Geometric Analysis of Gradient Flow Problems in Shape Optimization

Jan Sokolowski

Institut Elie Cartan
Systems Research Institute Polish Academy of Sciences

Joint work with Pavel I. Plotnikov

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Outline

Background

- Shape Optimization is used in Solid and Fluid Mechanics:
  - Linear Elasticity
  - Navier-Stokes Equations
- Mathematical results:
  - Existence of optimal shapes
  - The first order and the second order optimality conditions
  - Numerical methods of shape optimization and Applications
- Geometric Analysis of Properties of Gradient Flow Dynamical System for Shape Optimization

Geometric Aspects

Pavel I. Plotnikov and Jan Sokolowski:
Geometric Framework for Gradient Flow in Shape Optimization,
Springer Briefs, in preparation.

Structural Optimization

The compliance problem is one of the most important shape optimization problems in the elasticity theory. In the simplest case, it can be formulated as follows. It is assumed that the two-component material fills a domain $\Omega \subset \mathbb{R}^2$, which consists of two subdomains Ω^+ and $\Omega^- = \Omega \setminus \mathrm{cl}\ \Omega^+$ occupied by different components. The displacement field $u(x)$ satisfies the equations of the elasticity theory

$$\begin{aligned} \mathrm{div}\ \left(a(\nabla u + \nabla u^\top) + b\mathrm{div}\ u\,\mathbf{I}\right) &= 0 \text{ in } \Omega, \\ \left(a(\nabla u + \nabla u^\top) + b\mathrm{div}\ u\,\mathbf{I}\right) \cdot \mathbf{n} + g &= 0 \text{ on } \partial\Omega. \end{aligned} \tag{1}$$

Here, $a = a^\pm$ and $b = b^\pm$ in $\Omega^\pm$ are the elasticity coefficients, g is a given external force. The problem is to minimize the work of external force by optimally chosen shape of the inclusion Ω^+.

Compliance Problem

In this case, the cost function is determined by the expression

$$\mathcal{J}_c = \int_{\Omega} \left(a(\nabla u + \nabla u^{\top}) + b\text{div } u\,\mathbf{I} \right) : \nabla u \, dx.$$

Note that $\mathcal{J}_c$ is a Kohn-Vogelius type functional and its Hadamard gradient admits an effective integral representation. It seems interesting to extend the results of the forthcoming book (P.I. Plotnikov, J.S., Geometric Framework for Gradient Flow in Shape Optimization) to the case of the compliance problem. The possibility of such a generalization is due to the fact that transmission problem (1) is reduced to a boundary value problem for a pair of piece-wise holomorphic functions using the Kolosov-Muskhelishvili formulae. Therefore, we can apply the method developed in the book.

Shape Optimization

1. Direct method of calculus of variations: regularization and shape calculus;
2. State equation, cost functional, numerical method of solution: finite elements;
3. Phase field method for problems depending on characteristic functions: a possibility to use the homogenization method;
4. Level set method based on the shape gradent and/or on the topological derivative concept.

Shape Optimization

1. The convergence of simple gradient method for numerical solution of shape optimization problems is still not known.
2. We have a result on the convergence in two spatial dimensions for a model problem.
    - The regularization of the cost is required in order to assure the existence of an optimal shape.
    - The regularization term can be considered as a cost of manufacturing so the parameter is not small.
    - in numerical methods of shape optimization the discretization of the continuous gradient is exclusively used, the exact gradient is expensive.

In the simplest case, the transmission single measurement identification problem can be formulated as follows. Suppose that a material occupies the bounded simple connected region Ω in the space of points $x \in \mathbb{R}^2$. Without loss of generality, we can assume that the boundary of Ω is infinitely differentiable Jordan curve. The inclusion, which is unknown and must be determined together with the solution, occupies the simply connected subdomain $\Omega^+ \Subset \Omega$ with the Jordan boundary Γ. The equilibrium equations for the electric field potential $u : \Omega \to \mathbb{R}$ can be written as

$$\begin{aligned} \mathrm{div}\,(a_0 \nabla u) = 0 \quad \text{in} \quad \Omega, \\ \boldsymbol{\nu} \cdot \nabla u = q \quad \text{on} \quad \partial\Omega. \end{aligned} \tag{2}$$

Here q is a given distribution of the voltage, $\boldsymbol{\nu}$ is the outward normal to $\partial\Omega$. We will assume that

$$q \in L^2(\partial\Omega), \quad \int_{\partial\Omega} q\, ds = 0. \tag{3}$$

The conductivity a_0 is defined by the equalities

$$a_0 = 1 \quad \text{in} \quad \Omega^- = \Omega \setminus \overline{\Omega^+}, \quad a_0 = a \quad \text{in} \quad \Omega^+, \tag{4}$$

where $a \neq 1$ is a given positive constant.

