Controllability of parabolic problems by the moment method in higher dimension

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X Partial differential equations, optimal design and numerics, Benasque.

Collaboration with F. Ammar Khodja (Besançon), A. Benabdallah (Marseille), M. González-Burgos (Sevilla) & L. de Teresa (Mexico)

1. Moment method: the appropriate extension of biorthogonal families
   - General abstract setting
   - The moment method for a scalar control
   - Space-time biorthogonal families
2. A direct construction in cylindrical geometries: heat equation controlled from a hyperplane
   - Strategy of proof and related results
   - A nice biorthogonal family for the relaxed problem
   - The restriction operator
3. A direct construction in cylindrical geometries: dealing with spectral condensation
   - An example with condensation of eigenvalues
   - General result

Goal of this first section

Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain and consider the heat equation

$$\begin{cases} \partial_t y - \Delta y = \mathbf{1}_\omega u \\ y(t)_{|\partial\Omega} = 0 \\ y(0) = y_0 \end{cases}$$

Goal : "prove" null controllability i.e. for any $T > 0$ and y_0 there exists u such that $y(T) = 0$ using the moment method.

Abstract linear control problem

$$\begin{cases} y'(t) + \mathcal{A}y(t) = \mathcal{B}u(t), \quad t \in (0,T), \\ y(0) = y_0. \end{cases} \tag{S}$$

- $-\mathcal{A}$ generates a C^0-semigroup on the Hilbert space $(X, \|\cdot\|)$,
- The space of controls is the Hilbert space $(U, \|\cdot\|_U)$.
- The control operator $\mathcal{B} : U \to D(\mathcal{A}^*)'$. Assume (for simplicity) that

$$\int_0^T \left\| \mathcal{B}^* e^{-t\mathcal{A}^*} z \right\|_U^2 \mathrm{d}t \leq C\|z\|^2, \qquad \forall z \in D(\mathcal{A}^*).$$

Notion of solution

Wellposedness theorem

Let $T > 0$. For any $y_0 \in X$ and any $u \in L^2(0,T;U)$, there exists a unique solution $y \in C^0([0,T],X)$ characterized by

$$\langle y(t), z\rangle - \left\langle y_0, e^{-t\mathcal{A}^*} z \right\rangle = \int_0^t \left\langle u(\tau), \mathcal{B}^* e^{-(t-\tau)\mathcal{A}^*} z \right\rangle_U \, \mathrm{d}\tau,$$

for any $t \in [0,T]$, and any $z \in X$.
Moreover, there exists $C > 0$ such that for any such y_0, u, the solution satisfies

$$\|y(t)\| \leq C \left(\|y_0\| + \|u\|_{L^2(0,T;U)} \right), \quad \forall t \in [0,T].$$

Assumptions and moment problem

The setting

- Assume that the operator $\mathcal{A}^*$ admits a sequence of positive eigenvalues Λ.
- We denote by $(\phi_\lambda)_{\lambda\in\Lambda}$ the associated sequence of normalized eigenvectors and we assume that it forms a Hilbert basis of X.

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Definition of solutions: for all $\lambda \in \Lambda$,

$$\langle y(T), \phi_\lambda\rangle - \left\langle y_0, e^{-\lambda T}\phi_\lambda\right\rangle = \int_0^T e^{-\lambda(T-t)} \langle u(t), \mathcal{B}^*\phi_\lambda\rangle_U \,\mathrm{d}t.$$

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Hilbert basis of eigenvectors $(\phi_\lambda)_{\lambda\in\Lambda}$:

$$y(T) = 0 \quad \Longleftrightarrow \quad \int_0^T e^{-\lambda(T-t)} \langle u(t), \mathcal{B}^*\phi_\lambda \rangle_U \,\mathrm{d}t = -\left\langle y_0, e^{-\lambda T}\phi_\lambda \right\rangle, \; \forall \lambda \in \Lambda$$

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$$\iff \quad \int_0^T e^{-\lambda t} \langle v(t), \mathcal{B}^*\phi_\lambda \rangle_U \, \mathrm{d}t = -\left\langle y_0, e^{-\lambda T}\phi_\lambda \right\rangle, \ \forall \lambda \in \Lambda$$

with $v := u(T - \cdot)$.

Reduction to a moment problem when $\dim U = 1$

- Scalar control ($\dim U = 1$) with observable eigenvectors ($\mathcal{B}^*\phi_\lambda \neq 0$)

$$
\begin{aligned}
y(T) = 0 \quad &\Longleftrightarrow \quad \int_0^T e^{-\lambda t} \langle v(t), \mathcal{B}^*\phi_\lambda \rangle_U \,\mathrm{d}t = -\left\langle y_0, e^{-\lambda T}\phi_\lambda \right\rangle, \ \forall \lambda \in \Lambda \\
&\Longleftrightarrow \quad \mathcal{B}^*\phi_\lambda \int_0^T e^{-\lambda t} v(t)\mathrm{d}t = -\left\langle y_0, e^{-\lambda T}\phi_\lambda \right\rangle, \ \forall \lambda \in \Lambda \\
&\Longleftrightarrow \quad \boxed{\int_0^T e^{-\lambda t} v(t)\mathrm{d}t = -e^{-\lambda T}\left\langle y_0, \frac{\phi_\lambda}{\mathcal{B}^*\phi_\lambda} \right\rangle, \ \forall \lambda \in \Lambda}
\end{aligned}
$$

Resolution of the moment problem using a biorthogonal family

$$\text{Find } v \text{ such that } \int_0^T e^{-\lambda t} v(t) \mathrm{d}t = -e^{-\lambda T} \left\langle y_0, \frac{\phi_\lambda}{\mathcal{B}^* \phi_\lambda} \right\rangle, \; \forall \lambda \in \Lambda$$

