f : \mathbb{R}^d \to \mathbb{R}$.

- $\nu \succeq_c \mu \implies [\mu] = [\nu]$; $\nu \succeq_c \delta_{[\nu]}$ (Jensen inequality);
- **Strassen Theorem:** $\nu \succeq_c \mu$ iff there exits a $\mu-$ measurable map $x \mapsto p^x \in \mathcal{P}(\mathbb{R}^d)$ such that:
  1. $[p^x] = x \quad \mu-$a.e.
  2. $\nu(B) = \int p^x(B) \, \mu(dx)$ for any Borel set $B \subset \mathbb{R}^d$.
- a martingale transport from μ to ν is a pairing measure $\gamma \in \Pi(\mu, \nu)$ of the form

$$\langle \gamma, \varphi \rangle = \int \left(\int \varphi(x, y) \, p^x(dy) \right) \mu(dx) ,$$

where $[p^x] = x \quad \mu-$a.e.

How (OT3) is related to martingale OT ?

Let (μ, ν, ρ) denote the marginals of an admissible $\gamma \in \Sigma(\mu, \nu)$. We claim that:

$$M_1(\gamma) = 0 \implies \rho \succeq_c \mu \quad , \qquad M_2(\gamma) = 0 \implies \rho \succeq_c \nu$$

Indeed $M_1(\gamma) = 0$ implies that, for every $\psi \in C_0(\mathbb{R}^d; \mathbb{R}^d)$:

$(*)$
$$\iiint \langle z - x, \psi(x) \rangle \, \gamma(dxdydz) = \iint \langle z - x, \psi(x) \rangle \, \gamma_{1,3}(dxdz) = 0.$$

Since $(\mu, \rho) \in \Pi(\gamma_{1,3})$, it exists a family $\{p^x\}$ in $\mathcal{P}(\mathbb{R}^d))$ such that:

$$\gamma_{1,3}(dxdz) = \int \delta_x \otimes p^x(dz) \, \mu(dx) \ , \quad \rho(dz) = \int p^x(dz) \mu(dx).$$

Then $(*)$ implies that $\int \langle [p^x] - x, \psi \rangle \, d\mu = 0$, thus $[p^x] = x$ μ- a.e.

.

The set of admissible ρ

Lemma: Let $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$ and

$$\mathcal{A} := \{\rho \in \mathcal{P}_2(R^d) \ : \ \exists \gamma \in \Sigma(\mu, \nu) \text{ such that } \Pi_3^\sharp(\gamma) = \rho\}.$$

Then:

i) $\mathcal{A} = \{\rho \in \mathcal{P}_2(\mathbb{R}^d) : \rho \succeq \mu \ , \ \rho \succeq \nu\}$.

ii) $\forall R > 0$, $\{\rho \in \mathcal{A} : var(\rho) \leq R\}$ is weakly* compact.

Moreover

Proof.

The inclusion $\subset$ is already proved. The converse is obtained by constructing martingale pairings $\gamma_{1,3} \in \Pi(\mu, \rho), \gamma_{2,3} \in \Pi(\nu, \rho)$ (exists by Strassen Theorem), and then by gluing $\gamma_{1,3}$ to $\gamma_{3,2}$.

□

Looking for optimal ρ

For every $\gamma \in \Sigma(\mu, \nu)$ with third marginal ρ, one has:

$$\int c(x,y,z)\, d\gamma = \left(\int \frac{|x|^2}{2} d\mu + \int \frac{|y|^2}{2} d\nu\right) + \int |z|^2 d\rho - \int \langle x+y, z\rangle \, d\gamma$$
$$= \int |z|^2 d\rho - \frac{1}{2}\left(\int |x|^2 \, d\mu + \int |y|^2 \, d\nu\right),$$

where in the last line we used that $M_1(\gamma) = M_2(\gamma) = 0$. Thus, by minimizing over $\mathcal{A}$ with the heelp of previous Lemma, we infer that

$$\mathcal{I}(f) = \min(OT3) = \mathcal{V}(\mu, \nu) - \frac{1}{2}(var(\mu) + var(\nu)) \ ,$$

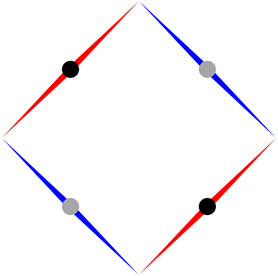
where:

$$\mathcal{V}(\mu, \nu) \ = \ \min \left\{ var(\rho) \ : \ \rho \succeq_c \mu \ , \ \rho \succeq_c \nu \right\}$$

4. Explicit examples

Example 1: from 2 Dirac masses to 2 Dirac masses

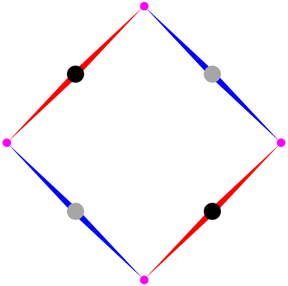
$\mu =$ equi-distributed on 2 points in black , ν equi-distributed on 2 points i



red= positive σ_{opt} , blue= negative σ_{opt}

The third marginal ρ

ρ equi-distributed on the 4 points in pink



Remark: $\mathrm{spt}(\overline{\sigma})$ and $\mathrm{spt}(\rho)$ are subsets of $\mathcal{B}(\mathrm{spt}(f))$.

Example 2: from a gaussian to a gaussian

Let μ, ν two centered gaussian laws with correlation matrices M, N:

$$M_{ij} = \int x_i x_j \, \mu(dx) \ , \ N_{ij} = \int y_i y_j \, \nu(dx).$$

As convex order for gaussians is equivalent to order between correlation tensors (as quadratic forms)

$$\mathcal{V}(\mu,\nu) = \min\{ Tr(X) \ : \ X \in \mathcal{S}^{d\times d} \ , \ X \geq M \ , \ X \geq N\}$$

which is reached for $\boxed{R = M + (N-M)_+}$,

where for a of spectral decomposition $N - M = \sum_{i=1}^{d} \lambda_i \, a_i \otimes a_i$, we set $(N-M)_+ := \sum_{i=1}^{d} (\lambda_i)_+ \, a_i \otimes a_i$. Thus:

- ρ solving $\mathcal{V}(\mu,\nu)$ is the gaussian with correlation matrice R;
- An optimal potential for $\mathcal{I}(\nu - \mu)$ is given by

$$u(x) = \sum_{i=1}^{d} \mathrm{sgn}(\lambda_i)(x|a_i)^2.$$

Example 3: case spt $\mu \perp$ spt ν.

Assume that $MN = NM = 0$. Then no need that μ and ν are gaussians !

- $R = M \vee N = M + N$;
- $\rho = \mu * \nu$;
- Same formula for the optimal potential u with

$$N - M := \sum_{i=1}^{d} \lambda_i \, a_i \otimes a_i \quad , \quad u(x) = \sum_{i=1}^{d} \mathrm{sgn}(\lambda_i)(x|a_i)^2.$$

Example 4: from Lebesgue to several Dirac masses

Optimal measures ρ consist of parts of dimension $0, 1, 2$

See forthcoming talk by Karol !

Thank you for listening