An optimal two-phase isolating material in the wall of a cavity

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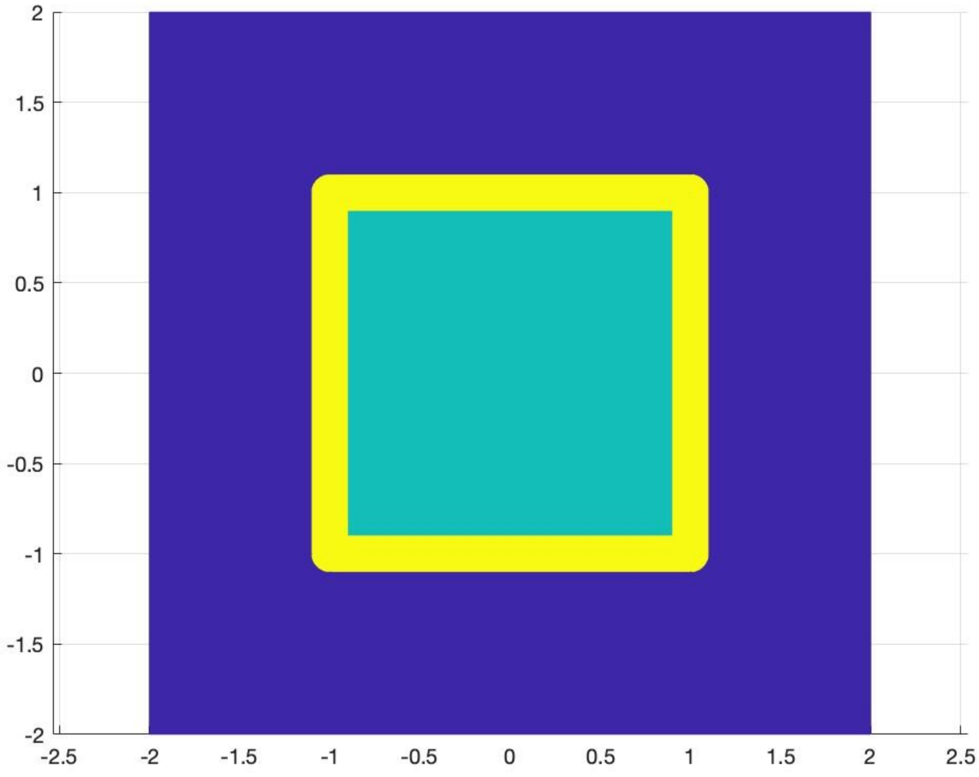
Problem: To get the optimal distribution of two isolating materials in the wall of a cavity.

Mathematical formulation:

$\Omega \subset \mathbb{R}^N$ bounded, open, $\partial\Omega$ is locally the graph of a Lipschitz function

$$\exists K \subset \partial\Omega \text{ closed}, \ \ |K|_{N-1} = 0, \ \ \partial\Omega \setminus K \ is \ C^1.$$

For $\varepsilon > 0$, take $\Gamma_\varepsilon = \{x \in \mathbb{R}^N, \ \mathrm{dist}(x, \partial\Omega) < \varepsilon\}$, $\Omega_\varepsilon = \Omega \cup \Gamma\_\varepsilon$



For $0 < \alpha < \beta_\varepsilon$ we take $\omega \subset \Gamma_\varepsilon$ measurable and a diffusion operator with diffusion coefficient

$$a_\varepsilon(x) = \begin{cases} 1 & \text{in } \Omega_\varepsilon \setminus \Gamma_\varepsilon \\ \varepsilon\alpha & \text{in } \omega \\ \varepsilon\beta_\varepsilon & \text{in } \Gamma_\varepsilon \setminus \omega. \end{cases}$$

Problem: To choose ω getting the best insulating material.

They are two cases

$$\beta_\varepsilon \to \beta \in (\alpha, \infty), \;\; \beta_\varepsilon \to \infty.$$

How to model this?