For every q satisfying condition (3), problem (2) admits a unique solution $u \in W^{1,2}(\Omega)$ satisfying the orthogonality condition

$$\int_{\partial\Omega} u\, ds = 0. \tag{5}$$

The problem on the identification of the inclusion Ω^+ is formulated as follows. For a given function $P : \partial\Omega \to \mathbb{R}$, it is necessary to find an inclusion Ω^+ such that the solution to problem (2) satisfies the extra boundary condition

$$u = P \quad \text{on} \quad \partial\Omega. \tag{6}$$

More generally, the problem of identification is to determine the shape of the inclusion by the additional boundary condition. This inverse problem is ill-posed and in general case has no solution. In practice, its approximate solution can be found by solving the variational problem

$$\min_{\Omega^+ \in \mathcal{O}} \mathcal{J}(\Omega^+)\,, \tag{7}$$

where the objective function $\mathcal{J}(\Omega^+)$ is a positive function that vanishes if and only if a solution to problem (2) satisfies condition (6), $\mathcal{O}$ is some class of admissible inclusions. Notice that the mapping $\Omega^+ \to u$, where u is a weak solution to problem (2), determines a nonlinear operator, which takes the set of admissible shapes $\mathcal{O}$ into $W^{1,2}(\Omega)$.

The most successful choice of the objective function is the Kohn-Vogelius energy functional, which is defined by the equality,

$$\mathcal{J}(\Omega^+) = \int_\Omega a_0 \nabla(u - U) \cdot \nabla(u - U)\, dx. \tag{8}$$

Here $u, U : \Omega \to \mathbb{R}$ satisfy the equations and boundary conditions

$$\begin{aligned} \operatorname{div}\,(a_0 \nabla u) &= 0, & \operatorname{div}\,(a_0 \nabla U) &= 0 & \text{in}\ \ &\Omega, \\ \nabla u \cdot \boldsymbol{\nu} &= q, & U &= P & \text{on}\ \ &\partial\Omega. \end{aligned} \tag{9}$$

Unfortunately, the identification problems as stated with no additional geometric constrains are ill-posed. Because in the absence of strong compactness of the minimizing sequences of designs, the optimal state should be attained by a fine mixture of different phases. The natural and widely used approach is to penalize the cost functional. In order to describe the penalization procedure and explain our strategy, it is convenient to introduce some standard notation, which will be used throughout of the book.

Denote by $\mathbb{S}^1$ the unit circle supplemented with the angle variable θ. We will consider 2π periodic functions $f : \mathbb{R} \to \mathbb{R}^d$ as mappings $f : \mathbb{S}^1 \to \mathbb{R}^d$. Hereinafter we will use the notation $\partial \equiv \partial_\theta$.

An immersion $f : \mathbb{S}^1 \to \mathbb{R}^d$ is a C^1 mapping satisfying the condition

$$0 < f^- \leq |\partial f(\theta)| \leq f^+ < \infty.$$

If an immersion is bijection, then we will say that f is an embedding.
For a given f, denote by $\Gamma \subset \mathbb{R}^d$ the curve $\Gamma = f(\mathbb{S}^1)$. The arc-length variable s on Γ and the length element ds of Γ are functions of θ. They are defined by the equalities

$$s(\theta) = \int_0^\theta \sqrt{g(\sigma)}\, d\sigma, \quad ds = \sqrt{g(\theta)}\, d\theta, \quad g = |\partial f|^2. \tag{10}$$

In this setting, the derivative $\partial_s = g^{-1/2}\partial := g^{-1/2}\partial_\theta$ with respect to the arc-length variable becomes a nonlinear differential operator.
The tangent vector $\boldsymbol{\tau}$ to Γ and the curvature vector $\mathbf{k}$ are given by the equalities

$$\boldsymbol{\tau}(\theta) = \partial_s f(\theta) := |\partial f|^{-1}\, \partial_\theta f(\theta), \quad \mathbf{k}(\theta) = \partial_s \boldsymbol{\tau}(\theta) = \partial_s^2 f(\theta). \tag{11}$$

