About optimal shape design of sensors

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(joint works with Enrique Zuazua)

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Plan of the presentation

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Initial problem and motivation

We address the issue of finding the optimal design of a park inside a given neighborhood in such a way to minimize the maximal distance from the park to all the citizen of the district.

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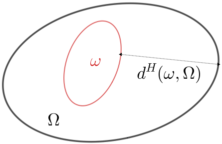
Given a set $\Omega \subset \mathbb{R}^2$, and a mass fraction $c \in (0, |\Omega|)$, the problem can be mathematically stated as follows

$$\inf\{\sup_{x\in\Omega} d(x,\omega) \mid |\omega| = c \text{ and } \omega \subset \Omega\},$$

where $d(x,\omega) := \inf_{y\in\omega} \|x-y\|$ is the minimal distance from x to ω.
The problem can be formulated via the **Hausdorff distance**

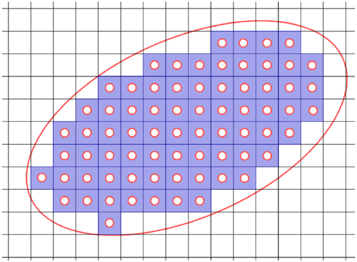
$$\inf\{d^H(\omega,\Omega) \mid |\omega| = c \text{ and } \omega \subset \Omega\}.$$



An additional (geometrical) constraint

We have:

$$\inf\{d^H(\omega,\Omega)\,|\,|\omega| = c \text{ and } \omega \subset \Omega\} = 0.$$



We are then going to assume that both ω and Ω are convex sets.

$$\inf\{d^H(\omega,\Omega)\,|\,\omega \text{ is convex, } |\omega| = c \text{ and } \omega \subset \Omega\}, \quad \text{where } c \in (0,|\Omega|).$$

We are interested in:

1. developing a numerical method to solve the problem.
2. Proving theoretical results on the problem.

A non-exhaustive list of related works

The problems involving the distance function have interested several authors:

- Asymptotique d′un problème de positionnement optimal (2002) by **G. Bouchitté, C. Jimenez and R. Mahadevan**.
- Optimal transportation problems with free Dirichlet regions (2002) by **G. Buttazzo, E. Oudet and Eugene Stepanov**.
- Minimization problems for average distance functionals (2004) by **G. Buttazzo and E. Stepanov**.
- Approximation of length minimization problems among compact connected sets (2015) by **M. Bonnivard, A. Lemenant, F. Santambrogio**.
- On convex sets that minimize the average distance (2010) by **A. Lemenant and E. Mainini**.

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An important lemma

Let J and H be two shape functionals and $\mathcal{C}$ a class of sets in $\mathbb{R}^n$.

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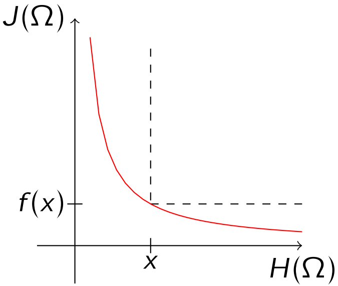
We consider

$$f : x \longmapsto \min\{J(\Omega) \mid \Omega \in \mathcal{C} \text{ and } H(\Omega) = x\}$$

Theorem (F., Lamboley (SIMA 2021))

Under some suitable assumptions on J, H and $\mathcal{C}$, the function f is continuous and strictly decreasing.
*Thus, the following problems **are equivalent**:*

- $\min\{J(\Omega) \mid \Omega \in \mathcal{C} \text{ and } H(\Omega) = x\}$,
- $\min\{J(\Omega) \mid \Omega \in \mathcal{C} \text{ and } H(\Omega) \leq x\}$,
- $\min\{H(\Omega) \mid \Omega \in \mathcal{C} \text{ and } J(\Omega) = f(x)\}$,
- $\min\{H(\Omega) \mid \Omega \in \mathcal{C} \text{ and } J(\Omega) \leq f(x)\}$.



