(Non)-synchronization of boundary observers for wave equations

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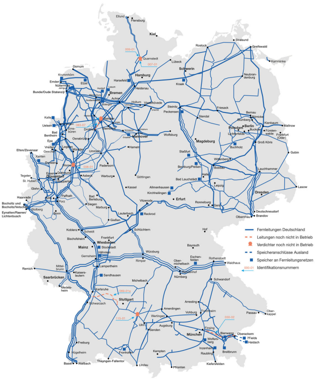
State estimation on networks using observers



- Estimate the current system state in gas/H2 networks (pressure, velocity in all pipes) to improve control decisions
- The state can only be measured at a certain number of points.
- Combine model/simulation and measurements.
- Construct observer system, i.e. IBVP that uses approximate initial data and nodal measurements.
- How many measurement points are needed so that we can guarantee synchronization for long times?

Setup

We consider the case

- Full state measurements on all boundary nodes (no inner nodes)
- No measurement errors
- Original system and observer system coincide
- Linear model: wave equation (without fricton)

On each pipe (edge on the graph) the model reads

$$\begin{pmatrix} R_+ \\ R_- \end{pmatrix}_t + \begin{pmatrix} c & 0 \\ 0 & -c \end{pmatrix} \begin{pmatrix} R_+ \\ R_- \end{pmatrix}_x = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

for some $c > 0$. Kirchhoff-type coupling conditions at inner nodes

$$\sum_{e \in \mathcal{E}(v)} (R^e_{\text{out}}(t, v) - R^e_{\text{in}}(t, v)) = 0, \quad R^e_{\text{out}}(t, v) + R^e_{\text{in}}(t, v) = R^f_{\text{out}}(t, v) + R^f_{\text{in}}(t, v) \; \forall e, f \in \mathcal{E}(v)$$

where $\mathcal{E}(v)$ is the set of edges adjacent to some node v. We prescribe R_{out} on each boundary node.

We can think of $R_\pm$ as Riemann invariants of a linearized Euler equation.

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$$R^e_{\text{out}}(t, v) = -R^e_{\text{in}}(t, v) + \frac{2}{|\mathcal{E}(v)|} \sum_{g \in \mathcal{E}(v)} R^g_{\text{in}}(t, v), \quad t \in (0, T), v \in \mathcal{V} \setminus \mathcal{V}_\partial$$

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Evolution equation of difference system

The difference $\delta_\pm$ between the solutions of original system and observer system satisfies

$$\begin{aligned}
&\partial_t \begin{pmatrix} \delta_+^e \\ \delta_-^e \end{pmatrix} + \begin{pmatrix} c & 0 \\ 0 & -c \end{pmatrix} \partial_x \begin{pmatrix} \delta_+^e \\ \delta_-^e \end{pmatrix} = 0, && e \in \mathcal{E}, \\
&\delta_\pm^e(0,x) = y_\pm^e(x) - z_\pm^e(x), && x \in (0,\ell^e), e \in \mathcal{E}, \\
&\delta_{\text{out}}^e(t,v) = 0, && t \in (0,T), v \in \mathcal{V}_\partial, e \in \mathcal{E}(v), \\
&\delta_{\text{out}}^e(t,v) = -\delta_{\text{in}}^e(t,v) + \tfrac{2}{|\mathcal{E}(v)|} \sum_{g \in \mathcal{E}(v)} \delta_{\text{in}}^g(t,v), && t \in (0,T), v \in \mathcal{V} \setminus \mathcal{V}_\partial.
\end{aligned}$$

where V_∂ is the set of boundary nodes and $y_\pm^e$ and $z_\pm^e$ are initial data of original system and observer system, respectively.

No synchronization for networks with inner cycles

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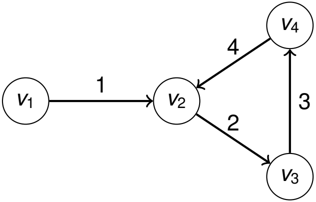
Lemma

If the graph $G = (\mathcal{V}, \mathcal{E})$ contains a cycle consisting of inner points then synchronization cannot be guaranteed, i.e. there exist initial data $y_\pm, z_\pm$ such that

$$\lim_{t \to \infty} \|(\delta_+(t), \delta_-(t))\|_{L^2(\mathcal{E})} \neq 0.$$

Proof.

We consider an example. The general case can be handled analogously.



For any $a \in \mathbb{R}$
$\delta^1_\pm = 0, \delta^j_+ = a, \delta^j_- = -a$ for $j \in 2, 3, 4$
is a stationary solution.

Observability inequality

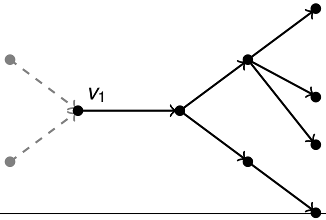
Lemma

Let $G = (\mathcal{V}, \mathcal{E})$ be a tree-shaped network with N inner nodes. Let ℓ_m denote the maximal length of a pipe in G. Then there exists a constant $C > 0$ such that for $T \geq N\frac{\ell_m}{c}$ and $t > T$ we have

$$\|(\delta_+, \delta_-)(t, \cdot)\|^2_{L^2(\mathcal{E})} \leq C \sum_{v \in \mathcal{V}_\partial} \|(\delta_+, \delta_-)(\cdot, v)\|^2_{L^2([t-T, t+T])}, \quad (1)$$

Proof.