Notice that $\mathbf{k}$ is orthogonal to $\boldsymbol{\tau}$. For every smooth vector field $\phi : \mathbb{S}^1 \times (0, T) \to \mathbb{R}^d$, the space and time normal connections ∇_s and ∇_t are defined by the relations

$$\nabla_s\, \phi = \partial_s \phi - (\partial_s \phi \cdot \boldsymbol{\tau})\, \boldsymbol{\tau}, \quad \nabla_t\, \phi = \partial_t \phi - (\partial_t \phi \cdot \boldsymbol{\tau})\, \boldsymbol{\tau}, \tag{12}$$

which can be written in the equivalent form

$$\nabla_s\, \phi = \Pi \partial_s \phi, \quad \nabla_t\, \phi = \Pi \partial_t \phi, \quad \Pi\phi = \phi - (\phi \cdot \boldsymbol{\tau})\, \boldsymbol{\tau}.$$

For planar curves, we assume that the point $f(\theta)$ moves around Γ in the positive counterclockwise direction while the parameter θ increases. In this case, the tangent vector $\boldsymbol{\tau}$ and the normal vector $\mathbf{n} = (-\tau_2, \tau_1)$ form the positive orthonormal frame moving along Γ.

The Euler elastica energy $\mathcal{E}_e$ and the length $\mathcal{P}$ of Γ are determined by the formulae

$$\mathcal{E}_e = \frac{1}{2}\int_\Gamma |\mathbf{k}|^2\, ds, \quad \mathcal{P} = \int_\Gamma ds. \tag{13}$$

The total energy $\mathcal{E}$ of the curve Γ is given by the equality

$$\mathcal{E} = \mathcal{E}_e + \mathcal{P} = \int_\Gamma \Big(\frac{1}{2}|\mathbf{k}|^2 + 1\Big)\, ds. \tag{14}$$

Assuming that the interface $\Gamma = \partial\Omega^+$, we can take the strong regularization of the Kohn-Vogelius functional in the form $\mathcal{E} + \mathcal{J}$.

With this notation we may consider the cost function $\mathcal{J}$, as a functional defined on the set of smooth embeddings $f : \mathbb{S}^1 \to \Omega$. Recall the definition of the Hadamard gradient of $\mathcal{J}(f)$.

Definition 1

A vector field $d\mathcal{J}(f) = \mathcal{B}\mathbf{n} : \mathbb{S}^1 \to \mathbb{R}^2$, $\mathcal{B} \in L^1(\mathbb{S}^1)$ is said to be the *Hadamard gradient* of $\mathcal{J}$ at the point f, if the integral identity

$$\lim_{t\to 0} \frac{1}{t}\big(\mathcal{J}(f + t\delta f) - \mathcal{J}(f)\big) = \int_\Gamma \mathcal{B}\mathbf{n}\cdot\delta f\, ds \equiv \int_0^{2\pi} \sqrt{g}\,\mathcal{B}\mathbf{n}\cdot\delta f\, d\theta \tag{15}$$

holds for every smooth vector field $\delta f : \mathbb{S}^1 \to \mathbb{R}^2$.

The explicit formulae for the Hadamard gradients of the Kohn-Vogelius type functionals are well known in literature. In particular, the Hadamard gradient of the Kohn-Vogelius functional (8) is given by

$$\begin{aligned} d\mathcal{J} &= 2\big(a_0\partial_n u\left[\partial_n u\right] - a_0\partial_n U\left[\partial_n U\right]\big)\,\mathbf{n} \\ &- \left[a_0\nabla u\cdot\nabla u - a_0\nabla U\cdot\nabla U\right]\mathbf{n}, \end{aligned} \tag{16}$$

where $\mathbf{n}$ is the inward normal to $\partial\Omega^+ = \Gamma$, $[\,\cdot\,]$ denotes the jump across Γ in the normal direction from Ω^- towards Ω^+, u and U are solutions to equations (9). The gradient of the cost functional can be regarded as a nonlinear operator acting on the periodic mapping $f : \mathbb{S}^1 \to \mathbb{R}^2$. We will denote this operator by

$$\mathcal{B}(f) := d\mathcal{J}(f) = \mathcal{B}\mathbf{n}.$$

It is a classic result that the Hadamard gradients $d\mathcal{E}_e$ and $d\mathcal{P}$ are defined by the equalities

$$d\mathcal{E}_e(f) = \nabla_s\nabla_s\,\mathbf{k} + \frac{1}{2}|\mathbf{k}|^2\,\mathbf{k}, \quad d\mathcal{P} = -\mathbf{k},$$

