Random Batch Method and Emergence of Domain Decomposition

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DYNAMICS, CONTROL, MACHINE LEARNING AND NUMERICS

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TRR 154

Overview

1. RBM for the heat equation
2. Application to PDEs on networks
3. Numerical implementations

RBM for the heat equation

Model

Let $T > 0$. Consider the heat equation

$$\begin{cases} \partial_t y - \partial_{xx} y = f & (x,t) \in (0,1) \times (0,T), \\ y(0,t) = y(1,t) = 0 & t \in (0,T), \\ y(x,0) = y_0(x) & x \in (0,1), \end{cases} \tag{1}$$

with initial condition $y_0 \in L^2(0,1)$ and source term $f \in L^2(0,T;0,1)$.

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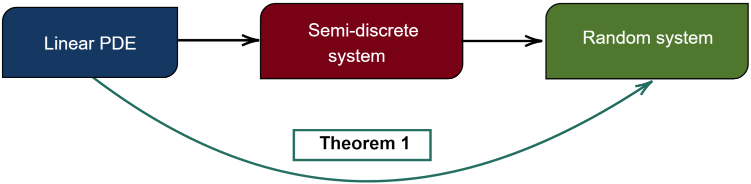

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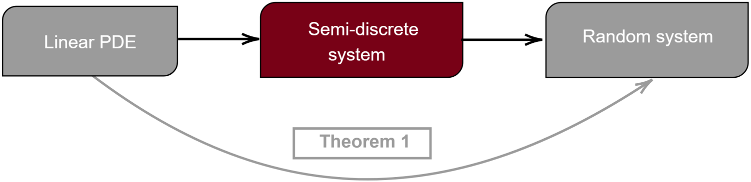

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Consider the finite-dimensional space V_h (with basis $\{\phi_j\}_{j=1}^{N}$).

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Semi-discrete problem: Find $y_h \in C^1(0, T; V_h)$ such that

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$$\begin{cases} E_h\partial_t\vec{y}_h + R_h\vec{y}_h = \vec{f}_h, \quad t\in(0,T), \\ \vec{y}_h(0) = \vec{y}_h^0, \end{cases} \tag{4}$$

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where $\vec{y}_h(t) = (y_1(t),\dots,y_N(t))$ and $\vec{f}_h$ is the vector with coefficients $(f,\phi_j)_{L^2}$. Here E_h, $R_h \in \mathbb{R}^{N\times N}$ are the so-called mass and stiffness matrices, respectively.

Since E_h is invertible, we can multiply the system

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by $B_h := E_h^{-1}$. We obtain

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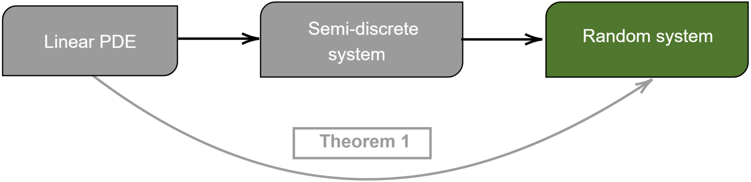

Random batch method

We need three ingredients:

- Let $K \in \mathbb{N}$. Consider $\delta_t = T/(K+1)$ and $I_k = [\delta_t(k-1), \delta_t k]$ for every $k \in \{1, \ldots, K\}$.

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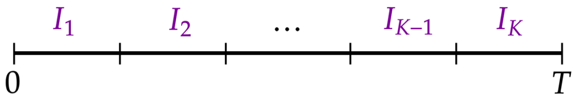

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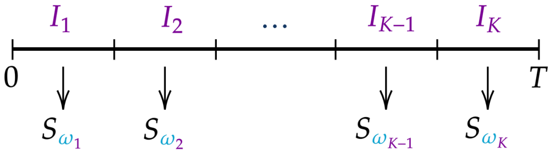

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In particular, this construction ensures that $\mathbb{E}[A_R^h(t)] = A_h$ for each $t \in (0, T)$.

