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Isothermal flow in gas networks: Synchronization of observer systems

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Outline

Why do we need observers?

Model for the flow in a natural gas pipe

The observer system and exponential synchronization:
Also here the exponential weights as used by Jean-Michel Coron et al. are very useful!

What about mixtures (hydrogen)? The problem of embrittlement - what is rainflow counting?

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The control system

Control devices (=controllers)

See the results of DFG CRC 154-3

TRR 154 | Mathematical Modelling, Simulation and Optimization Using the Example of Gas Networks

In the operation of gas networks,

the control acts at

1. compressor stations,
2. valves,
3. regulators (pressure reducers).

Thus the control acts only in certain points (as in *boundary control*).

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Pointwise Control: Nodal Control

We call this control action that is concentrated in a finite number of points in space **nodal control**.

The stabilization problem and observers

The Stabilization problem seeks

nodal control laws

(feedback laws)

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$$p_x(t, 0) = k_0 p_t(t, 0)$$

see *Limits of stabilizability for a semilinear model for gas pipeline flow*, M. Gugat, M. Herty, 2022.

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Why do we need observers?

In order to obtain an approximation of the **full current state**, an **observer system** can be used.

Optimal control problems and observers

Optimal Control

In the operation of gas networks, **pressure bounds** are essential state constraints!

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The compressor cost is a generic objective functional in the mathematical literature:

$$\int_0^T \sum_{v \in V_c} A_v \, q^v(t) \left[\left(\frac{p_{out,v}(t)}{p_{in,v}(t)} \right)^{R_v} - 1 \right] dt$$

(see e.g. *mixed integer approach for time-dependent gas network optimization*, 2009).

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This is usually not true!

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In order to *approximate the initial state* **observers** are needed!

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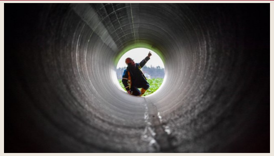
The isothermal Euler equations for ideal gas (see e.g. BANDA, HERTY, KLAR(2006) NHM)



For a horizontal pipe, we have

$$\begin{cases} \rho_t + q_x = 0 \\ q_t + \left(c^2\,\rho + \frac{q^2}{\rho}\right)_x = -\frac{1}{2}\,\theta\,\frac{q\,|q|}{\rho} \end{cases}$$

- ρ: density
- q: flow rate
- c: sound speed
- $\theta = \frac{f_g}{D}$
- f_g: friction, D: diameter

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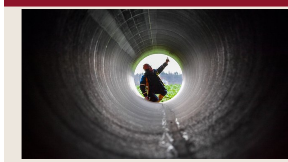

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Slow flow ($|q/\rho| << c$)

In the regular operation of gas pipelines, the flow is **subsonic**. See the results of DFG CRC 154-3

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Riemann invariants - the system in diagonal form

For a general pressure law $p(\rho)$ the **Riemann invariants** are given by

$$R_\pm(\rho, q) = \tilde{R}(\rho) \pm \frac{q}{\rho} \text{ with } \tilde{R}(\rho) = \int_1^\rho \frac{\sqrt{p'(r)}}{r}\, dr.$$

The **eigenvalues** are

$$\lambda_\pm = \tfrac{R_+ - R_-}{2} \pm \left(p'\left(\tilde{R}^{-1}\left(\tfrac{R_+ + R_-}{2}\right)\right)\right)^{1/2}.$$

In terms of the Riemann invariants, for an edge $e \in E$ with

$$\sigma^e(R_+^e, R_-^e) = \nu^e \left|R_+^e - R_-^e\right| (R_+^e - R_-^e)$$

the PDE has the **diagonal form**

$$\partial_t \begin{pmatrix} R_+^e \\ R_-^e \end{pmatrix} + \begin{pmatrix} \lambda_+^e & 0 \\ 0 & \lambda_-^e \end{pmatrix} \partial_x \begin{pmatrix} R_+^e \\ R_-^e \end{pmatrix} = \sigma^e(R_+^e, R_-^e) \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Model for the flow through pipe junctions: Node conditions

Node conditions

Our node conditions model

1. Conservation of mass

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1. Conservation of mass
2. Continuity of the pressure

Model for the flow through pipe junctions: Algebraic node conditions

Continuity of the pressure

For all $v \in V$ and $e, f \in E_0(v)$ we have

$$p^e(x^e(v)) = p^f(x^f(v)).$$

Also, *perfect mixing of the incoming entropies* has been suggested.