In particular, we have

$$d\mathcal{E}(f) = \nabla_s\nabla_s\,\mathbf{k} + \frac{1}{2}|\mathbf{k}|^2\,\mathbf{k} - \mathbf{k} \equiv \mathcal{A}(f). \tag{17}$$

Note that $\mathcal{A}(f)$ is a quasilinear fourth order differential operator.

We split it into the elastic part $\mathcal{A}_e$ and the perimeter part $\mathcal{A}_p$,

$$\mathcal{A} = \mathcal{A}_e + \mathcal{A}_p, \quad \mathcal{A}_e = \nabla_s \nabla_s \mathbf{k} + \frac{1}{2}|\mathbf{k}|^2 \mathbf{k}, \quad \mathcal{A}_p = -\mathbf{k}.$$

The most important question of the theory is the construction of a robust algorithm for the numerical study of shape optimization problems. The standard approach is to use the steepest descent method which can be described as follows. The Hadamard shape gradient $d\mathcal{J} = \mathcal{B}$ can be regarded as a normal vector field on Γ parallel to the normal vector field $\mathbf{n}$. If f is sufficiently smooth, for example $f \in C^{2+\alpha}$, then the mapping $f + \sigma d\mathcal{J}(f)$ defines an immersion of $\mathbb{S}^1$ into $\mathbb{R}^2$ for all sufficiently small $\sigma > 0$. In the steepest descent method, the optimal immersion f and the corresponding shape $\Gamma = f(\mathbb{S}^1)$ are determined as a limit of the sequence of immersions

$$f_{n+1} = f_n - \sigma\, d\mathcal{J}(f_n), \quad n \geq 0, \tag{18}$$

and the corresponding sequence of curves $\Gamma_n = f_n(\mathbb{S}^1)$. The steepest descent method for our problem have been investigated by many authors. Recurrent relation (18) can be regarded as the time discretization of the evolution problem

$$\partial_t f + \mathcal{B}(f) = 0 \;\text{ in }\; \mathbb{S}^1 \times [0, T), \quad f\Big|_{t=0} = f_0 \tag{19}$$

with a quasi-time t. This equation is nothing else but the gradient flow of the Kohn-Vogelius functional.

Similarly, the gradient flow of the regularized Kohn-Vogelius functional is defined by the equations

$$\partial_t f + \mathcal{A}(f) + \mathcal{B}(f) = 0 \quad \text{in} \quad \mathbb{S}^1 \times [0, T), \quad f\Big|_{t=0} = f_0. \tag{20}$$

It can be regarded as a perturbation of a well-known geometric equation

$$\partial_t f + \mathcal{A}(f) = 0 \quad \text{in} \quad \mathbb{S}^1 \times [0, T), \quad f\Big|_{t=0} = f_0 \tag{21}$$

called *the straightening equation* or *one-dimensional Willmore flow*. In this case, it is not necessary to consider Γ as boundary of some planar domain. Therefore, we will consider the straightening equation in the space of multidimensional immersions $f : \mathbb{S}^1 \times (0, T) \to \mathbb{R}^d$, $d \geq 2$. The straightening equation can be written in the form of the evolutionary nonlinear partial differential equation

$$\partial_t f + \nabla_s^2 \mathbf{k} + \frac{1}{2}|\mathbf{k}|^2 \mathbf{k} - \mathbf{k} = 0, \quad f(0) = f_0. \tag{22}$$

Results

1. The local solvability of equations (20) and (21) is shown.
2. The equations (20) and (21) (like e.g., the reaction-diffusion equations) have no type. The type is identified after linearization at a given solution. The tangential component of the equations is a nonlinear ODE.
3. The Nash-Moser type Newton method with smoothing is used for the proof.

References

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