Random system

Consider the random dynamical system

$$\begin{cases} \partial_t \vec{y}_R(\omega, t) + A_R^h(\omega, t)\vec{y}_R(\omega, t) = B\vec{f}, \\ \vec{y}_R(0) = \vec{y}_h^0. \end{cases} \tag{6}$$

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In the following, we write

$$y_R(x, t) = \sum_{j=1}^{N} (\vec{y}_R)_j \phi_j(x) \tag{7}$$

Theorem (1)

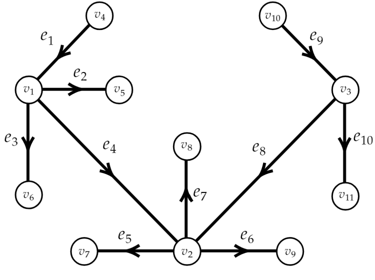
Let y_R be as in (7) and let y be the solution of the heat equation with initial condition $y^0 \in H_0^1(0, L)$ and $f \in L^2(0, T; H_0^1(0, L))$. Then,

$$\mathbb{E}[\|y_R(\cdot, t) - y(\cdot, t)\|_{L^2(0,L)}^2] \leq C\left(h^4 + \frac{\delta_t}{h^7} C(M, \omega)\right), \tag{8}$$

for every $t \in (0, T)$, where $C > 0$ and $C(M, \omega) > 0$ are independent of h and δ_t, with $C(M, \omega)$ depending on the chosen decomposition.

Application to PDEs on networks

Let us consider a graph $\mathcal{G} = (\mathcal{E}, \mathcal{V})$, bwhere $\mathcal{V}_0$ denotes its interior nodes, and $\mathcal{V}_b$ its boundary nodes.



$$\mathcal{V}_0 = \{v_1, v_2, v_3\}$$
$$\mathcal{V}_b = \{v_4, v_5, \ldots, v_{11}\}$$
$$\mathcal{E} = \{e_1, \ldots, e_{10}\}$$
$$\mathcal{E}(v_1) = \{e_1, e_2, e_3\}.$$

Heat equation on the network

On each e_i with $i \in \{1, \dots, 10\}$ we consider the heat equation

$$\begin{cases} \partial_t y^{e_i} - \partial_{xx} y^{e_i} = f^{e_i}, & (x,t) \in (0,L) \times (0,T), \\ y^{e_i}(x,0) = y^0_{e_i}(x), & x \in (0,L). \end{cases} \tag{9}$$

System (9) is complemented with boundary and coupling conditions

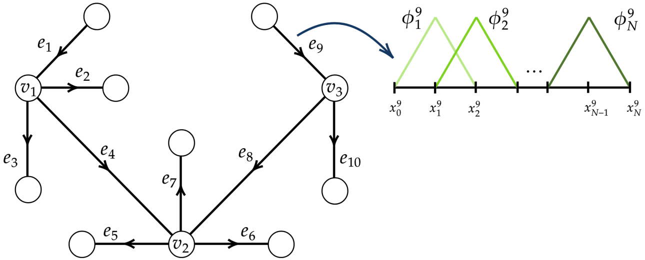
$$\begin{cases} y^e(v,t) = 0, & v \in \mathcal{V}_b,\ e \in \mathcal{E}(v), \\ y^{e_1}(v,t) = y^{e_2}(v,t), & v \in \mathcal{V}_0,\ e_1, e_2 \in \mathcal{E}(v), \\ \displaystyle\sum_{e \in \mathcal{E}(v)} \partial_x y^e(v,t) n_e(v) = 0, & v \in \mathcal{V}_0, \end{cases} \tag{10}$$

over the time interval $(0,T)$.

Let us introduce a semi-discrete system.

FEM discretization

On each e_i we define the basis functions $\{\phi_j^i\}_{j=1}^N$.



What about the interior vertices?

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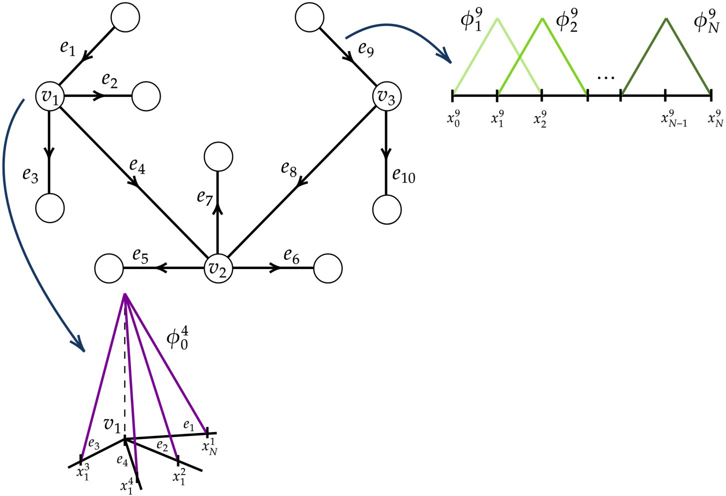

