A Steklov version of the torsional rigidity

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Bibliography

[1] L. Brasco, M. González and M. I. “A Steklov version of torsional rigidity”. *Communications in Contemporary Mathematics* (2024).

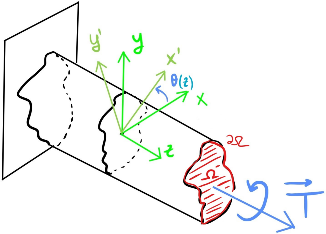
Motivation: Classical torsional rigidity

- The *stress-function* v_Ω:

$$\begin{cases} -\Delta v_\Omega = 1, & \text{in } \Omega, \\ v_\Omega = 0, & \text{on } \partial\Omega, \end{cases}$$

- The *torsional rigidity, $T(\Omega)$*:

$$\mathcal{T}(\Omega) = \int_\Omega v_\Omega \, dx = \sup_{\varphi|_{\partial\Omega}=0} \frac{\left(\int_\Omega \varphi \, dx\right)^2}{\int_\Omega |\nabla \varphi|^2 \, dx}.$$



Let $\lambda_1(\Omega)$ be the first *Dirichlet's eigenvalue*:

$$\begin{cases} -\Delta u = \lambda_1 u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

Saint-Venant inequality:

$$T(\Omega) \leq T(\Omega^*)$$

Faber-Krahn inequality:

$$\lambda_1(\Omega^*) \leq \lambda_1(\Omega)$$

Classic **Pólya**'s inequality

$$\frac{T(\Omega)\lambda_1(\Omega)}{|\Omega|} \leq 1.$$

$$|\Omega| = |\Omega^*|$$

Steklov eigenvalue

Posed by V. A. Steklov at the turn of the 20th century.

Steklov Eigenvalue Problem

$$\begin{cases} -\Delta u = 0, & \text{in } \Omega, \\ \partial_\nu u = \sigma\, u, & \text{on } \partial\Omega. \end{cases}$$

Compact embedding $W^{1,2}(\Omega) \hookrightarrow L^q(\partial\Omega)$
Applications: sloshing problem, electric impedance tomography, spectral shape optimization, Dirichlet-to-Neumann operator...

- **Brock-Weinstock** inequality: $\sigma(\Omega) \le \sigma(\Omega^*)$.

Question 1: Is there a *boundary torsional rigidity related to* σ*?*

Question 2: If so, is it meaningful?

Boundary torsional rigidity. Definition.

δ-Steklov eigenvalue:

$$\begin{cases} -\Delta u = \delta u, & \text{in } \Omega, \\ \partial_\nu u = \sigma_\delta\, u, & \text{on } \partial\Omega. \end{cases}$$

Boundary δ-torsional rigidity

$$\mathcal{T}(\Omega;\delta) = \sup \frac{\left(\int_{\partial\Omega} \varphi \, d\mathcal{H}^{N-1}\right)^2}{\int_\Omega |\nabla\varphi|^2 \, \mathrm{d}x + \delta^2 \int_\Omega \varphi^2 \, \mathrm{d}x},$$

Boundary δ-torsion function

$$\begin{cases} -\Delta u_{\Omega,\delta} + \delta^2 u_{\Omega,\delta} = 0, & \text{in } \Omega, \\ \partial_\nu u = 1, & \text{on } \partial\Omega. \end{cases}$$

Boundary δ-torsional rigidity

$$\mathcal{T}(\Omega;\delta) = \int_{\partial\Omega} u_{\Omega,\delta} \, d\mathcal{H}^{N-1}$$

Related to best constant in Sobolev embedding $W^{1,2}(\Omega) \hookrightarrow L^1(\partial\Omega)$.

Remark: Pòlya type inequality

$$\frac{\sigma(\Omega;\delta)\, T(\Omega;\delta)}{\mathcal{H}^{N-1}(\partial\Omega)} \leq 1,$$

As a limit...

Let $\Omega \subset \mathbb{R}^N$ with C^2 boundary $\partial\Omega$. We define the strip

$$\omega_\varepsilon = \{x - \alpha\nu_\Omega(x), x \in \partial\Omega, \alpha \in [0,\varepsilon)\}$$

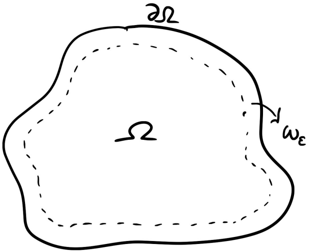
Consider:

$$\begin{cases} -\nabla\cdot(a(x)\nabla u^\varepsilon(x)) + \lambda u^\varepsilon(x) + c(x)u^\varepsilon = \frac{1}{\varepsilon}\chi_{\omega_\varepsilon} f_\varepsilon & \text{in } \Omega, \\ a(x)\langle\nabla u^\varepsilon, \nu_\Omega\rangle + b(x)u^\varepsilon = 0 & \text{on } \partial\Omega. \end{cases} \quad (1)$$

Theorem [J. M. Arrieta, A. Jiménez-Casas, A. Rodríguez-Bernal, 2008]

Set $a = 1$ $\quad b = c = 0 \quad \lambda = \delta^2$ and $\quad f_\varepsilon = 1$.
Taking $\varepsilon \to 0$, the unique solution u_ε of (1) converges to the solution of

$$\begin{cases} -\Delta u + \delta^2 u = 0, & \text{in } \Omega, \\ \langle\nabla u, \nu_\Omega\rangle = 1, & \text{on } \partial\Omega. \end{cases}$$



Geometric Estimates

- Lower bound in dim 2.
- Lower bound in dim N (Convex sets).
- Upper bound in dim N (Convex sets).