Null controllability in time T $\Longrightarrow$ existence of a biorthogonal family $(q_\lambda)_{\lambda \in \Lambda}$ to the exponentials associated with Λ in $L^2(0, T; \mathbb{R})$

$$\begin{cases} \displaystyle\int_0^T e^{-\mu t} q_\lambda(t) \mathrm{d}t = 0, \quad \forall \mu \in \Lambda \backslash \{\lambda\}, \\ \displaystyle\int_0^T e^{-\lambda t} q_\lambda(t) \mathrm{d}t = 1. \end{cases}$$

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Existence of such biorthogonal family $\overset{\text{Schwartz}}{\iff} \displaystyle\sum_{\lambda \in \Lambda} \frac{1}{\lambda} < +\infty.$

In this case,

$$u : t \in (0,T) \mapsto -\sum_{\lambda \in \Lambda} e^{-\lambda T} \left\langle y_0, \frac{\phi_\lambda}{\mathcal{B}^* \phi_\lambda} \right\rangle q_\lambda(T-t)$$

formally solves the moment problem.

Question: estimate $\mathcal{B}^* \phi_\lambda$ and $\|q_\lambda\|_{L^2(0,T;\mathbb{R})}$ to prove that the series converges in $L^2(0,T;\mathbb{R})$.

Some estimates on biorthogonal families

Under the gap condition ($|\lambda - \mu| > \rho, \quad \forall \lambda \neq \mu \in \Lambda$).

- H.O. Fattorini & D.L Russell (1974): $\|q_\lambda\|_{L^2(0,T;\mathbb{R})} \leq C_{\varepsilon,T} e^{\varepsilon\lambda}$.
  Uniform estimates with respect to Λ in a certain class.
- A. Benabdallah, F. Boyer, M. González Burgos & G. Olive (2014)
  Sharper estimates + dependency $/T$: $\|q_\lambda\|_{L^2(0,T;\mathbb{R})} \leq C e^{C/T} e^{C\sqrt{\lambda}}$.
- P. Cannarsa, P. Martinez & J. Vancostenoble (2020)
  Optimal estimates + dealing with asymptotic gap.

Under a weak gap condition (gap between blocks of bounded cardinality)

- N. Cîndea, S. Micu, I. Roventa & M.Tucsnak (2015)
  Union of two sequences with gap condition plus a non-condensation assumption
- A. Benabdallah, F. Boyer & M. M. (2020)
- M. González Burgos & L. Ouaili (2020)

Without any gap condition

- F. Ammar Khodja, A. Benabdallah, M. González Burgos & L. de Teresa (2014)
  Condensation index of the sequence.
- D. Allonsius, F. Boyer & M. Morancey (2021)
  "Local" gap for each λ.

The moment problem for the heat equation

Hilbert basis $(\phi_\lambda)_{\lambda\in\Lambda}$ of eigenvectors + definition of solutions:
for any y_0,

$$\begin{aligned} y(T)=0 \quad &\iff \quad \int_0^T \left\langle v(t), \mathcal{B}^* e^{-\lambda t}\phi_\lambda \right\rangle_U \mathrm{d}t = -\left\langle y_0, e^{-\lambda T}\phi_\lambda \right\rangle, \ \forall \lambda\in\Lambda \\ &\iff \quad \left\langle v, e^{-\lambda \cdot}\phi_\lambda \right\rangle_{L^2((0,T)\times\omega)} = -\left\langle y_0, e^{-\lambda T}\phi_\lambda \right\rangle, \ \forall \lambda\in\Lambda \end{aligned}$$

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Natural notion of biorthogonal family

$$\Big((t,x)\in(0,T)\times\omega \mapsto q_\lambda(t,x)\Big)_{\lambda\in\Lambda}$$

such that

$$\begin{cases} \displaystyle\int_0^T\int_\omega q_\lambda(t,x)e^{-\mu t}\phi_\mu(x)\mathrm{d}x\mathrm{d}t = 0, \quad \forall \mu\in\Lambda\backslash\{\lambda\}, \\ \displaystyle\int_0^T\int_\omega q_\lambda(t,x)e^{-\lambda t}\phi_\lambda(x)\mathrm{d}x\mathrm{d}t = 1. \end{cases}$$

With such a "space-time" biorthogonal family at hand, a formal solution of the control problem is given by

$$u : (t,x) \in (0,T) \times \omega \mapsto - \sum_{\lambda \in \Lambda} e^{-\lambda T} \langle y_0, \phi_\lambda \rangle \, q_\lambda(T-t,x).$$

From biorthogonal families to controllability

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Two very natural questions:

- why the hell should such a biorthogonal family exists ??
- and even if it exists, how can we estimate its norm to study convergence of the previous series ??

Existence of a space-time biorthogonal family (for the heat equation)

G. Lebeau & L. Robbiano (1995).
For any $\lambda \in \Lambda$, let $y_0 = -e^{\lambda T}\phi_\lambda$.
Let $q_\lambda(T-\cdot)$ be the control of minimal L^2 norm such that $y(T) = 0$.