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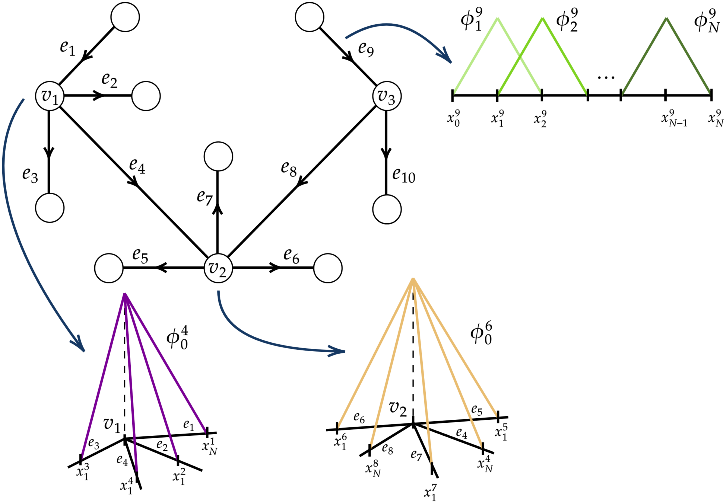

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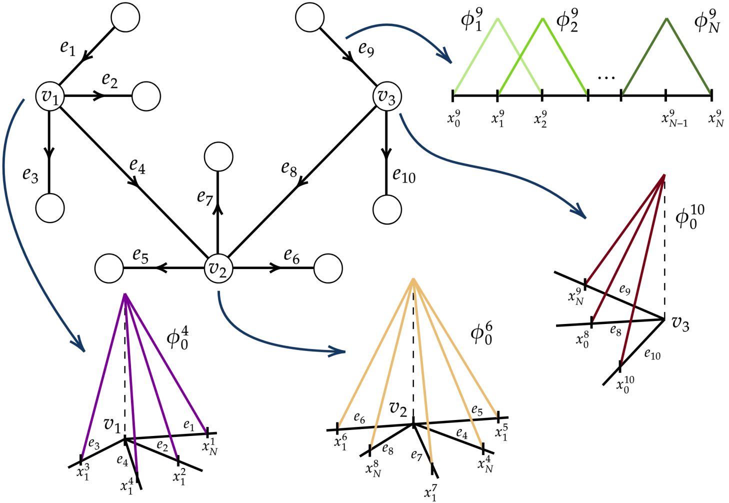

Semi-discrete equation

We can introduce the space $V_h^{\mathcal{E}}$ as the span of the functions $\left(\bigcup_{i=1}^{10}\bigcup_{j=1}^{N}\{\phi_j^i\}\right) \cup \{\phi_0^4, \phi_0^6, \phi_0^{10}\}$.

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$$\begin{cases} \displaystyle\sum_{i=1}^{10} \left(\int_0^L \partial_t y_h^{e_i}(x,t)\phi_j^i(x) + \partial_x y_h^{e_i}(x,t)\partial_x \phi_j^i(x)\, dx \right) = \sum_{i=1}^{10} \int_0^L f^{e_i}(x,t)\phi_j^i(x)\, dx \\ \displaystyle\int_0^L y_h^{e_i}(x,0)\phi_j^i(x) = \int_0^L y_{h,e_i}^0(x)\phi_j^i(x)\, dx, \end{cases}$$

