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A review: basic fractional nonlinear-wave models and solitons

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1. Introduction. The concept **of derivatives of fractional orders** was introduced, as a mathematical curiosity, by **Niels Henrik Abel** in **1823**:

N. H. Abel, *Oplösning af et par opgaver ved hjelp af bestemte integraler*. Magazin for Naturvidenskaberne, Aargang I, Bind 2, Christiania, 1823.

IV.

Oplösning af et Par Opgaver ved Hjelp af bestemte Integraler.

Af

N. H. Abel.

I.

Det er som bekiendt ofte Tilfældet, at man ved Hjelp af bestemte Integraler (intégrales définies) kan opløse mange Opgaver, som man paa anden Maade enten aldeles ikke eller dog meget vanskelig kan opløse, og især har man anvendt dem med Held paa Opløsningen af flere vanskelige Opgaver i Mechaniken, f. Ex. om Bevægelsen af en elastisk Flade, i Bolgetheorien &c. En anden Anvendelse af disse Integraler vil jeg vise i Opløsningen af følgende Opgave:

"Lad CB Tavle I, Fig. 4, være en horizontal Linie, A et "givet Punkt; AB lodret paa BC, AM en krum Linie, "hvis retvinklede Koordinater ere $AP = x$, $PM = y$. "Endvidere være $AB = a$ og $KM = s$. Tænker man "sig at et Legeme gjennemløber Buen CA med en

In that work, Abel had introduced a fractional-order derivative of function ***f*(*t*)**, which, in the modern literature, is usually called the ***Caputo derivative***, that was reintroduced in **1967**:

M. Caputo, Linear models of dissipation whose *Q* is almost frequency independent – II. *Geophysical Journal of the Royal Astronomical Society*, **13**, Issue 5 (1967), cited ca. **2,600** times.

Book: **Michele Caputo**, *Elasticitá e Dissipazione*. Zanichelli, Bologna, 1969.

$$D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-1-\alpha} f^{(n)}(\tau)\, d\tau,$$

where the **integer part** of α, $n \equiv [\alpha]$, is an integer closest to α, such that $n-1 < \alpha < n$.

In physics, the concept of fractal derivatives was introduced by **Nikolai Laskin** (University of Toronto, Canada) in **2000**, in the context of ***fractional quantum mechanics***:

N. Laskin, Fractional quantum mechanics and Lévy path integrals, Phys. Lett. A **268**, 298-305 (2000) (cited about **1,500 times**).

ELSEVIER

17 April 2000

Physics Letters A 268 (2000) 298–305

PHYSICS LETTERS A

www.elsevier.nl/locate/physleta

Fractional quantum mechanics and Lévy path integrals

Nikolai Laskin

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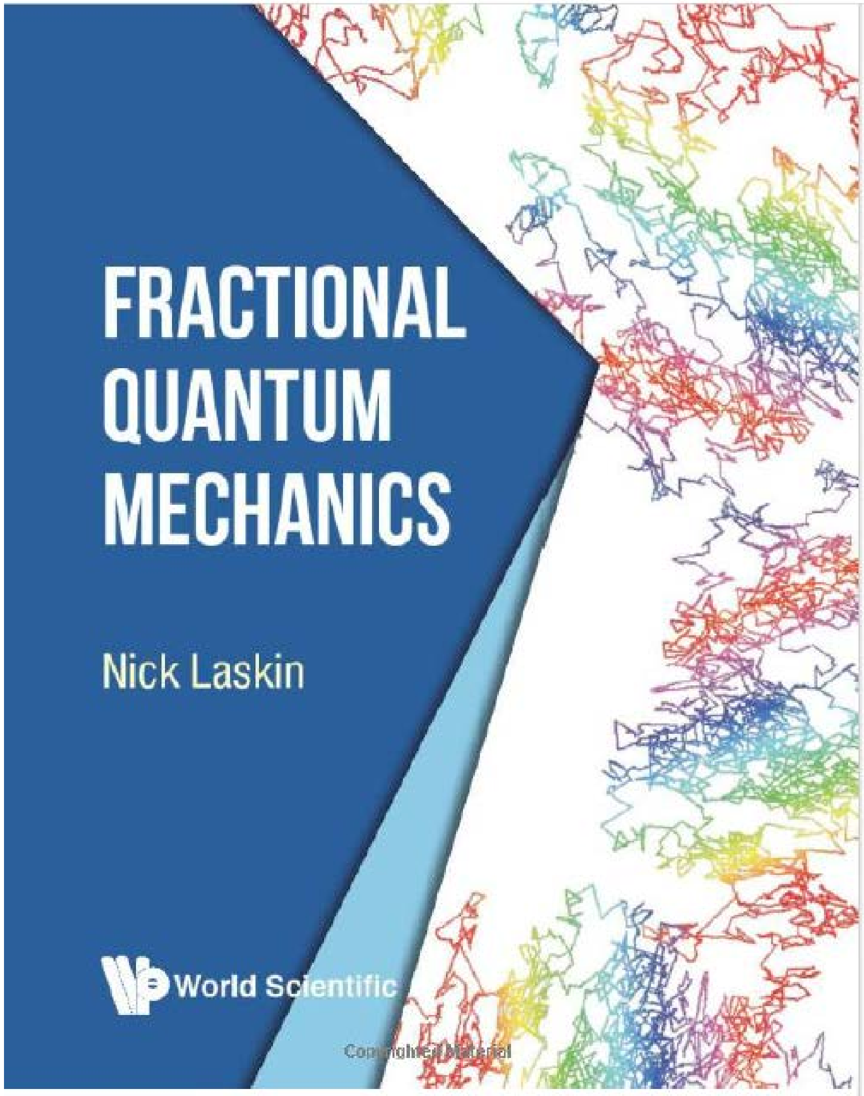
Communicated by A.R. Bishop