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Notation: Outer normal $\mathbf{n}(e, v)$

Let $G = (V, E)$. The pipes $e \in E$ correspond to intervals $[0, L^e]$. Define the 'outer normal'

$$\mathbf{n}(e, v) = \begin{cases} 0 & \text{if } e \notin E_0(v), \\ -1 & \text{if } e \in E_0(v) \text{ and } x^e(v) = 0, \\ 1 & \text{if } e \in E_0(v) \text{ and } x^e(v) = L^e. \end{cases}$$

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Conservation of mass

Define the mass flow $m^e = \frac{\pi}{4}(D^e)^2 q^e$.
Define the **incidence matrix**

$$A = (\mathbf{n}(e, v))_{v \in V,\, e \in E}.$$

The balance of mass is given by

$$A\, m = 0.$$

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$A\, m = 0$ (Conservation of mass).

See *On the Cauchy Problem for the p-System at a Junction*, COLOMBO GARAVELLO SIAM J. MATH. ANAL. 39 (2008), 1456-1471.

The system in diagonal form - (S)

$G = (V, E)$: graph of a pipeline network

- $E_0(v)$ denotes the set of edges in the graph that are incident to a vertex $v \in V$.
- $x^e(v) \in \{0, L^e\}$ denotes the end of the interval $[0, L^e]$ corresponding to $e \in E_0(v)$.
- For $v \in V$, let $\omega_v := 2 \left(\sum_{f \in E_0(v)} |D^f|^2 \right)^{-1}$.

Let the diagonal matrix $diag(S^e)$ contain $\lambda^e_\pm$; $\mu^v \in [-1, 1]$. The quasilinear system is

$$
(\mathbf{S}) \begin{cases}
S^e_+(0,x) = y^e_+(x),\ x \in (0, L^e),\ e \in E, \\
S^e_-(0,x) = y^e_-(x),\ x \in (0, L^e),\ e \in E, \\
\\
S^e_{out}(t, x^e(v)) = (1 - \mu^v)\, u^e(t) + \mu^v S^e_{in}(t, L^e),\ t \in (0, T), \ \text{if } |E_0(v)| = 1, \\
\\
S^e_{out}(t, x^e(v)) = -S^e_{in}(t,\ x^e(v)) + \omega_v \sum\limits_{g \in E_0(v)} (D^g)^2\, S^g_{in}(t,\ x^g(v)),\ t \in (0, T), \\
\hfill \text{if } |E_0(v)| \geq 2, \\
\partial_t \begin{pmatrix} S^e_+ \\ S^e_- \end{pmatrix} + diag(S^e)\, \partial_x \begin{pmatrix} S^e_+ \\ S^e_- \end{pmatrix} = \sigma^e(S^e_+,\ S^e_-) \begin{pmatrix} -1 \\ 1 \end{pmatrix} \ \text{on } [0, T] \times [0, L^e],\ e \in E.
\end{cases}
$$

Well–posedness of the system

Note that with **gas at rest**, we have a **steady state** with constant pressure and constant Riemann invariants

$$S_+^e = S_-^e = J.$$

Semi-global classical solutions (Ta-Tsien Li,...)

Let $T > 0$ and a real number J be given.
Then there exists $\varepsilon(T) > 0$ such that for **initial data** $y_\pm^e \in C^1(0,\ L^e)$ such that

$$\tilde{I} := \|y_\pm^e - J\|_{C^1(0,\ L^e)} \leq \varepsilon(T)$$

and **control functions** $u^e \in C^1(0,\ T)$ (e with $|E_0(v)| = 1$ for some $v \in V$) such that

$$\tilde{C} := \|u^e - J\|_{C^1(0,T)} \leq \varepsilon(T)$$

that satisfy the C^1*-compatibility conditions* for (**S**) there exists a

unique classical solution of (**S**) on $[0,\ T]$.

There exists a constant $\hat{C}_T > 0$ such that the solution satisfies the **a priori bound**

$$\max_{e\in E} \left\{ \|S_+^e - J\|_{C^1((0,\ T)\times(0,\ L^e))},\ \|S_-^e - J\|_{C^1((0,\ T)\times(0,\ L^e))} \right\} \leq \hat{C}_T\ \max\{\tilde{I},\ \tilde{C}\}. \qquad (1)$$

Inhalt

The observer system (R)

The idea of the observer system is to replace the node conditions for (**S**) at $v \in V$ in the observer system (**R**) by a **linear combination of two terms**:

- the physical node conditions as in the original system (**S**): $S_{out} = F^v(S_{in})$;
- a nudging term of LUENBERGER type where the *difference*

$$\delta = R - S$$

between the observer state and the measured value is fed into the observer system (**R**).

$$\boxed{R_{out} = S_{out} + \mu^V \left[F^v(R_{in}) - F^v(S_{in})\right] = \mu^V F^v(R_{in}) + (1 - \mu^v) S_{out}.}$$

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The choice of the nodal parameter μ^v allows to control the influence of the measured values that are prone to measurement errors.