Consider the problem

$$\begin{cases} \partial_t u - \operatorname{div}(a_\varepsilon(x)\nabla u) = 0 \ \text{ in } (0,\infty) \times \Omega_\varepsilon \\ u = 0 \ \text{ on } (0,\infty) \times \partial\Omega_\varepsilon \\ u(0,x) = u_0(x). \end{cases}$$

We know

$$\|u(t,.)\|_{L^2(\Omega_\varepsilon)} \leq \|u_0\|_{L^2(\Omega_\varepsilon)} e^{-\lambda_\varepsilon t}$$

with λ_ε the first eigenvalue of the operator

$$v \mapsto -\operatorname{div}(a_\varepsilon(x)\nabla v) \ \text{ with Dirichlet conditions.}$$

Optimal shape problem:

To choose $\omega \subset \Gamma_\varepsilon$ minimizing λ_ε, i.e. to solve

$$\min_{\substack{\omega\subset\Gamma_\varepsilon \\ \omega \text{ measurable}}} \min_{\|u\|_{H_0^1(\Omega)}\le 1} \int_{\Omega_\varepsilon} a_\varepsilon |\nabla u|^2 dx.$$

Written in this way the solution is the trivial one: $\omega = \Gamma_\varepsilon$.

We add the constraint $|\omega| \le \kappa\varepsilon$ with $\quad (|\Gamma_\varepsilon| \sim |\partial\Omega|_{N-1}\varepsilon)$

$$0 < \kappa < |\partial\Omega|_{N-1}.$$

We get

$$\min_{\substack{\omega\subset\Gamma_\varepsilon \\ |\omega|\le\kappa\varepsilon}} \min_{\|u\|_{H_0^1(\Omega)}\le 1} \int_{\Omega_\varepsilon} \left(\varepsilon\alpha\chi_\omega + \varepsilon\beta_\varepsilon\chi_{\Gamma_\varepsilon\setminus\omega} + \chi_{\Omega_\varepsilon\setminus\Gamma_\varepsilon}\right)|\nabla u|^2 dx.$$

The problem has no solution in general (F. Murat 1973, JCD 2015).

We need to use a relaxed formulation.

We recall (Spagnolo 1968) $A_n \in L^\infty(\Omega)^{N\times N}$ symmetric, $\lambda I \leq A_n \leq \gamma I$.

Then, for a subsequence, $\exists A \in L^\infty(\Omega)^{N\times N}$ symmetric, $\lambda I \leq A \leq \gamma I$ such that $\forall f \in H^{-1}(\Omega)$ the solution u_n of

$$\begin{cases} -\mathrm{div}(A_n \nabla u_n) = f \ in \ \Omega \\ u_n = 0 \ on \ \partial\Omega \end{cases}$$

satisfies

$$u_n \rightharpoonup u \ in \ H_0^1(\Omega), \qquad A_n \nabla u_n \rightharpoonup A\nabla u \ in \ L^2(\Omega)^N,$$

$$\begin{cases} -\mathrm{div}(A \nabla u) = f \ in \ \Omega \\ u = 0 \ on \ \partial.\Omega \end{cases}$$

(F. Murat, L. Tartar, 1974 the non-symmetric case)

Corollary: The k-th eigenvalue of $-\mathrm{div}(A_n \nabla)$ converges to the k-th eigenvalue of $-\mathrm{div}(A\nabla)$.

F. Murat, L. Tartar: If $\Omega = \Omega_\varepsilon$

$$A_n = \left(\varepsilon\alpha\chi_{\omega_n} + \varepsilon\beta_\varepsilon\chi_{\Gamma_\varepsilon\setminus\omega_n} + \chi_{\Omega_\varepsilon\setminus\Gamma_\varepsilon}\right)I, \qquad \omega_n \subset \Gamma_\varepsilon \text{ measurable}$$

$$\chi_{\omega_n} \underset{*}{\rightharpoonup} \theta \text{ in } L^\infty(\Omega), \qquad \theta \in L^\infty(\Omega;[0,1]).$$