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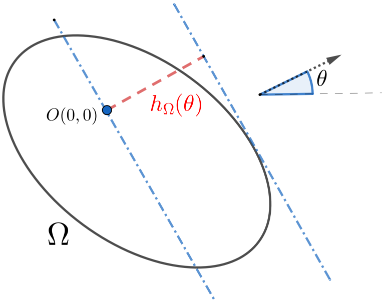
The support function of a convex set

Definition

Let Ω be a convex body. The support function h_Ω is defined on $[0, 2\pi]$ as

$$\forall \theta \in [0, 2\pi], \quad h_\Omega(\theta) := \sup_{x\in\Omega} \left\langle x, \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \right\rangle.$$



The support function of a convex set

Proposition

1. $$\Omega \text{ is convex} \Leftrightarrow \forall \theta \in [0, 2\pi], \quad h_\Omega''(\theta) + h_\Omega(\theta) \geq 0.$$

2. $$|\Omega| = \frac{1}{2}\int_0^{2\pi} \left(h_\Omega^2 - h_\Omega'^2\right) d\theta = \frac{1}{2}\int_0^{2\pi} h_\Omega (h_\Omega'' + h_\Omega) d\theta.$$

3. $$\Omega_1 \subset \Omega_2 \Longleftrightarrow h_{\Omega_1} \leq h_{\Omega_2}.$$

4. $$d^H(\Omega_1, \Omega_2) = \max_{\theta \in [0,2\pi]} |h_{\Omega_1}(\theta) - h_{\Omega_2}(\theta)| = \|h_{\Omega_1} - h_{\Omega_2}\|_\infty.$$

$\Rightarrow$ Idea: parametrize the shapes via the Fourier coefficients of its support function $h_\Omega(\theta) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta)$.

This idea has been introduced by **T. Bayen** and **D. Henrion** (2012) and used by different authors: **P. Antunes, B. Bogosel, I. F. ...**

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Equivalence between four problems

Theorem (F., Zuazua (J. Geom. Analysis 2023))

The function

$$f : c \in [0, |\Omega|] \longmapsto \inf\{d^H(\omega, \Omega) \mid \omega \text{ is convex}, |\omega| = c \text{ and } \omega \subset \Omega\}$$

is continuous and strictly decreasing. Moreover, for every $c \in [0, |\Omega|]$, the main problem admits solutions and is equivalent to the following shape optimization problems:

1. $\min\{d^H(\omega, \Omega) \mid \omega \text{ is convex}, |\omega| \le c \text{ and } \omega \subset \Omega\}$,
2. $\min\{\, |\omega| \mid \omega \text{ is convex},\ d^H(\omega, \Omega) = f(c) \text{ and } \omega \subset \Omega\}$,
3. $\min\{\, |\omega| \mid \omega \text{ is convex},\ d^H(\omega, \Omega) \le f(c) \text{ and } \omega \subset \Omega\}$,

in the sense that any solution of one of the problems also solves the other ones.

- As we shall see next, this equivalence result drastically simplifies the numerical resolution of the problem.

The initial problem

The problem

$$\min\left\{d^H(\omega,\Omega)\mid \omega \text{ is convex, } |\omega| = c \text{ and } \omega\subset\Omega\right\}$$

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The problem

$$\min \{d^H(\omega,\Omega) \mid \omega \text{ is convex}, |\omega| = c \text{ and } \omega \subset \Omega\}$$

is equivalent to the analytical one

$$\begin{cases} \min\limits_{h} \|h_\Omega - h\|_\infty, \\ h \le h_\Omega, \\ h'' + h \ge 0, \\ \frac{1}{2}\int_0^{2\pi} h(h''+h)d\theta = c. \end{cases}$$