Then, by definition of solutions

$$\langle y(T), \phi_\mu \rangle - \left\langle y_0, e^{-T\mathcal{A}^*}\phi_\mu \right\rangle = \int_0^T \left\langle q_\lambda(T-t,\cdot), \mathcal{B}^* e^{-(T-t)\mathcal{A}^*}\phi_\mu \right\rangle_U \mathrm{d}t, \quad \forall \mu \in \Lambda$$

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$$\iff \quad e^{(\lambda-\mu)T}\langle \phi_\lambda, \phi_\mu \rangle = \int_0^T \int_\omega q_\lambda(t,x) e^{-\mu t}\phi_\mu(x)\mathrm{d}x\mathrm{d}t, \quad \forall \mu \in \Lambda$$

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$$\iff \quad \int_0^T \int_\omega q_\lambda(t,x) e^{-\mu t}\phi_\mu(x)\mathrm{d}x\mathrm{d}t = \delta_{\lambda,\mu}.$$

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General "theorem":
Hilbert basis of eigenvectors + null controllability in time T_0

$$\implies \begin{cases} \textbf{existence of a biorthogonal family to } \left(e^{-\lambda\cdot}\mathcal{B}^*\phi_\lambda\right)_{\lambda\in\Lambda} \text{ in } L^2(0,T_0;U) \\ \textbf{biorthogonal element } = \text{ control that drives } e^{\lambda T_0}\phi_\lambda \text{ to 0 in time } T_0 \end{cases}$$

Estimates on biorthogonal families

- Null controllability in time T_0 with cost of controllability C_{T_0}.

$$\text{biorthogonal element } q_\lambda = \text{ control that drives } e^{\lambda T_0}\phi_\lambda \text{ to } 0$$
$$\implies \quad \|q_\lambda\|_{L^2(0,T_0;U)} \leq C_{T_0}\|y_0\| = C_{T_0}e^{\lambda T_0}$$

implies convergence of the series

$$u : (t,x) \in (0,T) \times \omega \mapsto -\sum_{\lambda\in\Lambda} e^{-\lambda T} \langle y_0, \phi_\lambda\rangle \, q_\lambda(T-t,x).$$

for any $T > T_0$: null controllability by the moment method in time $T > T_0$.

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  for any $T > T_0$: null controllability by the moment method in time $T > T_0$.

- Null controllability in arbitrary time. Let $T > 0$. For any $\varepsilon \in (0,T)$,

$$\|q_\lambda\|_{L^2(0,T;U)} \leq C_\varepsilon e^{\varepsilon\lambda}$$

  implies null controllability by the moment method in any time $T > 0$.

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$$\|q_\lambda\|_{L^2(0,T;U)} \leq C_\varepsilon e^{\varepsilon\lambda}$$

  implies null controllability by the moment method in any time $T > 0$.

- Null controllability in arbitrary time $T > 0$ with cost of controllability $Ce^{C/T}$ (M. González Burgos (private communication)). Let $T > 0$. Construction of q_λ on a time-interval depending on λ implies

$$\|q_\lambda\|_{L^2(0,T;U)} \leq Ce^{C/T}e^{C\sqrt{\lambda}}.$$

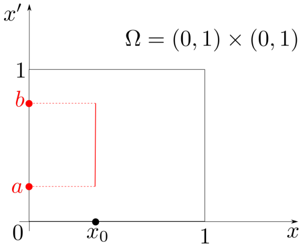
The big question

How to prove existence of such "space-time" biorthogonal family and estimate it (without using that the problem is null controllable) ??

2D heat equation controlled from a hyperplane

Let $\Omega = (0,1) \times (0,1)$.

$$\begin{cases} \partial_t y - \Delta y = \delta_{x_0} \mathbf{1}_{(a,b)}(x') u(t,x,x'), & t \in (0,T), (x,x') \in \Omega \\ y(t,\cdot)_{|\partial\Omega} = 0, & t \in (0,T), \\ y(0,x,x') = y_0(x,x'), & (x,x') \in \Omega. \end{cases}$$



Goal : find u such that $y(T) = 0$

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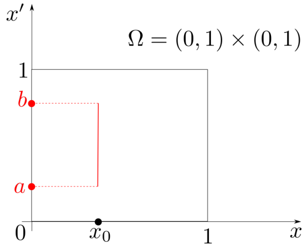
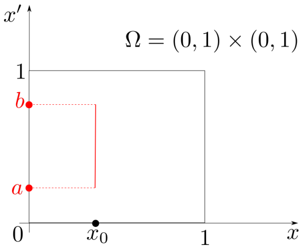


Goal : find u such that $y(T) = 0$
or rather design and estimate a biorthogonal family associated with this null controllability problem.

Strategy of proof

The **first idea** is inspired by H.O. Fattorini & D.L Russell (1974): solve a "relaxed" simpler problem and deduce the existence and estimates of biorthogonal families from an abstract restriction argument.



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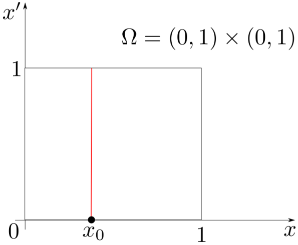

- Here, "relaxed" problem = biorthogonal family associated with control on $\{x_0\} \times (0,1)$.
- Then, use a restriction argument in the space variable x'.

Previous results on this example

- The 1D case by S. Dolecki (1973).

$$\begin{cases} \partial_t y - \partial_{xx} y = \delta_{x_0} u, \\ y(t,0) = y(t,1) = 0. \end{cases} \tag{$\star$}$$

Minimal null control time given by

$$T_0(x_0) = \limsup_{k \to +\infty} \frac{-\ln|\sin(k\pi x_0)|}{k^2\pi^2}, \qquad \text{with } T_0((0,1)) = [0,+\infty].$$

It is related to the competition between observation of eigenvectors $\big(\sin(k\pi x_0)\big)$ and dissipation $\left(e^{-k^2\pi^2 T}\right)$.

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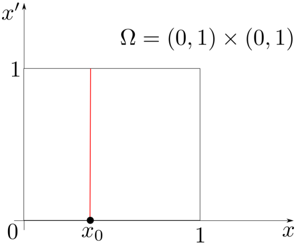
- The 2D case by E.H. Samb (2015).



Minimal null control time given by $T_0(x_0)$.