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Since $y_h^{e_i}$ can be written as $y_h^{e_i}(x,t) = \sum_{j=1}^{N} y_j^{e_i}(t)\phi_j^i(x)$, we consider the vector of coefficients $\boldsymbol{Y}_h(t) = \left(\vec{y}_h^{e_1}(t), \dots, \vec{y}_h^{e_{10}}(t)\right)^\top$. Then,

$$\begin{cases} \boldsymbol{E}_h\partial_t\boldsymbol{Y}_h + \boldsymbol{R}_h\boldsymbol{Y}_h = \boldsymbol{F}_h, \quad t \in (0,1), \\ \boldsymbol{Y}_h(0) = \boldsymbol{Y}_h^0. \end{cases} \tag{11}$$

Semi-discrete equation

Here we have that

$$R_h = \begin{pmatrix} R_h \; R_h \; R_h & c_1^4 & 0 & 0 & 0 & 0 & 0 \\ a_1^4 \; a_2^4 \; a_3^4 & 4/h & a_4^4 \; 0 & 0 & 0 & 0 & 0 \\ 0 & c_2^4 & R_h \; R_h & c_1^6 & 0 & 0 & 0 \\ 0 & 0 & a_1^2 \; a_2^2 & 5/h & a_6^6 \; a_7^6 \; a_8^6 \; 0 & 0 & 0 \\ 0 & 0 & 0 & c_2^6 & R_h \; R_h \; R_h \; R_h & c_1^{10} & 0 \\ 0 & 0 & 0 & 0 & 0 \; 0 \; a_8^{10} \; a_9^{10} & 3/h & a_{10}^{10} \\ 0 & 0 & 0 & 0 & 0 & c_2^{10} & R_h \end{pmatrix}$$

where $R_h \in \mathbb{R}^{N \times N}$ is the stiffness matrix of the $1-d$ heat equation.

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$$
\begin{array}{c}
\left[\; e_1 \;\; e_2 \;\; e_3 \qquad v_1 \qquad e_4 \;\; e_5 \qquad v_2 \qquad e_6 \;\; e_7 \;\; e_8 \;\; e_9 \qquad v_{10} \qquad e_{10} \;\right]
\end{array}
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0 & 0 & 0 & c_2^6 & R_h \;\; R_h \;\; R_h \;\; R_h & c_1^{10} & 0 \\
0 & 0 & 0 & 0 & 0 \;\; 0 \;\; a_8^{10} \;\; a_9^{10} & 3/h & a_{10}^{10} \\
0 & 0 & 0 & 0 & 0 & c_2^{10} & R_h
\end{pmatrix}
$$

where $R_h \in \mathbb{R}^{N\times N}$ is the stiffness matrix of the $1-d$ heat equation.

Semi-discrete equation

Here we have that

$$
\begin{array}{ccccccccccccc}
[\,e_1 & e_2 & e_3 & v_1 & e_4 & e_5 & v_2 & e_6 & e_7 & e_8 & e_9 & v_{10} & e_{10}\,]
\end{array}
$$

$$
R_h = \begin{pmatrix}
R_h & R_h & R_h & c_1^4 & 0 & & 0 & & & 0 & & 0 & 0 \\
a_1^4 & a_2^4 & a_3^4 & 4/h & a_4^4 & 0 & 0 & & & 0 & & 0 & 0 \\
 & 0 & & c_2^4 & R_h & R_h & c_1^6 & & & 0 & & 0 & 0 \\
 & 0 & & 0 & a_1^2 & a_2^2 & 5/h & a_6^6 & a_7^6 & a_8^6 & 0 & 0 & 0 \\
 & 0 & & 0 & 0 & & c_2^6 & R_h & R_h & R_h & R_h & c_1^{10} & 0 \\
 & 0 & & 0 & 0 & & 0 & 0 & 0 & a_8^{10} & a_9^{10} & 3/h & a_{10}^{10} \\
 & 0 & & 0 & 0 & & 0 & & & 0 & & c_2^{10} & R_h
\end{pmatrix}
$$

where $R_h \in \mathbb{R}^{N\times N}$ is the stiffness matrix of the $1-d$ heat equation.