If $|1 - \mu^v|$ (i.e. the weight of the Luenberger term) is small, the influence of these errors is damped.
Then the synchronization has the tendency to be slower.

The observer system (R)

For $v \in V$ let $\mu^v \in [0,1]$ control the coupling of the original system to the observer system ($\mu^v = 1$: (**R**) decouples from (**S**); $\mu^v = 0$: full flow of information from (**S**)).

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The observer system has a similar structure as (**S**):

$$
(\mathbf{R}) \begin{cases}
R_+^e(0,x) = z_+^e(x),\ x \in (0, L^e),\ e \in E, \\
R_-^e(0,x) = z_-^e(x),\ x \in (0, L^e),\ e \in E, \\
\\
R_{out}^e(t, x^e(v)) = (1-\mu^v)\, u^e(t) + \mu^v\, R_{in}^e(t, x^e(v)),\ t \in (0,T),\ \text{if } |E_0(v)| = 1, \\
\\
R_{out}^e(t, x^e(v)) = S_{out}^e(t,\, x^e(v)) - \mu^v \left[R_{in}^e(t, x^e(v)) - S_{in}^e(t,\, x^e(v)) \right] \\
\qquad\qquad + \mu^v\, \omega_v \sum\limits_{g \in E_0(v)} (D^g)^2 \left[R_{in}^g(t,\, x^g(v)) - S_{in}^g(t,\, x^g(v)) \right], \\
\qquad\qquad t \in (0,T),\ \text{if } |E_0(v)| \geq 2, \\
\\
\partial_t \begin{pmatrix} R_+^e \\ R_-^e \end{pmatrix} + diag(R^e)\, \partial_x \begin{pmatrix} R_+^e \\ R_-^e \end{pmatrix} = \sigma^e(R_+^e,\, R_-^e) \begin{pmatrix} -1 \\ 1 \end{pmatrix} \text{ on } [0,T] \times [0, L^e],\ e \in E.
\end{cases}
$$

The initial state (z_+^e, z_-^e) is an estimation of the initial state of the original system.

The difference system between (R) and (S)

For the proof of synchronization, we show that the solution of the difference system decays exponentially fast.

Let $\delta = R - S$. For the difference δ we obtain the system

$$
\textbf{(Diff)}\quad
\begin{cases}
\delta_+^e(0,x) = z_+^e(x) - y_+^e(x),\ x \in (0, L^e),\ e \in E,\\
\delta_-^e(0,x) = z_-^e(x) - y_-^e(x),\ x \in (0, L^e),\ e \in E,\\
\\
\delta_{out}^e(t, x^e(v)) = \mu^v\, \delta_{in}^e(t, x^e(v)),\ t \in (0,T),\ \text{if}\ \ |E_0(v)| = 1,\\
\\
\delta_{out}^e(t, x^e(v)) = \mu^v \left[-\delta_{in}^e(t,\, x^e(v)) + \omega_v \sum_{g \in E_0(v)} (D^g)^2\, \delta_{in}^g(t,\, x^g(v)) \right],\\
\qquad\qquad\qquad\qquad\qquad\qquad t \in (0,T),\ \text{if}\ \ |E_0(v)| \geq 2,\\
\partial_t \begin{pmatrix} \delta_+^e \\ \delta_-^e \end{pmatrix} + diag(S^e + \delta^e)\, \partial_x \begin{pmatrix} \delta_+^e \\ \delta_-^e \end{pmatrix} + [diag(S^e + \delta^e) - diag(S^e)]\, \partial_x \begin{pmatrix} S_+^e \\ S_-^e \end{pmatrix}\\
= \left[\sigma^e(\delta_+^e + S_+^e,\ \delta_-^e + S_-^e) - \sigma^e(S_+^e,\ S_-^e)\right] \begin{pmatrix} -1 \\ 1 \end{pmatrix}\\
\quad \text{on}\ \ [0,T] \times [0, L^e],\ e \in E.
\end{cases}
$$

Result on Exponential Synchronization

Result on Exponential Synchronization Let $T > 0$ be given.

- Assume that for each node $v \in V$, $|\mu^v|$ is sufficiently small.
- Assume that on $[0,\ T]$ there exists a classical solution of (**Diff**) that satisfies a priori bounds for the C^1-norm and hence for the eigenvalues.
- Assume that the solution of (**S**) and the initial state of (**R**) satisfy appropriate smallness conditions in C^1.