A book by the same author:

N. Laskin, *Fractional Quantum Mechanics*

(World Scientific, Singapore, 2018)



In these works, the **fractal Schrödinger equation** was derived, by means of the **Feynman's integrals**, alias **path integrals** (~ $\int$**exp(i*S*)d(path)**), for a quantum particle whose classical stochastic motion, with **action *S***, does not follow the usual **Brownian law**, but proceeds through **random jumps** (***Lévy flights***).

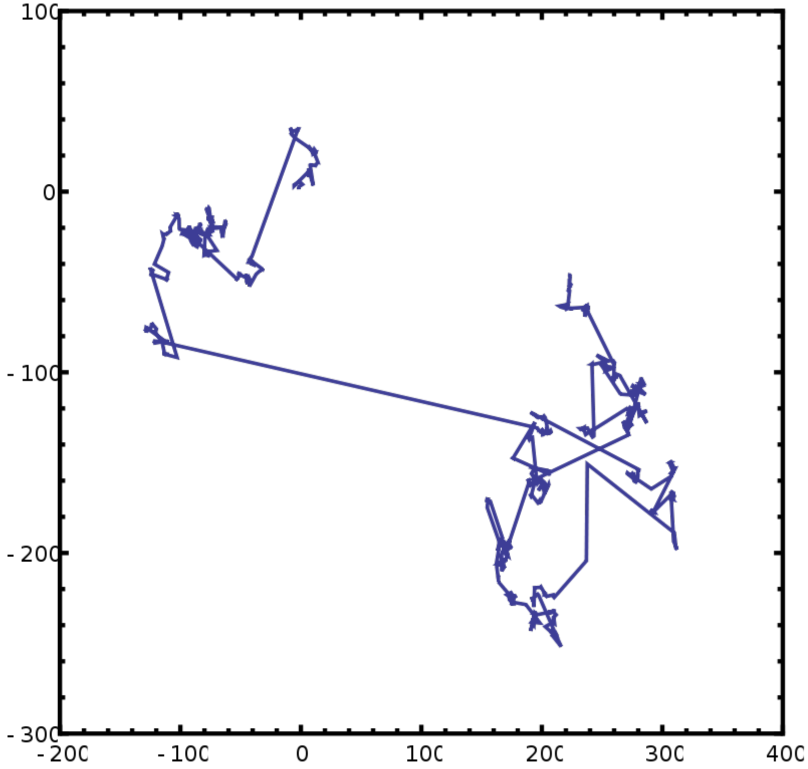
The term "Lévy flights" was coined by **Benoît Mandelbrot** (the author of the concept of ***fractals***). The average distance from the initial position of a classical particle moving by Lévy flights (along axis ***x***) grows with time as

$$\langle |x| \rangle \sim t^{1/\alpha}, \text{where } \alpha \leq 2 \text{ is called the } Levy\ \ index.$$