Then

$$A = \varepsilon C\chi_{\Gamma_\varepsilon} + \chi_{\Omega_\varepsilon\setminus\Gamma_\varepsilon}I, \;\; \mu_\varepsilon^-(\theta)I \le C \le \mu_\varepsilon^+(\theta)I,$$

with

$$\mu_\varepsilon^-(\theta) = \left(\frac{\theta}{\alpha} + \frac{1-\theta}{\beta_\varepsilon}\right)^{-1}, \;\; \mu_\varepsilon^+(\theta) = \theta\alpha + (1-\theta)\beta_\varepsilon.$$

Theorem: A relaxation of

$$\inf_{\substack{\omega\subset\Gamma_\varepsilon \\ |\omega|\le\kappa\varepsilon}} \min_{\|u\|_{H_0^1(\Omega)}\le 1} \int_{\Omega_\varepsilon} (\varepsilon\alpha\chi_\omega + \varepsilon\beta_\varepsilon\chi_{\Gamma_\varepsilon\setminus\omega} + \chi_{\Omega_\varepsilon\setminus\Gamma_\varepsilon})|\nabla u|^2 dx$$

is given by $\displaystyle \min_{\substack{\theta\in L^\infty(\Gamma_\varepsilon,[0,1]) \\ \int_{\Gamma_\varepsilon}\theta dx\le\kappa\varepsilon}} \lambda_\varepsilon(\theta).$

with $\displaystyle \lambda_\varepsilon(\theta) = \min_{\|u\|_{H_0^1(\Omega)}\le 1} \left(\varepsilon\int_{\Gamma_\varepsilon} \mu_\varepsilon^-(\theta)|\nabla u|^2 dx + \int_{\Omega_\varepsilon\setminus\Gamma_\varepsilon} |\nabla u|^2 dx\right)$

first eigenvalue of div $\left((\varepsilon\mu_\varepsilon^-(\theta)\chi_{\Gamma_\varepsilon} + \chi_{\Omega_\varepsilon\setminus\Gamma_\varepsilon})\nabla\right)$ with Dirichlet conditions

Let us assume

$$\beta_\varepsilon \to \beta \in (\alpha,\infty].$$

Theorem: Assume $(\theta_\varepsilon, u_\varepsilon)$ a solution of

$$\min_{\substack{\theta\in L^\infty(\Gamma_\varepsilon,[0,1]) \\ \int_{\Gamma_\varepsilon}\theta dx\le\kappa\varepsilon}} \min_{\|u\|_{H_0^1(\Omega)}\le 1} \left(\varepsilon\int_{\Gamma_\varepsilon}\mu_\varepsilon^-(\theta)|\nabla u|^2 dx + \int_{\Omega_\varepsilon\setminus\Gamma_\varepsilon}|\nabla u|^2 dx\right).$$

Then, for a subsequence, $\exists\theta \in L^\infty(\partial\Omega,[0,1]), u \in H_0^1(\Omega)$

$$\frac{\theta_\varepsilon}{\varepsilon}\chi_{\Gamma_\varepsilon} \underset{*}{\rightharpoonup} \theta_{H_{N-1}\llcorner\partial\Omega} \text{ in } \mathcal{M}(\mathbb{R}^N), \qquad u_\varepsilon \rightharpoonup u \text{ in } H^1(\mathbb{R}^N)$$

$$\varepsilon\int_{\Gamma_\varepsilon}\mu_\varepsilon^-(\theta)|\nabla u|^2 dx + \int_{\Omega_\varepsilon\setminus\Gamma_\varepsilon}|\nabla u|^2 dx \to \int_{\partial\Omega}\mu^-(\theta)|u|^2 d\sigma(x) + \int_\Omega |\nabla u|^2 dx$$

with (θ, u) a solution of

$$\min_{\substack{\theta\in L^\infty(\partial\Omega,[0,1]) \\ \int_{\partial\Omega}\theta d\sigma(x)\le\kappa}} \min_{\|u\|_{H_0^1(\Omega)}\le 1} \left(\int_{\partial\Omega}\mu^-(\theta)|u|^2 d\sigma(x) + \int_\Omega |\nabla u|^2 dx\right)$$

$$\mu^-(\theta) = \left(\frac{\theta}{\alpha} + \frac{1-\theta}{\beta}\right)^{-1}$$