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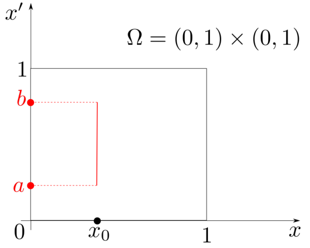
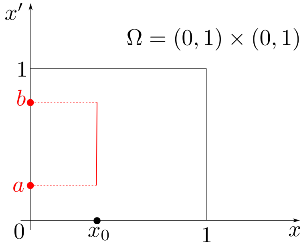
Under assumptions on x_0 that imply
- $T_0(x_0) = 0$
- the cost of controllability in small time behaves like $Ce^{C/T}$

null controllability in any time



Space restriction in a favorable case: a Lebeau-Robbiano type strategy



Strategy of proof used in E.H. Samb (2015).

A. Benabdallah, Y. Dermenjian & J. Le Rousseau (2007),
K. Beauchard, P. Cannarsa & R. Guglielmi (2014),
A. Benabdallah, F. Boyer, M. González-Burgos & G. Olive (2014).

See also L. Miller (2010).

The restriction in space strategy relies on

- null controllability of the 1D problem **in arbitrary time** and cost of controllability like $Ce^{C/T}$
- spectral inequality in the other direction

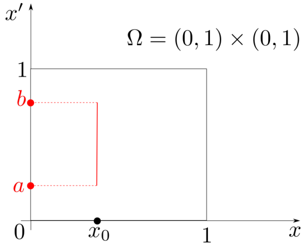
The proof uses a Lebeau-Robbiano type strategy: succession of steps of control of low frequencies and dissipation.

Space restriction in a favorable case: a Lebeau-Robbiano type strategy

Strategy of proof used in E.H. Samb (2015).

A. Benabdallah, Y. Dermenjian & J. Le Rousseau (2007),
K. Beauchard, P. Cannarsa & R. Guglielmi (2014),
A. Benabdallah, F. Boyer, M. González-Burgos & G. Olive (2014).

See also L. Miller (2010).



The restriction in space strategy relies on

- null controllability of the 1D problem **in arbitrary time** and cost of controllability like $Ce^{C/T}$
- spectral inequality in the other direction

The proof uses a Lebeau-Robbiano type strategy: succession of steps of control of low frequencies and dissipation.

What if x_0 is such that $T_0(x_0) > 0$?

Goal of this section

In the rest, we focus on space-time biorthogonal families for any time T.
Even if the problem is not null controllable in time T!

We will use

- a nice biorthogonal family for the 1D problem
- spectral inequality in the other direction

Setting

- Eigenelements

$$\lambda_{k,m} = k^2\pi^2 + m^2\pi^2, \qquad \phi_{k,m}(x,x') = \sin(k\pi x)\sin(m\pi x').$$

$$\big(\mathcal{B}^*\phi_{k,m}\big)(x') = \sin(k\pi x_0)\,\mathbf{1}_{(a,b)}(x')\sin(m\pi x').$$

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- Moment problem: find $u \in L^2((0,T)\times(a,b))$ such that for all $k, m \geq 1$,

$$\sin(k\pi x_0)\int_0^T\int_a^b e^{-\lambda_{k,m}(T-t)}\sin(m\pi x')u(t,x')\mathrm{d}x'\mathrm{d}t = -e^{-\lambda_{k,m}T}\left\langle y_0, \phi_{k,m}\right\rangle.$$

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- Look for a biorthogonal family in $L^2((0,T)\times(a,b))$ to

$$F_{k,m} : (t,x') \mapsto e^{-\lambda_{k,m}t}\sin(m\pi x'), \qquad \forall k,m \geq 1.$$

First step: a nice biorthogonal family in $L^2((0,T)\times(0,1))$

- As $\lambda_{k,m} = k^2\pi^2 + m^2\pi^2$, for any **fixed** $m \geq 1$, biorthogonal family $\left(q_{k,m}\right)$ in $L^2(0,T;\mathbb{R})$ to

  $$t \in (0,T) \mapsto e^{-\lambda_{k,m} t}, \quad k \geq 1,$$

  with estimate

  $$\|q_{k,m}\| \leq Ce^{C/T} e^{C\sqrt{\lambda_{k,m}}}, \qquad \forall k, m \geq 1.$$

  For instance, A. Benabdallah, F. Boyer & M. M. (2020) and refined estimates F. Boyer & M. M. (2023).

First step: a nice biorthogonal family in $L^2((0,T)\times(0,1))$

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For instance, A. Benabdallah, F. Boyer & M. M. (2020) and refined estimates F. Boyer & M. M. (2023).
- Orthogonality in $L^2((0,1),\mathbb{R})$ of $(\sin(m\pi\cdot))_{m\geq 1}$ implies that
$$Q_{k,m} : (t,x') \mapsto q_{k,m}(t)\sin(m\pi x')$$
forms a biorthogonal family in $L^2((0,T)\times(0,1))$ to
$$F_{k,m} : (t,x') \mapsto e^{-\lambda_{k,m} t}\sin(m\pi x'), \qquad \forall k, m \geq 1$$
with estimate
$$\|Q_{k,m}\|_{L^2((0,T)\times(0,1))} \leq C e^{C/T} e^{C\sqrt{\lambda_{k,m}}}, \qquad \forall k, m \geq 1.$$

Same construction as F. Boyer & G. Olive (2023).