That can be discretized as follows ($\theta_k := \frac{2k\pi}{M}$, $k \in [\![1, M]\!]$)

$$\begin{cases} \min_{(a_0,a_1,\dots,a_N,b_1,\dots,b_N)\in\mathbb{R}^{2N+1}} \left(\max\limits_{\theta\in[0,2\pi]} \left(h_\Omega(\theta) - a_0 - \sum\limits_{j=1}^{N} \left(a_j \cos(j\theta) + b_j \sin(j\theta) \right) \right) \right), \\ \forall k \in [\![1, M]\!], \quad a_0 + \sum\limits_{j=1}^{N} \left(a_j \cos(j\theta_k) + b_j \sin(j\theta_k) \right) \le h_\Omega(\theta_k), \\ \forall k \in [\![1, M]\!], \quad a_0 + \sum\limits_{j=1}^{N} \left((1-j^2)\cos(j\theta_k)a_j + (1-j^2)\sin(j\theta_k)b_j \right) \ge 0, \\ \pi a_0^2 + \frac{\pi}{2}\sum\limits_{j=1}^{N} (1-j^2)(a_j^2+b_j^2) = c. \end{cases}$$

The equivalent problem

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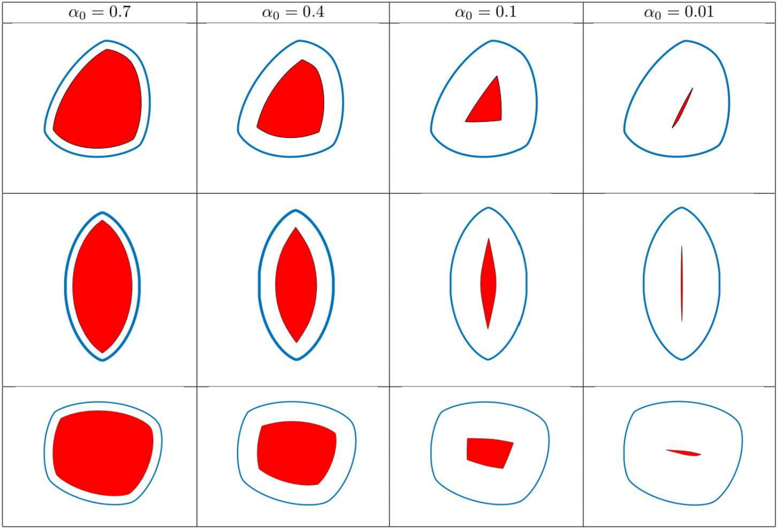
$$\begin{cases} \min\limits_h \frac{1}{2}\int_0^{2\pi} h(h''+h)d\theta, \\ h''+h \ge 0, \\ h \le h_\Omega, \\ h_\Omega - h \le \|h_\Omega - h\|_\infty \le d. \end{cases} \Longleftrightarrow \begin{cases} \min\limits_h \frac{1}{2}\int_0^{2\pi} h(h''+h)d\theta, \\ h''+h \ge 0, \\ h_\Omega - d \le h \le h_\Omega. \end{cases}$$

That can be discretized as follows ($\theta_k := \frac{2k\pi}{M}$, $k \in [\![1, M]\!]$)

$$\begin{cases} \min\limits_{(a_0,a_1,\dots,a_N,b_1,\dots,b_N)\in\mathbb{R}^{2N+1}} \pi a_0^2 + \frac{\pi}{2}\sum\limits_{j=1}^N (1-j^2)(a_j^2+b_j^2), \\ \forall k \in [\![1,M]\!], \quad h_\Omega(\theta_k) - d \le a_0 + \sum\limits_{j=1}^N \Big(a_j\cos(j\theta_k) + b_j\sin(j\theta_k)\Big) \le h_\Omega(\theta_k), \\ \forall k \in [\![1,M]\!], \quad a_0 + \sum\limits_{j=1}^N \Big((1-j^2)\cos(j\theta_k)a_j + (1-j^2)\sin(j\theta_k)b_j\Big) \ge 0. \end{cases}$$

This is a trivial numerical problem !