Moreover, if (θ, u) is a solution of

$$\min_{\substack{\theta\in L^\infty(\partial\Omega,[0,1]) \\ \int_{\partial\Omega}\theta d\sigma(x)\le\kappa}} \min_{\|u\|_{H_0^1(\Omega)}\le 1} \left(\int_{\partial\Omega}\mu^-(\theta)|u|^2 d\sigma(x) + \int_\Omega |\nabla u|^2 dx\right)$$

then, there exists $(\theta_\varepsilon, u_\varepsilon) \in L^\infty(\Gamma_\varepsilon, [0,1]) \times H_0^1(\Omega_\varepsilon)$ with

$$\frac{\theta_\varepsilon}{\varepsilon}\chi_{\Gamma_\varepsilon} \underset{*}{\rightharpoonup} \theta_{H_{N-1}\llcorner\partial\Omega} \text{ in } \mathcal{M}(\mathbb{R}^N), \qquad u_\varepsilon \rightharpoonup u \text{ in } H_0^1(\Omega)$$

$$\varepsilon\int_{\Gamma_\varepsilon}\mu_\varepsilon^-(\theta)|\nabla u|^2 dx + \int_{\Omega_\varepsilon\setminus\Gamma_\varepsilon}|\nabla u|^2 dx \to \int_{\partial\Omega}\mu^-(\theta)|u|^2 d\sigma(x) + \int_\Omega|\nabla u|^2 dx$$

Remark: If $\beta = \infty$ then $\mu^{-}(\theta) = \frac{\alpha}{\theta}$ is singular at $\theta = 0$.

In this case

$$\mu^{-}(0)|u|^2 := \begin{cases} +\infty & \text{if } u \neq 0 \\ 0 & \text{if } u = 0. \end{cases}$$

Remark:

$$\lambda_0(\theta) := \min_{\|u\|_{H_0^1(\Omega)} \leq 1} \left(\int_{\partial\Omega} \mu^{-}(\theta)|u|^2 d\sigma(x) + \int_{\Omega} |\nabla u|^2 dx \right)$$

is the first eigenvalue of

$$\begin{cases} -\Delta u = \lambda u \ \text{ in } \Omega \\ \dfrac{\partial u}{\partial \nu} + \mu^{-}(\theta) u = 0 \ \text{ on } \partial\Omega. \end{cases}$$

Optimality conditions: (θ, u) solution of

$$\min_{\substack{\theta \in L^\infty(\partial\Omega,[0,1]) \\ \int_{\partial\Omega} \theta d\sigma(x) \le \kappa}} \min_{\|u\|_{H_0^1(\Omega)} \le 1} \left(\int_{\partial\Omega} \mu^-(\theta)|u|^2 d\sigma(x) + \int_\Omega |\nabla u|^2 dx \right).$$

$u > 0$ in Ω, then $\exists \gamma > 0$ such that

$$\text{if } \beta < \infty \qquad \theta(x) = \begin{cases} 0 & \text{if } u < \gamma\alpha \\ \dfrac{u - \gamma\alpha}{\gamma(\beta - \alpha)} & \text{if } \gamma\alpha \le u \le \gamma\beta \\ 1 & \text{if } \gamma\beta < u \end{cases}$$

$$\text{if } \beta = \infty \qquad \theta(x) = \begin{cases} \dfrac{u}{\gamma} & \text{if } u \le \gamma \\ 1 & \text{if } \gamma < u \end{cases}$$

$$\int_{\partial\Omega} \theta(x) d\sigma(x) = k\,, \qquad \begin{cases} -\Delta u = \lambda_0(\theta) u \ \text{ in } \Omega \\ \dfrac{\partial u}{\partial \nu} + \mu^-(\theta) u = 0 \ \text{ on } \partial\Omega. \end{cases}$$