From $(0,1)$ to (a,b)

- Prove that the restriction in space operator

$$\begin{array}{rcl} \mathcal{R} : \overline{\mathrm{Span}\{F_{k,m}\ ;\ k,m \geq 1\}}^{L^2((0,T)\times(0,1))} & \to & \overline{\mathrm{Span}\{F_{k,m}\ ;\ k,m \geq 1\}}^{L^2((0,T)\times(a,b))} \\ F & \mapsto & F_{|(a,b)} \end{array}$$

is an isomorphism where

$$F_{k,m} : (t,x') \mapsto e^{-\lambda_{k,m} t} \sin(m\pi x'), \qquad \forall k,m \geq 1$$

From $(0,1)$ to (a,b)

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$$\mathcal{R} : \overline{\mathrm{Span}\{F_{k,m}\ ;\ k,m \geq 1\}}^{L^2((0,T)\times(0,1))} \to \overline{\mathrm{Span}\{F_{k,m}\ ;\ k,m \geq 1\}}^{L^2((0,T)\times(a,b))}$$
$$F \mapsto F_{|(a,b)}$$

is an isomorphism where

$$F_{k,m} : (t,x') \mapsto e^{-\lambda_{k,m} t} \sin(m\pi x'), \qquad \forall k,m \geq 1$$

- Projection of the biorthogonal family $Q_{k,m}$ onto $\overline{\mathrm{Span}\{F_{k,m}\ ;\ k,m \geq 1\}}^{L^2((0,T)\times(0,1))}$ then apply $(\mathcal{R}^*)^{-1}$

$\longrightarrow$ family $(\widetilde{Q}_{k,m})_{k,m}$ such that

$$\int_0^T \int_a^b \widetilde{Q}_{k,m}(t,x') F_{j,l}(t,x') \mathrm{d}x' \mathrm{d}t = \int_0^T \int_0^1 Q_{k,m}(t,x') F_{j,l}(t,x') \mathrm{d}x' \mathrm{d}t.$$

Properties of the restriction operator

To prove that the restriction operator is bi-continuous, the key point is

$$\int_0^T \int_0^1 \left|P_N(t,x')\right|^2 \mathrm{d}x'\mathrm{d}t \leq C \int_0^T \int_a^b \left|P_N(t,x')\right|^2 \mathrm{d}x'\mathrm{d}t$$

for any

$$P_N(t,x') = \sum_{k=1}^{N} \sum_{m=1}^{N} a_{k,m} F_{k,m}(t,x').$$

Integrated observability inequality with constant cost: not much hope...

Properties of the restriction operator: weighted spaces

We prove that, for $\alpha > 0$ sufficiently large,

$$\int_0^T \int_0^1 e^{-\frac{\alpha\beta}{t}} \left|P_N(t,x')\right|^2 \mathrm{d}x'\mathrm{d}t \leq C \int_0^T \int_a^b \left|P_N(t,x')\right|^2 \mathrm{d}x'\mathrm{d}t \qquad (\star\star)$$

for any

$$P_N(t,x') = \sum_{k=1}^{N} \sum_{m=1}^{N} a_{k,m} F_{k,m}(t,x')$$

where

$$F_{k,m} : (t,x') \mapsto e^{-\lambda_{k,m} t} \sin(m\pi x'), \qquad \forall k,m \geq 1$$

and $\beta > 0$ is the constant appearing in the 1D spectral inequality

$$\int_0^1 \left| \sum_{m \leq \lambda} c_m \sin(m\pi x') \right|^2 \mathrm{d}x' \leq e^{\beta\lambda} \int_a^b \left| \sum_{m \leq \lambda} c_m \sin(m\pi x') \right|^2 \mathrm{d}x'.$$

Introduction of a weight function leads to some modifications in the announced strategy. For instance, adapt the construction of $Q_{k,m}$ such that it vanishes near $t = 0$.

Proof of $(\star\star)$: frequency cut-off and spectral inequality

$$P_N(t,x') = \sum_{k=1}^{N} \sum_{m=1}^{N} a_{k,m} e^{-\lambda_{k,m} t} \sin(m\pi x')$$

We separate the study for $t \in \left(0, \frac{\alpha}{N}\right)$ and $t \in \left(\frac{\alpha}{N}, T\right)$.
Inspired by Miller (2010).

Let $t \in \left(0, \frac{\alpha}{N}\right) \iff N < \frac{\alpha}{t}$. Then, the spectral inequality implies

$$\begin{aligned}
\int_0^1 \left|P_N(t,x')\right|^2 \mathrm{d}x' &= \int_0^1 \left| \sum_{m=1}^{N} \left(\sum_{k=1}^{N} a_{k,m} e^{-\lambda_{k,m} t} \right) \sin(m\pi x') \right|^2 \mathrm{d}x' \\
&\leq e^{\beta N} \int_a^b \left| \sum_{m=1}^{N} \left(\sum_{k=1}^{N} a_{k,m} e^{-\lambda_{k,m} t} \right) \sin(m\pi x') \right|^2 \mathrm{d}x' \\
&\leq e^{\frac{\alpha\beta}{t}} \int_a^b \left|P_N(t,x')\right|^2 \mathrm{d}x'
\end{aligned}$$

which gives $(\star\star)$ when $T < \frac{\alpha}{N}$.

Proof of ($\star\star$): dissipation wins over cost of biorthogonal family

Let $t \in \left(\frac{\alpha}{N}, T\right)$. Recall that $P_N(t,x') = \sum_{k=1}^{N} \sum_{m=1}^{N} a_{k,m} e^{-\lambda_{k,m} t} \sin(m\pi x')$

$$\int_0^1 \left|P_N(t,x')\right|^2 \mathrm{d}x' = \int_0^1 \left| \sum_{m \leq \frac{\alpha}{t}} \left(\sum_{k=1}^{N} a_{k,m} e^{-\lambda_{k,m} t} \right) \sin(m\pi x') \right|^2 \mathrm{d}x' + \int_0^1 \left| \sum_{m > \frac{\alpha}{t}} \left(\sum_{k=1}^{N} a_{k,m} e^{-\lambda_{k,m} t} \right) \sin(m\pi x') \right|^2 \mathrm{d}x'$$

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$$\int_0^1 \left|P_N(t,x')\right|^2 \mathrm{d}x' = \int_0^1 \left| \sum_{m \le \frac{\alpha}{t}} \left(\sum_{k=1}^{N} a_{k,m} e^{-\lambda_{k,m} t} \right) \sin(m\pi x') \right|^2 \mathrm{d}x'$$
$$+ \int_0^1 \left| \sum_{m > \frac{\alpha}{t}} \left(\sum_{k=1}^{N} a_{k,m} e^{-\lambda_{k,m} t} \right) \sin(m\pi x') \right|^2 \mathrm{d}x'$$