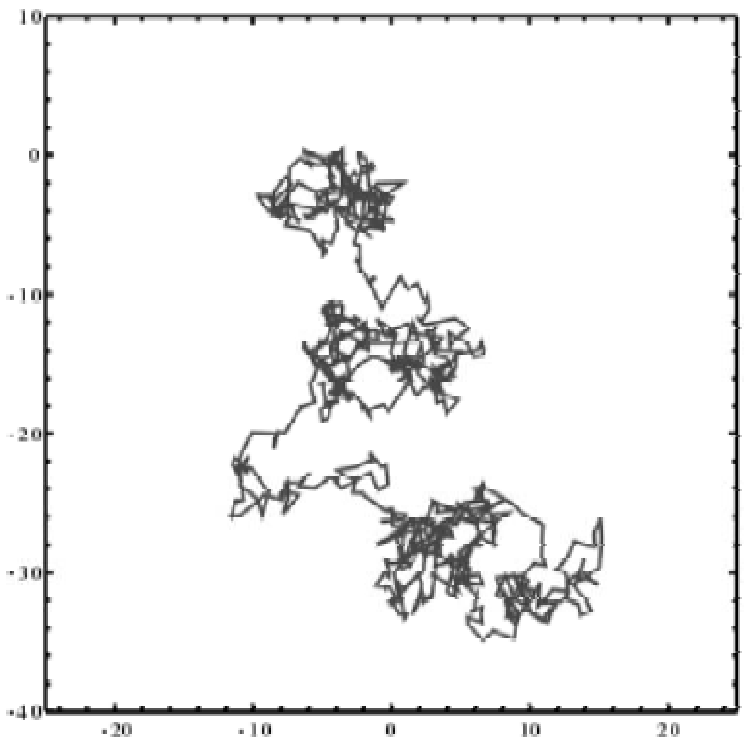
That is, in the case of $\boldsymbol{\alpha < 2}$, the stochastic motion of the Lévy particle (at $\boldsymbol{t} \to \boldsymbol{\infty}$) is ***faster*** than the classical **random (Brownian) walk**, which corresponds to $\boldsymbol{\alpha = 2}$, i.e., $\boldsymbol{\langle x^2 \rangle \sim t}$.

A typical example of the trajectory built of **1000** **stochastic Lévy flights** of a particle, corresponding to ***α* = 1** in two dimensions [**e.g., a shark in the search of food in the ocean** (even if a shark can scarcely be considered as a prototype of a quantum particle); the picture is borrowed from Wikipedia]. For comparison, a trajectory built of **1000** random steps of the usual Brownian particle (***α* = 2**) is shown too (right) (note the **difference in the spatial scales**):

α = 1 (Lévy flights)



α = 2 (Brownian motion)



The Schrödinger equation derived by Laskin for the quantum particle moving by the Lévy flights, written in a scaled form, is

$$i\frac{\partial\psi}{\partial t}=\frac{1}{2}\left(-\frac{\partial^2}{\partial x^2}\right)^{\alpha/2}\psi+U(x)\psi,$$

where $U(x)$ is an external potential, and the kinetic-energy operator, $\left(-\partial^2/\partial x^2\right)^{\alpha/2}$, is represented by the *Riesz derivative* (named after Marcel Riesz), which is defined as follows: take the Fourier transform of ψ, with wavenumber k; in the Fourier space, the action of operator $\left(-\partial^2/\partial x^2\right)^{\alpha/2}$ amounts to the multiplication by $|k|^\alpha$; after that, return from the Fourier space back to the coordinate space, applying the inverse Fourier transform.

Thus, the fractional differential operator, which represents the kinetic energy in the one-dimensional version of **fractional quantum mechanics**, is actually an integral operator, generated by the juxtaposition of the direct and inverse Fourier transforms:

$$\left(-\frac{\partial^2}{\partial x^2}\right)^{\alpha/2}\psi=\int_{-\infty}^{+\infty}dk\,|\,k\,|^{\alpha}\int_{-\infty}^{+\infty}d\xi e^{ik(x-\xi)}\psi(\xi).$$

Similarly, the kinetic-energy operator appearing in the Schroedinger equation for the two-dimensional quantum Levy particle takes the following integral form:

$$\left(-\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}\right)^{\alpha/2}\psi=\iint dkdq\left(p^2+q^2\right)^{\alpha/2}\iint d\xi d\eta e^{ik(x-\xi)+iq(y-\eta)}\psi\left(\xi,\eta\right).$$

The structure of the talk

2. A proposal to **emulate** the fractional Schrödinger equation **in optics**.

3. **Experimental realization** of the temporal fractional group-velocity dispersion in fiber optics.