Obtained numerical results



Plan of the presentation

The case of the square

Theorem (F., Zuazua (J. Geom. Analysis 2023))

Let $\Omega = [0,1] \times [0,1]$ *be the unit square. There exists a threshold* $c_0 \in (0,1)$ *such that:*

- *If* $c \in [c_0, 1]$, *then the solution of the main problem is given by the square of area* c *and same axes of symmetry as* Ω.
- *If* $c \in [0, c_0)$, *then the solution of the main problem is given by a suitable rectangle.*

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Ingredients of the proof:

- Ω is a polygon of N sides $\Rightarrow$ ω is a polygon of at most N sides.
- The main problem is equivalent to $\min\{|\omega| \mid \omega$ is a quadrilateral s.t. $d^H(\omega, \Omega) = \delta\}$.
- $a + b \geq 2\sqrt{ab}$ with equality if and only if $a = b$.
- Basic calculus.

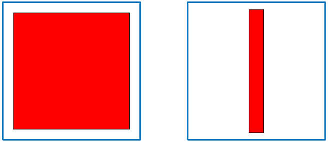
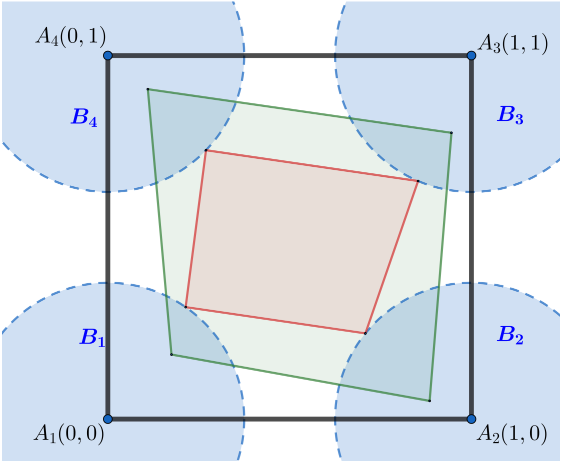


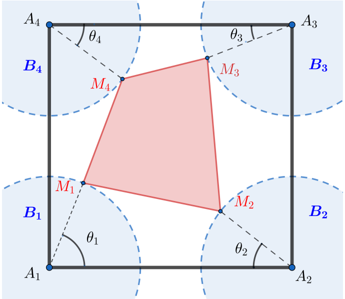
Figure: The optimal sets for mass fractions 70 and 10 percents.

Step 1: The vertices of the optimal set are on the blue spheres

$$\min\{|\omega| \mid \omega \text{ is a } \underline{\text{quadrilateral}} \text{ s.t. } d^H(\omega, \Omega) = \delta\}.$$



Step 2: Parametrization of the problem



We have

$$|\omega| = \frac{1}{2}\sum_{k=1}^{4}(x_k y_{k+1} - x_{k+1} y_k) = 1 - \frac{1}{2}\delta\sum_{k=1}^{4}\cos\theta_k - \frac{1}{2}\delta\sum_{k=1}^{4}\sin\theta_k + \frac{1}{2}\delta^2\sum_{k=1}^{4}(\cos\theta_k\sin\theta_{k+1} + \cos\theta_{k+1}\sin\theta_k),$$

that can be factorized as follows

$$|\omega| = \frac{1}{2}\Big((1-\delta\cos\theta_1-\delta\cos\theta_3)(1-\delta\sin\theta_2-\delta\sin\theta_4)+(1-\delta\cos\theta_2-\delta\cos\theta_4)(1-\delta\sin\theta_1-\delta\sin\theta_3)\Big).$$