Spectral inequality

$$\int_0^1 \left| \sum_{m \le \frac{\alpha}{t}} \dots \right|^2 \mathrm{d}x' \le e^{\frac{\alpha\beta}{t}} \int_a^b \left| \sum_{m \le \frac{\alpha}{t}} \dots \right|^2 \mathrm{d}x' = e^{\frac{\alpha\beta}{t}} \int_a^b \left| \left(\sum_{m \le N} - \sum_{m > \frac{\alpha}{t}} \right) \dots \right|^2 \mathrm{d}x'$$
$$\le 2e^{\frac{\alpha\beta}{t}} \int_a^b \left|P_N(t,x')\right|^2 \mathrm{d}x' + 2e^{\frac{\alpha\beta}{t}} \int_0^1 \left| \sum_{m > \frac{\alpha}{t}} \dots \right|^2 \mathrm{d}x'$$

Proof of $(\star\star)$: dissipation wins over cost of biorthogonal family

Let $t \in \left(\frac{\alpha}{N}, T\right)$. Recall that $P_N(t,x') = \sum_{k=1}^{N}\sum_{m=1}^{N} a_{k,m} e^{-\lambda_{k,m} t} \sin(m\pi x')$

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Thus,

$$e^{-\frac{\alpha\beta}{t}} \int_0^1 \left|P_N(t,x')\right|^2 \mathrm{d}x' \leq 2 \int_a^b \left|P_N(t,x')\right|^2 \mathrm{d}x' + 3 \sum_{m > \frac{\alpha}{t}} \left| \left(\sum_{k=1}^{N} a_{k,m} e^{-\lambda_{k,m} t} \right) \right|^2$$

Proof of (⋆⋆): dissipation wins over cost of biorthogonal family

$$e^{-\frac{\alpha\beta}{t}} \int_0^1 \left|P_N(t,x')\right|^2 \mathrm{d}x' \leq 2\int_a^b \left|P_N(t,x')\right|^2 \mathrm{d}x' + 3\sum_{m>\frac{\alpha}{t}} \left| \left(\sum_{k=1}^{N} a_{k,m} e^{-k^2\pi^2 t} \right) e^{-m^2\pi^2 t} \right|^2$$

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Estimate of the coefficients $a_{k,m}$

$$|a_{k,m}| \leq C e^{C/T} e^{\frac{\alpha\beta}{2\varepsilon}} e^{C\pi\sqrt{k^2+m^2}} e^{\varepsilon(k^2+m^2)\pi^2} \left(\int_0^T \int_0^1 e^{-\frac{\alpha\beta}{t}} \left|P_N(t,x')\right|^2 \mathrm{d}x'\mathrm{d}t\right)^{1/2}.$$

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Proof: for any $\varepsilon > 0$, the biorthogonal family $Q^\varepsilon_{k,m}$ to $F_{k,m}$ in $L^2((0,T)\times(0,1))$ such that $Q^\varepsilon_{k,m}(t,\cdot) = 0$ for $t \in (0,\varepsilon)$ satisfies

$$\|Q^\varepsilon_{k,m}\|_{L^2((0,T)\times(0,1))} \leq C e^{C/T} e^{C\sqrt{\lambda_{k,m}}} e^{\varepsilon \lambda_{k,m}}$$

and

$$a_{k,m} = \left\langle Q^\varepsilon_{k,m}, P_N \right\rangle = \left\langle e^{\frac{\alpha\beta}{2\cdot}} Q^\varepsilon_{k,m}, e^{-\frac{\alpha\beta}{2\cdot}} P_N \right\rangle.$$

Key point: the estimate of $a_{k,m}$ is not exponential.

Proof of $(\star\star)$: dissipation wins over cost of biorthogonal family

$$e^{-\frac{\alpha\beta}{t}} \int_0^1 \left|P_N(t,x')\right|^2 \mathrm{d}x' \leq 2\int_a^b \left|P_N(t,x')\right|^2 \mathrm{d}x' + 3 \sum_{m>\frac{\alpha}{t}} \left| \left(\sum_{k=1}^N a_{k,m} e^{-k^2\pi^2 t} \right) e^{-m^2\pi^2 t} \right|^2$$

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Use

$$e^{C\pi\sqrt{k^2+m^2}} \leq \exp\left(\frac{C}{2t}\right) \exp\left(\frac{t}{2}(k^2+m^2)\pi^2\right),$$

choice of ε depending on t and estimate of the rest of the series

$$\sum_{m>\frac{\alpha}{t}} e^{-m^2\tau} \leq \frac{C}{\sqrt{\tau}} e^{-\frac{\alpha^2}{t^2}\tau}$$

Proof of ($\star\star$): dissipation wins over cost of biorthogonal family

$$e^{-\frac{\alpha\beta}{t}} \int_0^1 \left|P_N(t,x')\right|^2 \mathrm{d}x' \leq 2 \int_a^b \left|P_N(t,x')\right|^2 \mathrm{d}x' + 3 \sum_{m>\frac{\alpha}{t}} \left| \left(\sum_{k=1}^N a_{k,m} e^{-k^2\pi^2 t} \right) e^{-m^2\pi^2 t} \right|^2$$

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$$|a_{k,m}| \leq C e^{C/T} e^{\frac{\alpha\beta}{2\varepsilon}} e^{C\pi\sqrt{k^2+m^2}} e^{\varepsilon(k^2+m^2)\pi^2} \left(\int_0^T \int_0^1 e^{-\frac{\alpha\beta}{t}} \left|P_N(t,x')\right|^2 \mathrm{d}x'\mathrm{d}t \right)^{1/2}.$$

Use

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choice of ε depending on t and estimate of the rest of the series

$$\sum_{m>\frac{\alpha}{t}} e^{-m^2\tau} \leq \frac{C}{\sqrt{\tau}} e^{-\frac{\alpha^2}{t^2}\tau}$$

imply

$$\sum_{m>\frac{\alpha}{t}} |\dots|^2 \leq \frac{C_T}{t^3} \exp\left(\frac{C+\alpha\beta-\alpha^2}{t}\right) \int_0^T \int_0^1 e^{-\frac{\alpha\beta}{t}} \left|P_N(t,x')\right|^2 \mathrm{d}x'\mathrm{d}t.$$

Choice of α sufficiently large and integration in the variable t gives the estimate ($\star\star$).