Corollary: If (θ, u) is a solution

$$\min_{\substack{\theta \in L^\infty(\partial\Omega,[0,1]) \\ \int_{\partial\Omega} \theta d\sigma(x) \le \kappa}} \min_{\|u\|_{H_0^1(\Omega)} \le 1} \left(\int_{\partial\Omega} \mu^-(\theta)|u|^2 d\sigma(x) + \int_\Omega |\nabla u|^2 dx \right),$$

then $u \in C^\infty(\Omega) \cap L^\infty(\partial\Omega)$, $\theta \in H^{\frac{1}{2}}(\partial\Omega)$. Moreover, if $\Omega \in C^{1,1}, \beta < \infty$, then $u \in W^{2,p}(\Omega)$, $\forall p > 1$, $\theta \in W^{r,\infty}(\Omega)$, $\forall\, r < 2$.

Corollary: If $\partial\Omega$ is connected, a solution of the limit problem is never a characteristic function.

Numerical algorithm: Choose

$$\theta_0 \in L^\infty(\partial\Omega; [0,1]), \quad \int_{\partial\Omega} \theta_0(x) d\sigma(x) \leq \kappa.$$

Given θ_n in the above conditions solve (e.g. use power method)

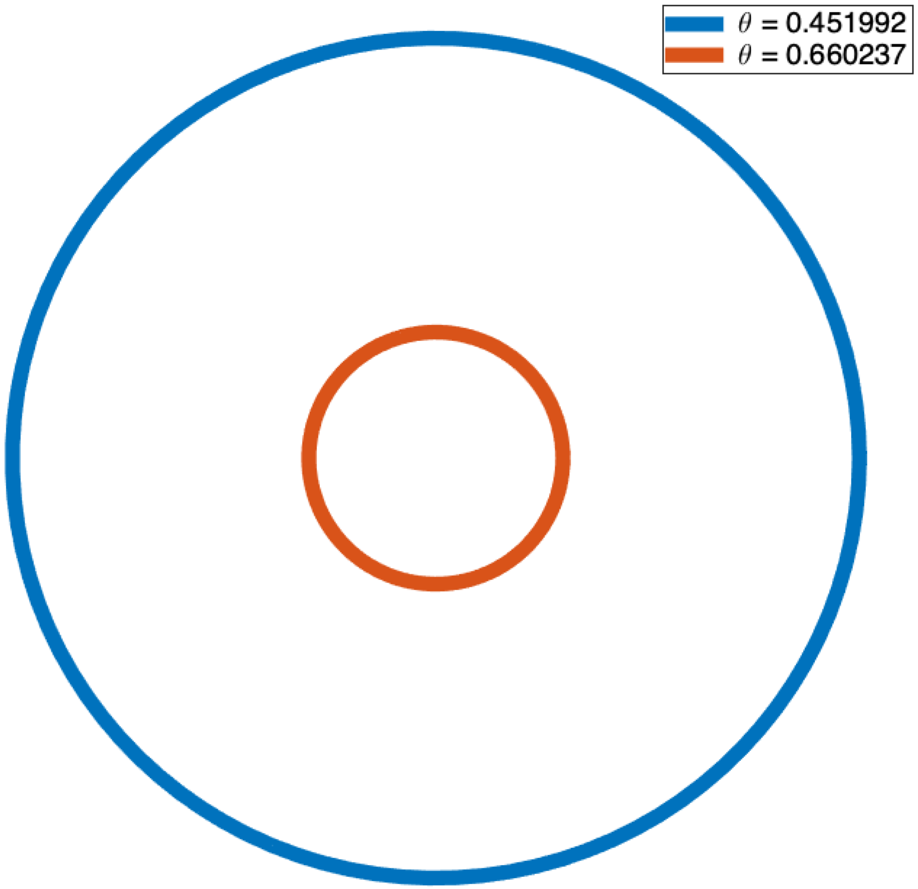
$$\begin{cases} -\Delta u_n = \lambda_0(\theta_n) u_n \ \text{ in } \Omega \\ \dfrac{\partial u_n}{\partial \nu} + \mu^-(\theta_n) u_n = 0 \ \text{ on } \partial\Omega, \end{cases} \qquad u_n > 0 \ in \ \Omega, \quad \int_{\partial\Omega} u_n d\sigma(x) = 1.$$