Here we have that

$$
\begin{array}{ccccccccccccc}
[\,e_1 & e_2 & e_3 & v_1 & e_4 & e_5 & v_2 & e_6 & e_7 & e_8 & e_9 & v_{10} & e_{10}\,]
\end{array}
$$

$$
R_h = \begin{pmatrix}
R_h & R_h & R_h & c_1^4 & 0 & & 0 & & & 0 & & 0 & 0 \\
a_1^4 & a_2^4 & a_3^4 & 4/h & a_4^4 & 0 & 0 & & & 0 & & 0 & 0 \\
& 0 & & c_2^4 & R_h & R_h & c_1^6 & & & 0 & & 0 & 0 \\
& 0 & & 0 & a_1^2 & a_2^2 & 5/h & a_6^6 & a_7^6 & a_8^6 & 0 & 0 & 0 \\
& 0 & & 0 & 0 & & c_2^6 & R_h & R_h & R_h & R_h & c_1^{10} & 0 \\
& 0 & & 0 & 0 & & 0 & 0 & 0 & a_8^{10} & a_9^{10} & 3/h & a_{10}^{10} \\
& 0 & & 0 & 0 & & 0 & & & 0 & & c_2^{10} & R_h
\end{pmatrix}
$$

where $R_h \in \mathbb{R}^{N \times N}$ is the stiffness matrix of the $1-d$ heat equation.

Matrix $\boldsymbol{E}_h$ has a similar structure.

Here we have that

$$[\, e_1 \;\; e_2 \;\; e_3 \;\; v_1 \;\; e_4 \;\; e_5 \;\; v_2 \;\; e_6 \;\; e_7 \;\; e_8 \;\; e_9 \;\; v_{10} \;\; e_{10} \,]$$

$$R_h = \begin{pmatrix} R_h \;\; R_h \;\; R_h & c_1^4 & 0 & 0 & 0 & 0 & 0 \\ a_1^4 \;\; a_2^4 \;\; a_3^4 & 4/h & a_4^4 \;\; 0 & 0 & 0 & 0 & 0 \\ 0 & c_2^4 & R_h \;\; R_h & c_1^6 & 0 & 0 & 0 \\ 0 & 0 & a_1^2 \;\; a_2^2 & 5/h & a_6^6 \;\; a_7^6 \;\; a_8^6 \;\; 0 & 0 & 0 \\ 0 & 0 & 0 & c_2^6 & R_h \;\; R_h \;\; R_h \;\; R_h & c_1^{10} & 0 \\ 0 & 0 & 0 & 0 & 0 \;\; 0 \;\; a_8^{10} \;\; a_9^{10} & 3/h & a_{10}^{10} \\ 0 & 0 & 0 & 0 & 0 & c_2^{10} & R_h \end{pmatrix}$$

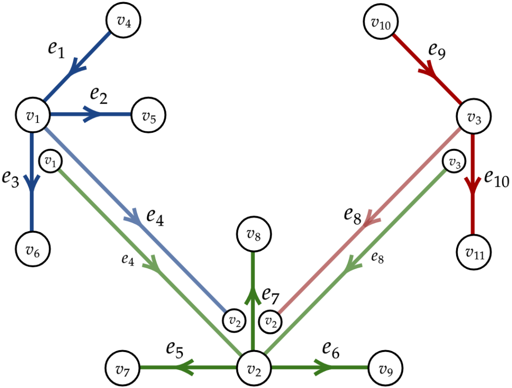
where $R_h \in \mathbb{R}^{N \times N}$ is the stiffness matrix of the $1-d$ heat equation.

Matrix $\boldsymbol{E}_h$ has a similar structure. Observe that

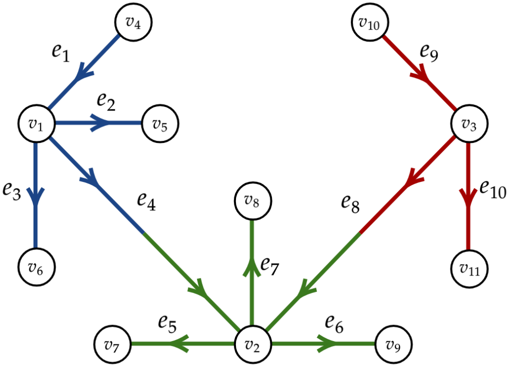
$$\boldsymbol{R}_h = \begin{pmatrix} B_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & B_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & B_3 \end{pmatrix} = \boldsymbol{R}_1 + \boldsymbol{R}_2 + \boldsymbol{R}_3$$

A different decomposition of $\boldsymbol{R}_h$ induces a different domain decomposition.