Step 3: Using $a + b \geq 2\sqrt{ab}$

We then use the inequality $a + b \geq 2\sqrt{ab}$, where the equality holds if and only of $a = b$, and obtain

$$\begin{cases} 1 - \delta\cos\theta_1 - \delta\cos\theta_3 = (\frac{1}{2} - \delta\cos\theta_1) + (\frac{1}{2} - \delta\cos\theta_3) \geq 2\sqrt{(\frac{1}{2} - \delta\cos\theta_1)(\frac{1}{2} - \delta\cos\theta_3)}, \\ 1 - \delta\sin\theta_2 - \delta\sin\theta_4 = (\frac{1}{2} - \delta\sin\theta_2) + (\frac{1}{2} - \delta\sin\theta_4) \geq 2\sqrt{(\frac{1}{2} - \delta\sin\theta_2)(\frac{1}{2} - \delta\sin\theta_4)}, \\ 1 - \delta\cos\theta_2 - \delta\cos\theta_4 = (\frac{1}{2} - \delta\cos\theta_2) + (\frac{1}{2} - \delta\cos\theta_4) \geq 2\sqrt{(\frac{1}{2} - \delta\cos\theta_2)(\frac{1}{2} - \delta\cos\theta_4)}, \\ 1 - \delta\sin\theta_1 - \delta\sin\theta_3 = (\frac{1}{2} - \delta\sin\theta_1) + (\frac{1}{2} - \delta\sin\theta_3) \geq 2\sqrt{(\frac{1}{2} - \delta\sin\theta_1)(\frac{1}{2} - \delta\sin\theta_3)}, \end{cases}$$

with equality if and only if

$$\theta_1 = \theta_3 \quad \text{and} \quad \theta_2 = \theta_4.$$

We then write

$$\begin{aligned} |\omega| \geq & \sqrt{(\frac{1}{2} - \delta\cos\theta_1)(\frac{1}{2} - \delta\cos\theta_3)}\sqrt{(\frac{1}{2} - \delta\sin\theta_2)(\frac{1}{2} - \delta\sin\theta_4)} \\ & + \sqrt{(\frac{1}{2} - \delta\cos\theta_2)(\frac{1}{2} - \delta\cos\theta_4)}\sqrt{(\frac{1}{2} - \delta\sin\theta_1)(\frac{1}{2} - \delta\sin\theta_3)} \end{aligned}$$

Step 3: Using (again) $a + b \geq 2\sqrt{ab}$

- We use again the inequality $a + b \geq 2\sqrt{ab}$ to obtain another lower bound of $|\omega|$ for which the equality holds if and only if one has

$$(\frac{1}{2} - \delta\cos\theta_1)(\frac{1}{2} - \delta\cos\theta_3)(\frac{1}{2} - \delta\sin\theta_2)(\frac{1}{2} - \delta\sin\theta_4) =$$

$$(\frac{1}{2} - \delta\cos\theta_2)(\frac{1}{2} - \delta\cos\theta_4)(\frac{1}{2} - \delta\sin\theta_1)(\frac{1}{2} - \delta\sin\theta_3).$$

- By combining the equality conditions, we show that $|\omega|$ is minimal, if and only if $\theta_1 = \theta_3, \theta_2 = \theta_4$ and

$$\frac{\frac{1}{2} - \delta\cos\theta_1}{\frac{1}{2} - \delta\sin\theta_1} = \frac{\frac{1}{2} - \delta\cos\theta_2}{\frac{1}{2} - \delta\sin\theta_2},$$

which holds if and only if $\theta_1 = \theta_2$, because the function $\theta \longmapsto \frac{\frac{1}{2}-\delta\cos\theta}{\frac{1}{2}-\delta\sin\theta}$ is a bijection.

- Thus, $|\omega|$ is minimal if and only if $\theta_1 = \theta_2 = \theta_3 = \theta_4 \Rightarrow$ ω^* **is a rectangle.**

Step 4: Which rectangle is optimal ?

The optimal set is a rectangle that corresponds to the value of θ_δ that minimizes the function

$$f_\delta : \theta \in [0, \frac{\pi}{2}] \longmapsto \left(\frac{1}{2} - \delta\cos\theta\right)\left(\frac{1}{2} - \delta\sin\theta\right).$$

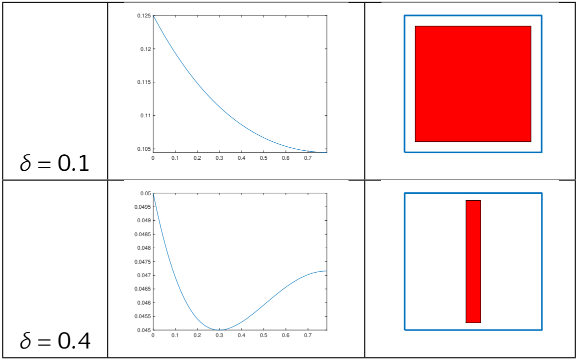


Figure: Optimal sets for different values of δ.