Geometric Properties in dim 2

Let $\mathbb{D} = \{x \in \mathbb{R}^2 \,:\, |x| < 1\}$, $\Omega \subsetneq \mathbb{R}^2$ simply connected, $x_0 \in \Omega$:

- *Riemann Mapping Theorem:* $\exists!$ holomorphic isomorphism

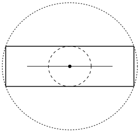
$$f_{x_0} : \mathbb{D} \to \Omega, \qquad \text{with } f_{x_0}(0) = x_0.$$

  Furthermore, when $\partial\Omega$ is $C^{1,\alpha}$, we know that $f_{x_0} \in C^1$ in $\overline{\mathbb{D}}$ and

$$f'_{x_0}(x) \neq 0, \qquad \text{for every } x \in \partial\mathbb{D}.$$

- *Boundary distortion radius* of Ω:

$$\dot{\mathcal{R}}_\Omega := \inf_{x_0 \in \Omega} \left(\frac{1}{2\pi} \int_{\partial\mathbb{D}} |f'_{x_0}|^2 \, d\mathcal{H}^1 \right)^{\frac{1}{2}},$$



- *Inradius*: $r_\Omega := \sup_{x \in \Omega} d_\Omega(x)$

- *High ridge set*: $M(\Omega) := \left\{ x \in \Omega \,:\, B_{r_\Omega}(x) \subset \Omega \right\}$

- *Proximal radius*: $L_\Omega := \inf\{R > 0 : \exists x_0 \in M(\Omega) \text{ s.t. } \Omega \subset B_R(x_0)\}$

Theorem (Lower bound in $\Omega \subset \mathbb{R}^2$)

Let $\delta > 0$ and let $\Omega \subset \mathbb{R}^2$ be a bounded simply connected open set, with $C^{1,\alpha}$ boundary, for some $0 < \alpha \leq 1$. Then,

$$\left(\frac{\mathcal{H}^1(\partial\Omega)}{2\,\pi}\right)^2 \frac{T\left(B_{\dot{\mathcal{R}}_\Omega};\delta\right)}{\dot{\mathcal{R}}_\Omega^2} \leq T(\Omega;\delta).$$

Moreover, equality holds if and only if Ω is a disk.

Geometric Lemma

Let $\Omega \subset \mathbb{R}^2$ be a bounded simply connected open set, with $\partial\Omega \in C^{1,\alpha}$. Then,

$$|\Omega| \leq \pi\,\dot{\mathcal{R}}_\Omega^2.$$

In particular, if Ω is a disk of radius R, we have $\dot{\mathcal{R}}_\Omega = R$ and equality holds.

Proof

Assume $\dot{\mathcal{R}}_\Omega = 1$ and define $\widetilde{u} = u \circ h_{x_0}$, where $h_{x_0} := f_{x_0}^{-1} : \overline{\Omega} \to \overline{\mathbb{D}}$.

Sketch of the proof. First, test the variational definition of $\mathcal{T}$:

$$\frac{1}{T(\Omega;\delta)} \leq \frac{\int_\Omega |\nabla \widetilde{u}|^2\,dx + \delta^2 \int_\Omega \widetilde{u}^2\,dx}{\left(\int_{\partial\Omega} \widetilde{u}\,d\mathcal{H}^1\right)^2} = \left(\frac{\delta\,I_1(\delta)}{I_0(\delta)}\right)^2 \frac{\int_{\mathbb{D}} |\nabla u|^2\,dw + \overbrace{\delta^2 \int_{\mathbb{D}} u^2\,|f'_{x_0}(w)|^2\,dw}^{I}}{(\mathcal{H}^1(\partial\Omega))^2},$$

In order to estimate (I) we set

$$\Phi(\varrho) = \frac{1}{2\pi\varrho}\int_{\{|w|=\varrho\}} |f'_{x_0}|^2\,d\mathcal{H}^1,$$

By monotonicity of $\varrho \mapsto \Phi(\varrho)$,

We indicate the solution in $\mathbb{D}$ as

$$u(x) = \mathcal{U}_\delta(\varrho) = \frac{\varrho^{1-N/2}\,I_{N/2-1}(\delta\,\varrho)}{\delta\,I_{N/2}(\delta)}$$

where $\varrho = |x|$.

$$\int_{B_1(0)} u^2\,|f'_{x_0}(w)|^2\,dw = 2\pi\int_0^1 u^2\,\Phi(\varrho)\,\varrho\,d\varrho \leq \left(2\pi\int_0^1 u^2\,\varrho\,d\varrho\right)\Phi(1) < (1+\varepsilon)\int_{\mathbb{D}} u^2\,dx.$$

Finally, recall u is optimal for the disk.