By induction in N
Full graph G in gray and black
Reduced graph G_1 in black



Observability inequality: induction step

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For $N = 1$ the graph is star shaped and the result is known.

For $N > 1$ there exists an inner node v_1 that has only one edge connected to another inner node. Let G_1 be the sub-graph obtained by removing from G all edges connecting v_1 to boundary nodes $\Longrightarrow$ observation inequality holds on G_1.

Note $\mathcal{V}_\partial(G_1) \subset \{v_1\} \cup \mathcal{V}_\partial(G)$. Thus, we need to control

$$\|(\delta_+, \delta_-)(\cdot, v_1)\|_{L^2(t-T,t+T)} \leq C \sum_{v \in \mathcal{V}_\partial(G)} \|(\delta_+, \delta_-)(\cdot, v)\|_{L^2(t-T-c,t+T+c)}$$

In a next step, we control the L^2-in-space norm on the 'removed' edges by $\sum_{v \in \mathcal{V}_\partial(G)} \|(\delta_+, \delta_-)(\cdot, v)\|_{L^2(t-T-c,t+T+c)}$.

Synchronization

Theorem

Let $G = (\mathcal{V}, \mathcal{E})$ be a tree-shaped network. Then there exist constants $\mu > 0$, $C_1 > 0$ such that

$$\|(\delta_+, \delta_-)(t, \cdot)\|^2_{L^2(\mathcal{E})} \leq C_1 \exp(-\mu t) \quad \forall t > 0.$$

Sketch of Proof:

$$\|(\delta_+, \delta_-)(t + \tilde{t}, \cdot)\|^2_{L^2(\mathcal{E})} - \|(\delta_+, \delta_-)(t - \tilde{t}, \cdot)\|^2_{L^2(\mathcal{E})}$$

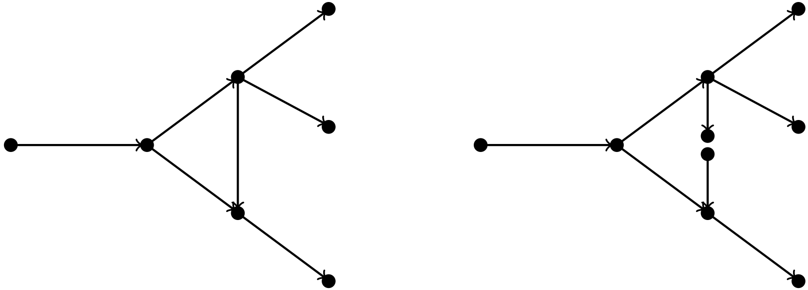
$$\leq (-c) \int_{t-\tilde{t}}^{t+\tilde{t}} \sum_{v \in \mathcal{V}_\partial} \sum_{e \in \mathcal{E}(v)} \left(|\delta^e_{\text{out}}(s, v)|^2 + |\delta^e_{\text{in}}(s, v)|^2 \right) ds$$

$$= (-c) \sum_{v \in \mathcal{V}_\partial} \|(\delta_+, \delta_-)(\cdot, v)\|^2_{L^2([t-\tilde{t}, t+\tilde{t}])}.$$

Apply observability and modified Gronwall lemma.

What to do about general networks?

The effect of inserting a full state measurement in a pipe (edge) on the observer corresponds to splitting that edge and adding a boundary node for each half. If we add one measurement per cycle, we end up with a tree shaped graph.



What about finite time synchronization?

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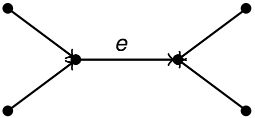
For start shaped networks without friction there is finite time synchronization.

If there is one inner pipe whose end-nodes have more than two adjacent pipes this is no longer true, due to reflection at nodes with more than two adjacent pipes:



If we start with $\delta^e_\pm(t=0) = 1$ and $\delta_\pm = 0$ on all other edges then for any $n \in \mathbb{N}$

$$\delta^e_\pm(t = n\frac{|e|}{c}) = \frac{1}{3^n}, \quad \delta^f_-(t = n\frac{|e|}{c}) = \frac{2}{3^n} \text{ for } f \neq e.$$

Summary

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- Exponential synchronization of boundary observers for linear wave equations with full state measurements for networks without cycles. (This is different from measuring amplitudes only in string networks)
- This is optimal in the sense that finite time synchronization does not hold in general.
- in general, there is no synchronization for networks with cycles $\longrightarrow$ one needs to add one measurement per cycle to ensure synchronization.
- Analogous results hold in the case with linear friction. Technical challenge: Riemann invariants are no longer constant along characteristics and interact constantly.
- We conjecture that analogous results also hold for non-linear friction – as long as the friction law is Lipschitz, and non-linear wave equations as long as solutions are subsonic.

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Thank you for your attention!