Plan of the presentation

We consider the problem

$$\min\{d^H(\cup_{i=1}^N B_i, \Omega) \mid \forall i \in \{1,\dots,N\}, \quad B_i \subset \Omega\},$$

where B_i are spherical sensors of radius $r > 0$.

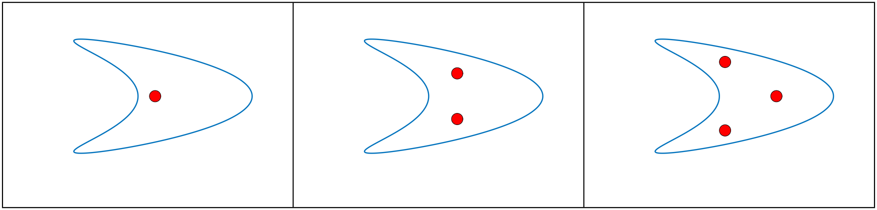


Figure: Optimal placement of $N \in \{1, 2, 3\}$ sensors.

Remark

This problem is related to the classical problem of finding the Chebyshev centers.

Formulation via distance functions

We recall that

$$d^H(\cup_{i=1}^N B_i, \Omega) = \max_{x\in\Omega} |d(x, \cup_{i=1}^N B_i)| =: \|d(\cdot, \cup_{i=1}^N B_i)\|_\infty.$$

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We have two challenges:

1. Computation of the distance function.
2. Dealing with the infinity norm.

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1. Computation of the distance function.
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We propose:

1. Using a Varadhan's result for the approximation of the distance.
2. Approximating $\|\cdot\|_\infty$ with $\|\cdot\|_p$ for p large.

Varadhan's result

Theorem (Varadhan, 69′)

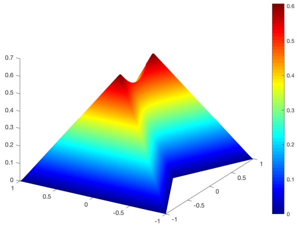
Let Ω be an open subset of $\mathbb{R}^n$ and $\varepsilon > 0$, we consider the problem

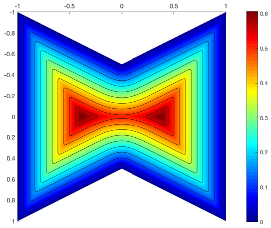
$$\begin{cases} w_\varepsilon - \varepsilon \Delta w_\varepsilon = 0 & \text{in } \Omega, \\ w_\varepsilon = 1 & \text{on } \partial\Omega. \end{cases}$$

We take $v_\varepsilon = -\sqrt{\varepsilon} \log w_\varepsilon(x)$. We have

$$\lim_{\varepsilon \to 0} v_\varepsilon = d(x, \partial\Omega) := \inf_{y \in \partial\Omega} \|x - y\|,$$

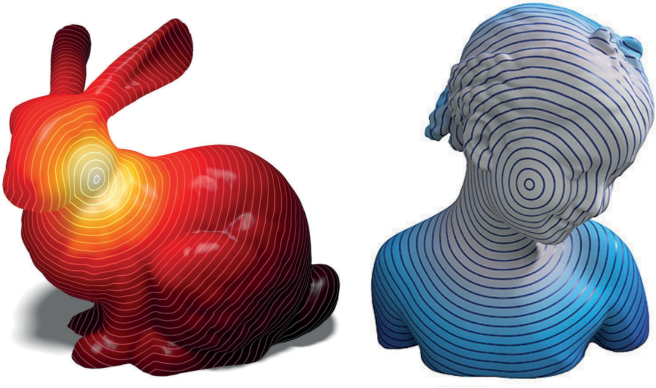
uniformly over compact subsets of Ω.