1) Overlapping decomposition:



2) Non-overlapping decomposition: A different election of B_1, B_2 and B_3 can introduce a non-overlapping decomposition.



Numerical implementations

We define

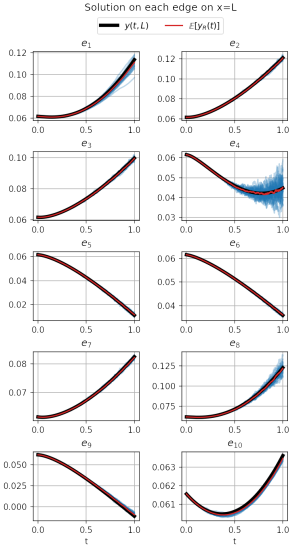
$$\begin{aligned} \boldsymbol{A}_h &= (\boldsymbol{E}_h)^{-1}\boldsymbol{R}_h \\ &= (\boldsymbol{E}_h)^{-1}\boldsymbol{R}_1 + (\boldsymbol{E}_h)^{-1}\boldsymbol{R}_2 + (\boldsymbol{E}_h)^{-1}\boldsymbol{R}_3 \\ &= \boldsymbol{A}_1^h + \boldsymbol{A}_2^h + \boldsymbol{A}_3^h \end{aligned}$$

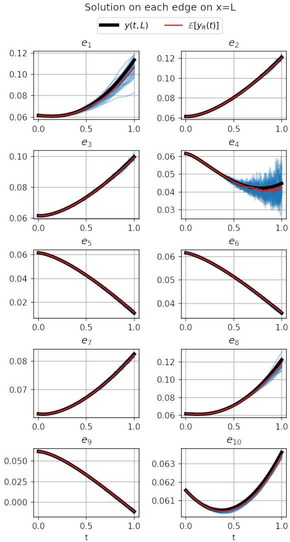
With the previous decomposition we can define a random matrix $\boldsymbol{A}_R^h(\boldsymbol{\omega})$. Then, the random system associated is given by

$$\begin{cases} \partial_t \boldsymbol{Y}_R(\boldsymbol{\omega}, t) + \boldsymbol{A}_R^h(\boldsymbol{\omega}, t)\boldsymbol{Y}_R(\boldsymbol{\omega}, t) = \boldsymbol{F}, \\ \boldsymbol{Y}_R(0) = \boldsymbol{Y}_h^0. \end{cases} \tag{12}$$

Numerical results

Thus, comparing the numerical solution of the heat and random equations, we obtain (left overlapping, right non-overlapping).





Comments and next steps

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1. In this work, we have extend this results to the linear quadratic regular for the heat equation.

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3. We are working on introducing a "random" operator splitting scheme of parabolic operators.

Thanks for your attention.

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Main theorems used to prove Theorem 1

The proof is based on two crucial theorems.

Theorem 2 (FEM error): Let us assume that $y^0 \in H_0^1(0, L)$ and $f \in L^2(0, T; L^2(0, L))$. Denote by y the solution of the heat equation and y_h the solution of the semi-discrete system. Then, there exists a constant $C > 0$, independent of $h > 0$, such that

$$\|y(t) - y_h(t)\|_{L^2(0,L)}^2 \leq Ch^4(\|y_0\|_{H_0^1(0,L)}^2 + \|f\|_{L^2(0,T;L^2(0,L))}^2), \qquad (13)$$

for every $t \in (0, T)$

Theorem 3 (RBM for ODEs): Denote by $\vec{y}_h$ and $\vec{y}_R$ the solutions of the semi-discrete and random equations, respectively. Then, for every $t \in (0, T)$, we have

$$\mathbb{E}[\|\vec{y}_h(t) - \vec{y}_R(t)\|^2] \leq (\|A_h\|T^2 + 2T)(\|\vec{y}_h^0\| + \|B_h \vec{f}_h\|_{L^1(0,T;\mathbb{R}^N)})^2 \mathrm{Var}[A_R]\Delta t,$$

where

$$\mathrm{Var}[A_R] := \sum_{i=1}^{2^M} \left(\left\| A_h - \sum_{m \in S_i} \frac{A_m^h}{\pi_m} \right\|^2 p_i \right).$$