Then the solution of (**Diff**) decays **exponentially fast** in the sense that

there exist constants $C_1 > 0$ and $\chi > 0$ (indep. of T) such that for all $t \in [0,\ T]$ we have $\sum_{e\in E} \int_0^{L^e} |\delta_+^e(t,\ x)|^2 + \left|\delta_-^e(t,\ x)\right|^2 \, dx \leq C_1 \mathrm{e}^{-\chi\, t} \sum_{e\in E} \int_0^{L^e} |\delta_+^e(0,\ x)|^2 + \left|\delta_-^e(0,\ x)\right|^2 \, dx.$

Discussion

- Due to the a priori bound, the smallness condition can be achieved for sufficiently small (in C^1) initial and boundary data for (**S**) and for initial data for (**R**) that are sufficiently close to the initial data of (**S**).

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For the proof we use **exponential weights**

$$h_{\pm}(x) = \exp(\mp\psi\, x)$$

(e.g. CORON, BASTIN, D'ANDREA-NOVEL, HAYAT and for boundary observers CASTILLO, WITRANT, PRIEUR, DUGARD)

in the **Lyapunov function**

$$\mathcal{E}_0^e(t) = \int_0^{L^e} h_+(x)\,|\delta_+^e(t,\,x)|^2 + h_-(x)\,|\delta_-^e(t,\,x)|^2\,dx.$$

We show that

$$\frac{d}{dt}\mathcal{E}_0(t) \le -\chi\,\mathcal{E}_0(t) - Q$$

with a positive real number $\chi > 0$ and

$$Q = \sum_{e\in E} h_+(x)\,\lambda_+^e(S+\delta)\,|\delta_+^e|^2(t,\,x) + h_-(x)\,\lambda_-^e(S+\delta)\,|\delta_-^e|^2(t,\,x)\Big|_{x=0}^{L^e}.$$

The proofs

For the proof, it suffices to show that for $|\mu^v|$ sufficiently small, we have $Q = \sum_{v \in V} Q^v \geq 0$,

where Q_v contains the terms in Q that correspond to a vertex $v \in V$.
Then **Gronwall's Lemma** implies

$$\mathcal{E}_0(t) \leq \exp(-\chi\, t)\, \mathcal{E}_0(0)$$

and the result follows.

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- For an H^2-Lyapunov function, we need additional assumptions since in the source term we have $r \mapsto r|r|$ which is C^1 but not C^2.
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The exponential synchronization of *an Observer for Pipeline Flow with Hydrogen Blending in Gas Networks*

has been studied by GUGAT, GIESSELMANN IN *SIAM Journal on Control and OptimizationVol. 62, Iss. 4 (2024)10.1137/23M1563840*.
TO MODEL THE FLOW OF THE MIXTURE, WE NEED A 3×3 SYSTEM.

C^1-decay?

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- Keep track of the evolution of the Riemann invariants along the characteristic curves!
- Define a time t_0 that is sufficiently large such that each characteristic curves **passes at least once trough a 'damping node'** where the factor μ^v decreases the C^1-norm.
- Show that after the time t_0, the C^1-norm is decreased by a certain fixed factor! It is given as the product of μ^v and an upper bound for the **possible amplification**.

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- Show that after the time t_0, the C^1-norm is decreased by a certain fixed factor! It is given as the product of μ^v and an upper bound for the **possible amplification.**
- This yields the desired result, similar as in GUGAT, M., WEILAND, S. (2021). *Nodal Stabilization of the Flow in a Network with a Cycle*. JODEA 29(2), 1-23.

C^1-decay?

Exponential decay of the C^1-norm can also be analyzed in the spirit of **geometric optics**:

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- However: In the control literature, LYAPUNOV FUNCTIONS are a preferred tool, even for finite-time stabilization!

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We expect that it works if the sensors are placed in such a way that if the graph is cut at all the sensors, only trees remain!

Inhalt

Why do we need observers?

Model for the flow in a natural gas pipe

The observer system and exponential synchronization:
Also here the exponential weights as used by Jean-Michel Coron et al. are very useful!

What about mixtures (hydrogen)? The problem of embrittlement - what is rainflow counting?

Extension Slide: Hydrogen, Embrittlement

We extended the observer to a model for a **mixture** of **natural gas and hydrogen**.

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- In general *Riemann invariants do not exist*. Instead we use *characteristic variables*.
- There are *two time-scales*: Pressure waves are fast and concentration travels much more slowly.