Summary of the construction of a biorthogonal family

We have used

- a nice biorthogonal family in $L^2((0,T)\times(0,1))$ to
$$F_{k,m} : (t,x') \mapsto e^{-\lambda_{k,m} t}\sin(m\pi x'), \qquad \forall k,m \geq 1.$$
It mostly comes from the biorthogonal family $(q_{k,m})$ to the time exponentials $\left(t \mapsto e^{-\lambda_{k,m} t}\right)_{k\geq 1}$ and orthogonality of $\left(\sin(m\pi\cdot)\right)_{m\geq 1}$ on $(0,1)$.
- The isomorphism property of the restriction operator from $(0,1)$ to (a,b) in the x' variable between appropriate (weighted) spaces.
It mostly comes from the non-exponential estimate of $q_{k,m}$ and the spectral inequality
$$\int_0^1 \left|\sum_{m\leq\lambda} c_m \sin(m\pi x')\right|^2 \mathrm{d}x' \leq e^{\beta\lambda}\int_a^b \left|\sum_{m\leq\lambda} c_m \sin(m\pi x')\right|^2 \mathrm{d}x'.$$

This gives a biorthogonal family $G_{k,m}$ to $F_{k,m}$ in $L^2((0,T)\times(a,b))$ satisfying
$$\|G_{k,m}\|_{L^2((0,T)\times(a,b))} \leq Ce^{C/T}e^{C\sqrt{\lambda_{k,m}}}.$$

Application to the study of null controllability

We have a biorthogonal family $G_{k,m}$ to $F_{k,m}$ in $L^2((0,T)\times(a,b))$ satisfying

$$\|G_{k,m}\|_{L^2((0,T)\times(a,b))} \leq Ce^{C/T}e^{C\sqrt{\lambda_{k,m}}}.$$

- Moment problem: find $u \in L^2((0,T)\times(a,b))$ such that for all $k,m \geq 1$,

$$\sin(k\pi x_0)\int_0^T\int_a^b e^{-\lambda_{k,m}(T-t)}\sin(m\pi y)u(t,x')\mathrm{d}x'\mathrm{d}t = -e^{-\lambda_{k,m}T}\left\langle y_0, \phi_{k,m}\right\rangle.$$

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- Moment problem: find $u \in L^2((0,T)\times(a,b))$ such that for all $k, m \geq 1$,

$$\sin(k\pi x_0)\int_0^T\int_a^b e^{-\lambda_{k,m}(T-t)}\sin(m\pi y)u(t,x')\mathrm{d}x'\mathrm{d}t = -e^{-\lambda_{k,m}T}\left\langle y_0, \phi_{k,m}\right\rangle.$$

2D heat equation controlled on $\{x_0\}\times(a,b)$ has minimal null control time

$$T_0(x_0) = \limsup_{k\to+\infty}\frac{-\ln|\sin(k\pi x_0)|}{k^2\pi^2}.$$

- When $T > T_0(x_0)$, null controllability follows from the convergence of the series

$$u(t,x') = \sum_{k=1}^{+\infty}\sum_{m=1}^{+\infty}\frac{e^{-\lambda_{k,m}T}}{\sin(k\pi x_0)}\left\langle y_0, \phi_{k,m}\right\rangle G_{k,m}(T-t,x').$$

- Lack of null controllability when $T < T_0(x_0)$: tensorization of the 1D counterexample.

A different example

Simultaneous controllability on $\Omega = (0,1) \times (0,1)$.

$$\begin{cases} \partial_t y + \begin{pmatrix} -\Delta & 0 \\ 0 & -\Delta + p(x) \end{pmatrix} y = 0, \\ y_{|\partial\Omega} = \begin{pmatrix} \mathbf{1}_\Gamma u \\ \mathbf{1}_\Gamma u \end{pmatrix}. \end{cases}$$

The function p satisfies $\partial_{x'} p = 0$.

- L. Ouaili (2019). 1D setting: minimal null control time (Dirichlet boundary condition at $x = 0$) given by the condensation index of the eigenvalues

$$T_0(p) = \limsup_{k \to +\infty} \frac{-\ln |k^2\pi^2 - \lambda_k(p)|}{k^2\pi^2}.$$

- 2D setting: same minimal time with $\Gamma = \{0\} \times (a,b)$.
  Eigenvalues

$$\Lambda = \left\{k^2\pi^2 + m^2\pi^2 \; ; \; k, m \geq 1\right\} \cup \left\{\lambda_k(p) + m^2\pi^2 \; ; \; k, m \geq 1\right\}.$$

Applying the previous strategy does not work well...

$$P_N(t,x') = \sum_{k=1}^{N} \sum_{m=1}^{N} \left(a_{k,m,1} e^{-(k^2+m^2)\pi^2 t} + a_{k,m,2} e^{-(\lambda_k(p)+m^2\pi^2)t} \right) \sin(m\pi x')$$