Define θ_{n+1} by

$$\text{if } \beta < \infty \qquad \theta_{n+1}(x) = \begin{cases} 0 & \text{if } u_n < \gamma\alpha \\ \dfrac{u_n - \gamma\alpha}{\gamma(\beta - \alpha)} & \text{if } \gamma\alpha \leq u_n \leq \gamma\beta \\ 1 & \text{if } \gamma\beta < u_n \end{cases}$$

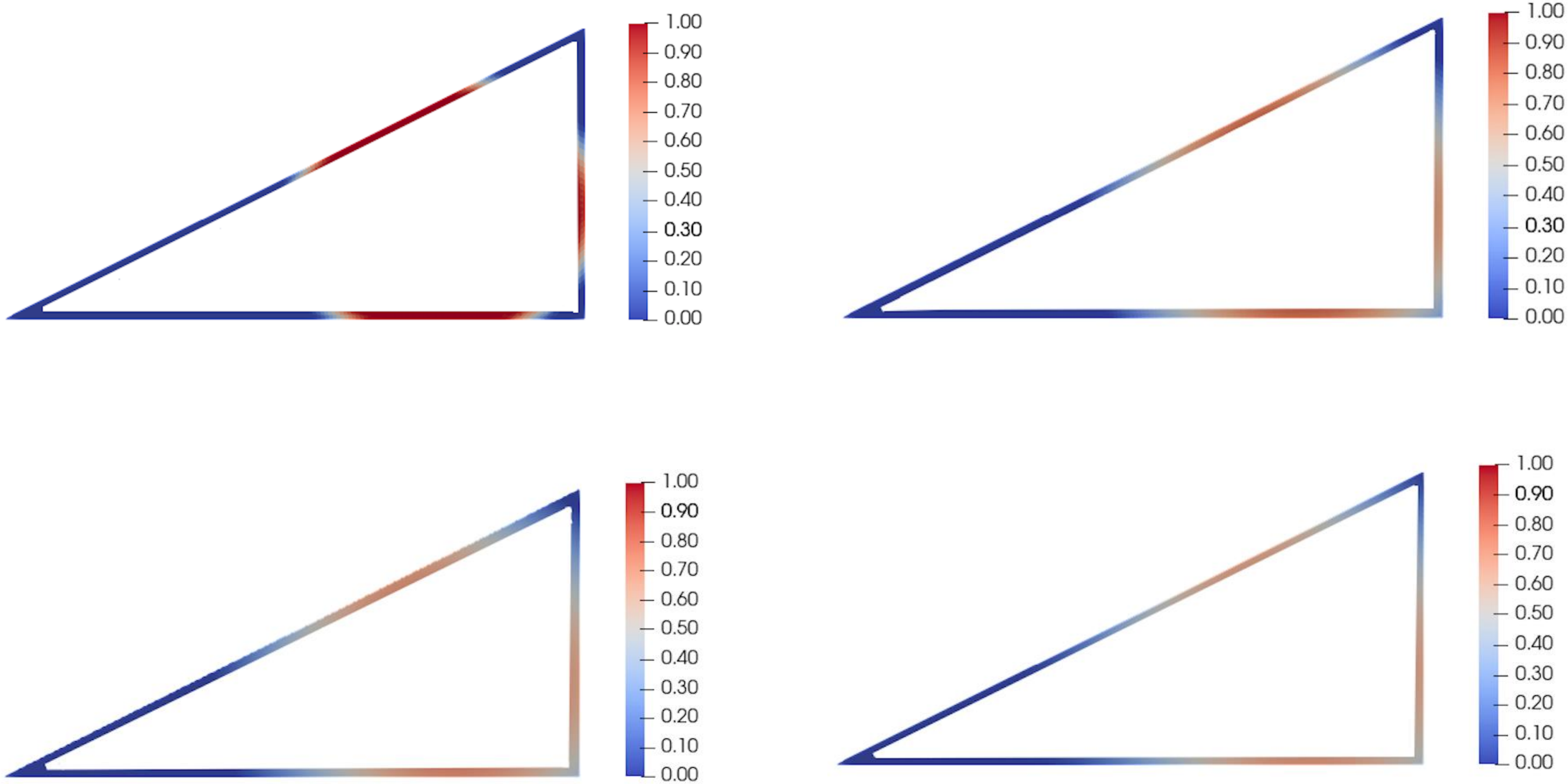
$$\text{if } \beta = \infty \qquad \theta_{n+1}(x) = \begin{cases} \dfrac{u_n}{\gamma} & \text{if } u_n \leq \gamma \\ 1 & \text{if } \gamma < u_n \end{cases}$$