The Heat Method for Distance Computation

- K. Crane, C. Weischedel, and M. Wardetzky (2017)



Approximated problems

We then perform the following approximations

$$\|d\|_\infty \approx \int_\Omega d^p(x)dx \approx \int_\Omega v_\varepsilon^p(x)dx =: J_{\varepsilon,p}(B_1,\cdots,B_n)$$

and consider the family of problems

$$(\mathcal{P}_{\varepsilon,p}) \qquad \min_{B_1,\cdots,b_n\subset\Omega} \quad J_{\varepsilon,p}(B_1,\cdots,b_n).$$

Theorem (F., Zuazua, 2024)

We have the following Γ*-convergence results*

$$(\mathcal{P}_{\varepsilon,p}) \underset{\varepsilon\to 0}{\longrightarrow} (\mathcal{P}_{0,p}) \underset{p\to+\infty}{\longrightarrow} (\mathcal{P}_{0,\infty}).$$

We then want to solve

$$(\mathcal{P}_{\varepsilon,p}) \qquad \min_{x_1,\cdots,x_N\in\Omega_{-r}} \quad J_{\varepsilon,p}(x_1,\cdots,x_N),$$

where x_i are the centers of the B_i.

Obtained results

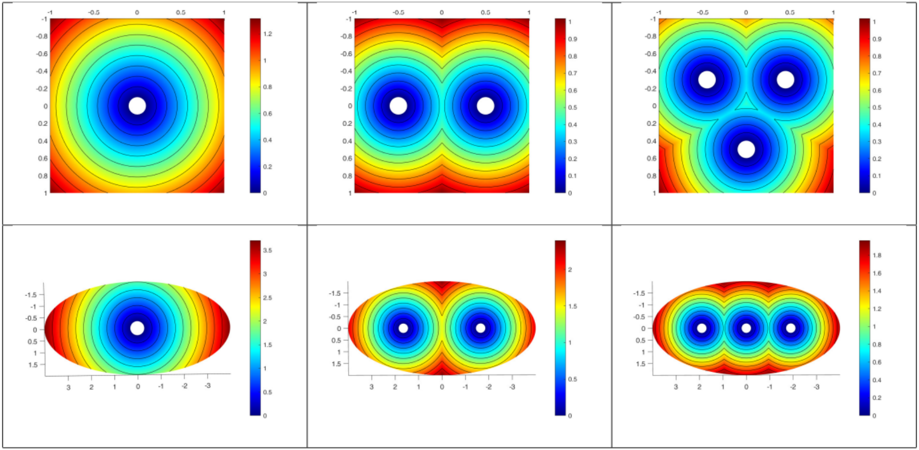
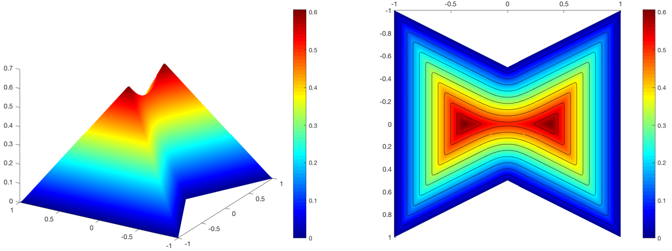


Figure: Optimal placement of sensors.

Plan of the presentation

- Considering the problem of the p-distance between convex sets (paper in collaboration with Zakaria Fattah and Enrique Zuazua).
- Proving qualitative properties on the optimal placement of sensors.
- Using other PDE approximations of the distance function:
    - B. Kawohl: $-\Delta_p u_p = 1$ in Ω, $u_p = 0$ on $\partial\Omega \Longrightarrow u_p \underset{p\to\infty}{\longrightarrow} d_{\partial\Omega}$.



THANK YOU