- Spectral condensation $\Longrightarrow$ biorthogonal family to the time exponentials

$$\|q_{k,m}\|_{L^2(0,T;\mathbb{R})} \simeq e^{(k^2+m^2)\pi^2 T_0(p)}$$

F. Ammar Khodja, A. Benabdallah, M. González Burgos & L. de Teresa (2014)

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$$P_N(t,x') = \sum_{k=1}^{N}\sum_{m=1}^{N}\left(a_{k,m,1}e^{-(k^2+m^2)\pi^2 t} + a_{k,m,2}e^{-(\lambda_k(p)+m^2\pi^2)t}\right)\sin(m\pi x')$$

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- estimate of $|a_{k,m,i}|$ will be of exponential-type

Applying the previous strategy does not work well...

$$P_N(t,x') = \sum_{k=1}^{N}\sum_{m=1}^{N}\left(a_{k,m,1}e^{-(k^2+m^2)\pi^2 t} + a_{k,m,2}e^{-(\lambda_k(p)+m^2\pi^2)t}\right)\sin(m\pi x')$$

- Spectral condensation $\Longrightarrow$ biorthogonal family to the time exponentials

$$\|q_{k,m}\|_{L^2(0,T;\mathbb{R})} \simeq e^{(k^2+m^2)\pi^2 T_0(p)}$$

F. Ammar Khodja, A. Benabdallah, M. González Burgos & L. de Teresa (2014)

- estimate of $|a_{k,m,i}|$ will be of exponential-type
- and thus is not sufficient to prove convergence of the series

$$\sum_{k,m}\left(a_{k,m,1}e^{-(k^2+m^2)\pi^2 t} + a_{k,m,2}e^{-(\lambda_k(p)+m^2\pi^2)t}\right)\sin(m\pi x').$$

A (not only) commercial break: block moment problems

- A. Benabdallah, F. Boyer & M. M. (2020)
  Scalar control, complete family of observable eigenvectors, weak-gap condition, $\sum_{\lambda \in \Lambda} \frac{1}{\lambda} < +\infty$.
  Resolution and study of the cost of resolution of block moment problems

$$\begin{cases} \displaystyle\int_0^T e^{-\lambda_{k,j} t} v_k(t) \mathrm{d}t = \omega_{k,j}, & \forall 1 \leq j \leq g_k, \\ \displaystyle\int_0^T e^{-\lambda t} v_k(t) \mathrm{d}t = 0, & \forall \lambda \in \Lambda \backslash G_k. \end{cases}$$

  Application to the characterization of the minimal null control time

- F. Boyer & M. M. (2023)
  Generalization to any admissible control operator.

Patch the proof (inspired by A. Benabdallah, F. Boyer & M. M. (2020))

$$P_N(t,x') = \sum_{k=1}^{N}\sum_{m=1}^{N}\left(a_{k,m,1}e^{-(k^2+m^2)\pi^2 t} + a_{k,m,2}e^{-(\lambda_k(p)+m^2\pi^2)t}\right)\sin(m\pi x')$$

Let $t \in (0,T)$ and $q_{k,m}^t$ be the solution for $m \geq 1$ fixed of the block moment problem

$$\begin{cases} \displaystyle\int_0^T q_{k,m}^t(s)e^{-(k^2+m^2)\pi^2 s}\mathrm{d}s = e^{-(k^2+m^2)\pi^2 t}, \\ \displaystyle\int_0^T q_{k,m}^t(s)e^{-(\lambda_k(p)+m^2\pi^2)s}\mathrm{d}s = e^{-(\lambda_k(p)+m^2\pi^2)t}, \\ \displaystyle\int_0^T q_{k,m}^t(s)e^{-(\nu_j+m^2\pi^2)s}\mathrm{d}s = 0, \quad \nu_j \in \{j^2\pi^2, \lambda_j(p)\},\ j \geq 1. \end{cases}$$

Then,

$$\left\langle q_{k,m}^t \sin(m\pi\cdot), P_N \right\rangle = a_{k,m,1}e^{-(k^2+m^2)\pi^2 t} + a_{k,m,2}e^{-(\lambda_k(p)+m^2\pi^2)t}$$

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and (see A. Benabdallah, F. Boyer & M. M. (2020))

$$\|q_{k,m}^t\|_{L^2(0,T;\mathbb{R})} \leq Ce^{C/T}e^{C\sqrt{k^2+m^2}}e^{-(k^2+m^2)\pi^2 t}$$

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$$\|q_{k,m}^t\|_{L^2(0,T;\mathbb{R})} \leq C e^{C/T} e^{C\sqrt{k^2+m^2}} e^{-(k^2+m^2)\pi^2 t}$$

This implies

$$\begin{aligned} &\left| a_{k,m,1} e^{-(k^2+m^2)\pi^2 t} + a_{k,m,2} e^{-(\lambda_k(p)+m^2\pi^2)t} \right| \\ &\leq C e^{C/T} e^{C\sqrt{k^2+m^2}} e^{-(k^2+m^2)\pi^2 t} \left(\int_0^T \int_0^1 \left| P_N(t,x') \right|^2 \mathrm{d}x' \mathrm{d}t \right)^{1/2}. \end{aligned}$$

The rest of the proof follows as previously using estimates of such blocks instead of estimates of $a_{k,m,i}$.

A general result

"Theorem"

- Cylindrical geometry and tensorized operators
- $\Lambda = \left\{ \lambda_k + \mu_m \; ; \; k, m \geq 1 \right\}$
- On the direction associated with λ_k: nice 1D assumptions (to solve block moment problems) on the eigenvalues. Allow geometrically multiple eigenvalues.
- On the direction associated with μ_m: asymptotic of μ_m + Riesz-basis property for the eigenvectors + spectral inequality for the eigenvectors.

$\Longrightarrow$ construction and estimate of a space-time biorthogonal family for any time $T > 0$.

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Conclusion:

- this construction of space-time biorthogonal families allows to study controllability in some cylindrical geometric configurations even in the presence of a positive minimal null control time ;
- the moment method can be applied in much more general geometric settings but a lot remains to be done...

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Thank you for your attention