$$\int_{\partial\Omega} \theta_{n+1}(x) d\sigma(x) = \kappa$$

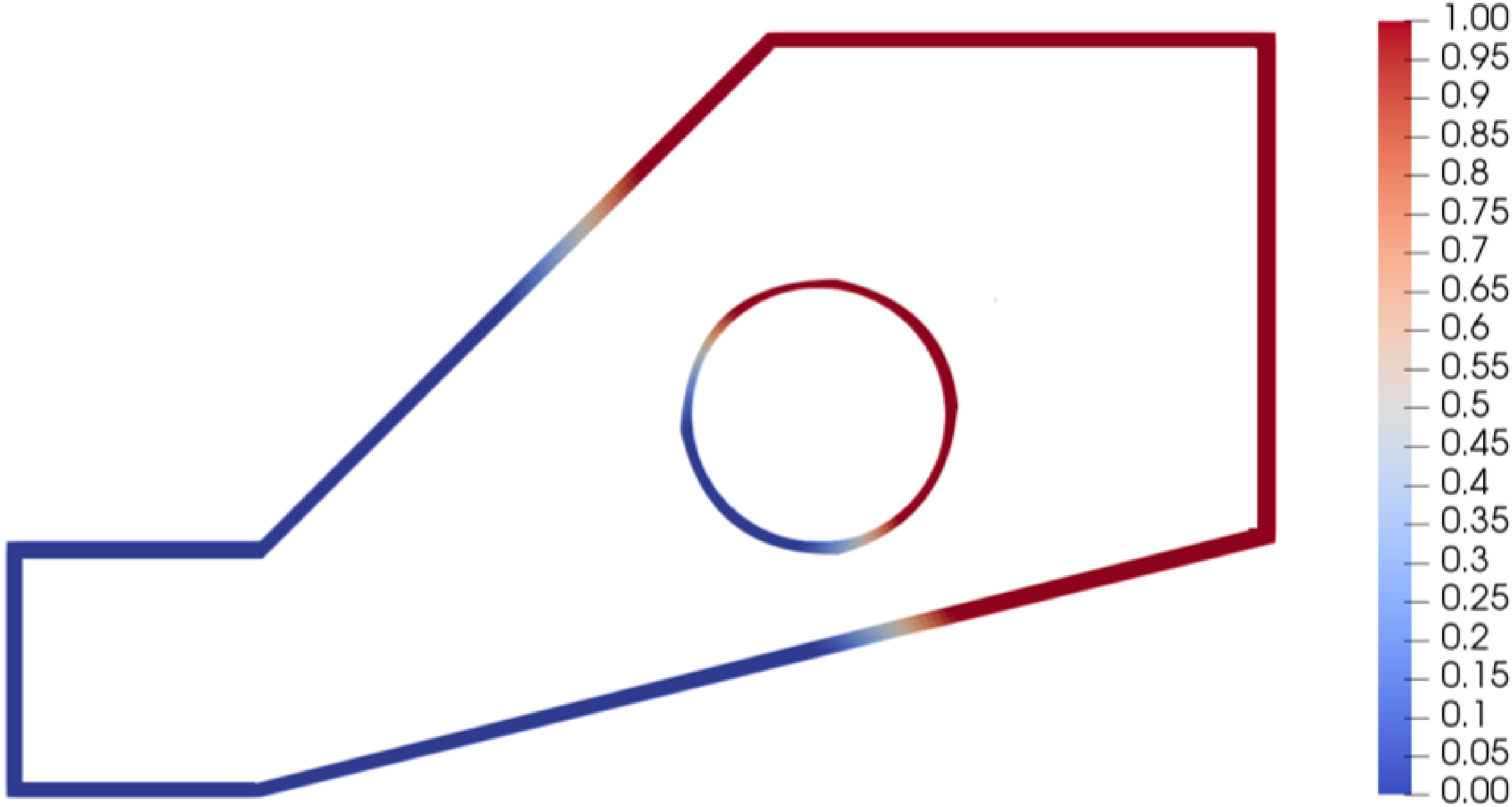


$$\Omega = \{(x, y):\ 0.3^2 < x^2 + y^2 < 1\}$$

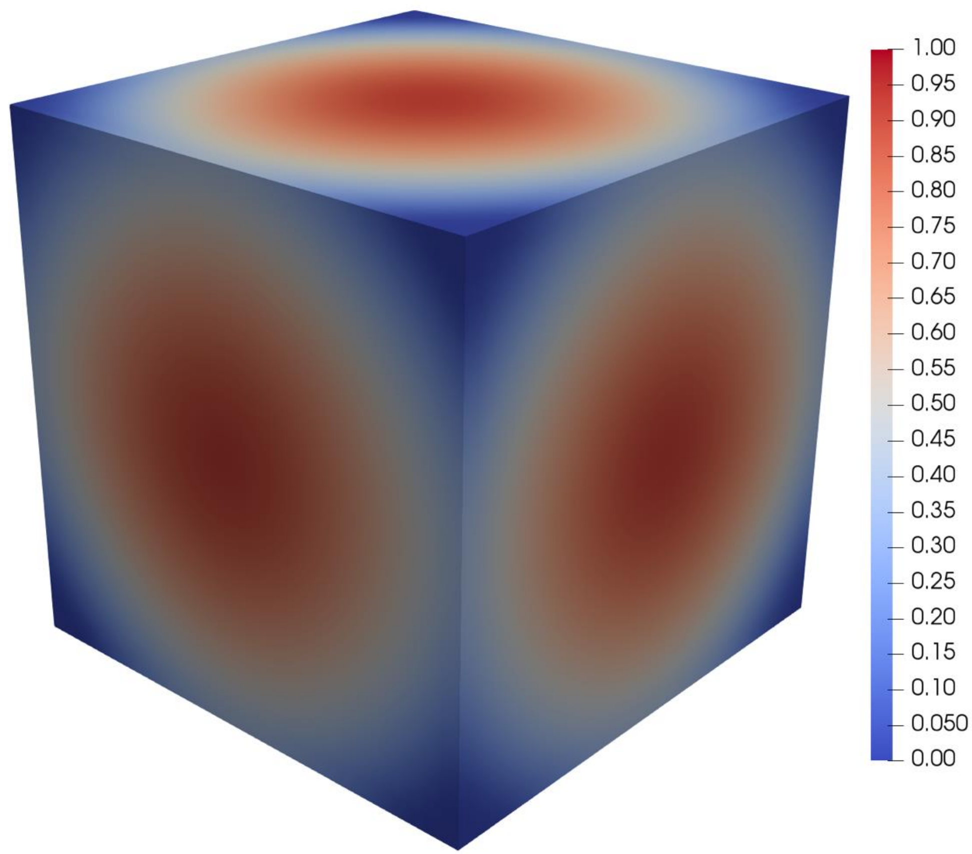
$$\alpha = 1, \quad \beta = 2, \quad \kappa = 0.5|\partial\Omega|_1,$$



$$\alpha = 1, \quad \kappa = 0.3|\partial\Omega|_1, \quad \beta = 1.2, \quad \beta = 2, \quad \beta = 5, \quad \beta = \infty$$



$\alpha = 1, \quad \beta = 5, \quad \kappa = 0.5|\partial\Omega|_1,$



$$\alpha = 1, \quad \beta = 2, \quad \kappa = 0.5|\partial\Omega|_1,$$