Lower bound, N-dim Convex sets

Theorem

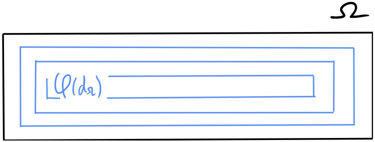
Let $\delta > 0$ and let $\Omega \subset \mathbb{R}^N$ be an open bounded **convex** set. Then,

$$T(\Omega;\delta) > \frac{\mathcal{H}^{N-1}(\partial\Omega)}{\delta\ \tanh\left(\delta\, r_\Omega\right)}.$$

Moreover, the estimate is sharp in the following sense: we have

$$\lim_{n\to\infty} \frac{T(\Omega_n;\delta)\ \tanh(\delta\, r_{\Omega_n})}{\mathcal{H}^{N-1}(\partial\Omega_n)} = \frac{1}{\delta}, \qquad \text{where } \Omega_n := (-n,n)^{N-1} \times (-1,1).$$

Method of interior parallels [E. Makai, 1954][G. Pòlya, 1960]



Proof

Assume $r_\Omega = 1$ and define $\varphi(x) = u_I(d_\Omega(x))$

$$T(\Omega;\delta) \geq \frac{\left(\int_{\partial\Omega} \varphi \, d\mathcal{H}^{N-1}\right)^2}{\int_\Omega |\nabla\varphi|^2 \, dx + \delta^2 \int_\Omega \varphi^2 \, dx} = \frac{(u_I(0))^2 \, (\mathcal{H}^{N-1}(\partial\Omega))^2}{\int_0^1 \left[|u_I'(t)|^2 + \delta^2 \, (u_I(t))^2\right] \mathcal{H}^{N-1}(\partial\Omega_t) \, dt}$$

$$\geq \frac{(u_I(0))^2}{\int_0^1 \left[|u_I'(t)|^2 + \delta^2 \, (u_I(t))^2\right] dt} \mathcal{H}^{N-1}(\partial\Omega).$$

- Coarea formula
- $|\nabla d_\Omega| = 1$
- Lemma

- $\mathcal{H}^{N-1}(\partial\Omega_t) \leq \mathcal{H}^{N-1}(\partial\Omega)$ for $t \in (0, r_\Omega)$, strict for an open bounded convex set

Lemma (test function)

For every $\delta > 0$, we have

$$\alpha(\delta) := \sup_{\varphi \in W^{1,2}(I)} \frac{(\varphi(0))^2}{\displaystyle\int_I |\varphi'|^2\,dt + \delta^2 \int_I \varphi^2\,dt} = \frac{1}{\delta\,\tanh(\delta)}.$$

Moreover, the maximum is attained by

$$u_I(t) = \frac{1}{\delta}\left(\frac{\cosh(\delta\,t)}{\tanh(\delta)} - \sinh(\delta\,t)\right),$$

Proof. We rephrase the maximization problem to

$$\alpha(\delta) = \sup_{\varphi \in W^{1,2}(I)} \left\{2\,\varphi(0) - \int_I |\varphi'|^2\,dt - \delta^2 \int_I \varphi^2\,dt\right\},$$

which is the weak formulation of the following ODE:

$$\begin{cases} -\varphi'' + \delta^2\,\varphi = 0, & \text{in } I, \\ \varphi'(0) = -1, & \\ \varphi'(1) = 0. & \end{cases}$$

Upper bound N-dim convex sets

Dual formulation

We set

$$\mathcal{A}^+(\Omega) = \left\{ (\phi, g) \in L^2(\Omega; \mathbb{R}^N) \times L^2(\Omega) \; : \; \begin{array}{rr} -\operatorname{div} \phi + \delta^2 g \geq 0, & \text{in } \Omega \\ \langle \phi, \nu_\Omega \rangle \geq 1, & \text{on } \partial\Omega \end{array} \right\},$$

whith the conditions intended in weak sense. Then, we have

$$T(\Omega; \delta) = \min_{(\phi,g)\in\mathcal{A}^+(\Omega)} \left\{ \int_\Omega |\phi|^2 \, dx + \delta^2 \int_\Omega g^2 \, dx \right\}, \tag{2}$$

and the minimum is uniquely attained by the pair $(\nabla u_{\Omega,\delta}, u_{\Omega,\delta})$.

Theorem: Upper bound N-dim

Let $\delta > 0$ and let $\Omega \subset \mathbb{R}^N$ be an open bounded **convex** set. Then,

$$T(\Omega; \delta) \leq \left(\frac{r_\Omega}{L_\Omega} \right)^{N-2} \left(\frac{I_{N/2}(\delta \, L_\Omega)}{I_{N/2}(\delta \, r_\Omega)} \right)^2 T(B_{L_\Omega}; \delta).$$