4. Adding **nonlinearity** to fractional systems.

5. An example: a **domain wall** in a system of two coupled fractional nonlinear Schrödinger equations.

6. Fractional 2D matter-wave solitons stabilized by the **spin-orbits coupling**.

7. Two-component solitons in the fractional system with the **second-harmonic generation.**

8. Quasi-solitons in the fractional **Lugiato-Lefever** system.

9. Conclusion.

2. A proposal to *emulate* the fractional Schrödinger equation in optics

The fractional quantum mechanics has not been, as yet, realized experimentally. Making use of the commonly known fact that the quantum-mechanical Schrödinger equation is tantamount to the classical equation for the paraxial propagation of light, it was proposed to ***emulate*** the fractional Schrödinger equation in optical cavities (this paper was cited ca. **300 times**):

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Fractional Schrödinger equation in optics

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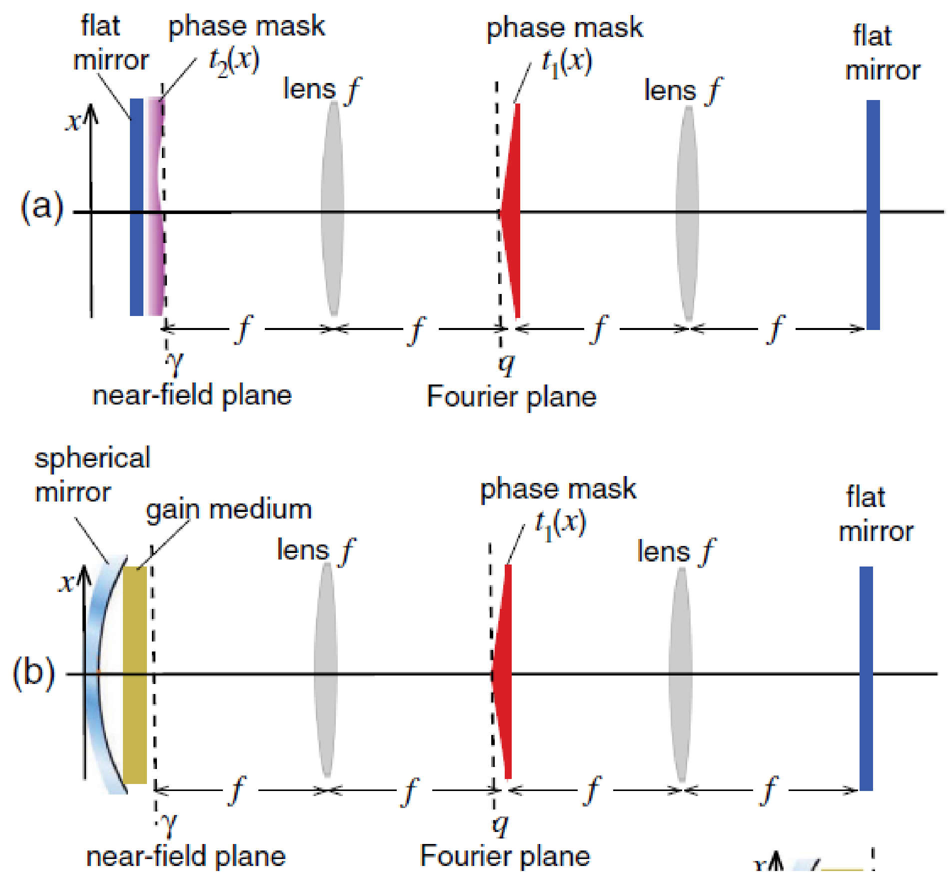
The proposal aimed to emulate ***the fractional diffraction*** in an optical **4*f* setup**. The transverse structure of a spatial light beam is converted into the ***Fourier form*** by a lens, then an appropriately designed ***phase mask*** adds phase shifts to different ***spatially separated*** Fourier components. The phase shifts are ***the same*** as would be produced by the **fractional Riesz derivative**. Finally, another lens casts the optical field back into the form of a parallel-propagating beam (the bottom scheme realizes the fractional Schrödinger equation including the harmonic-oscillator trapping potential, $U(x) = \text{const}\cdot x^2$:



Circulation of light in this optical cavity is governed by the effective averaged fractional Schrödinger equation, which ***emulates*** the corresponding equation in quantum mechanics:

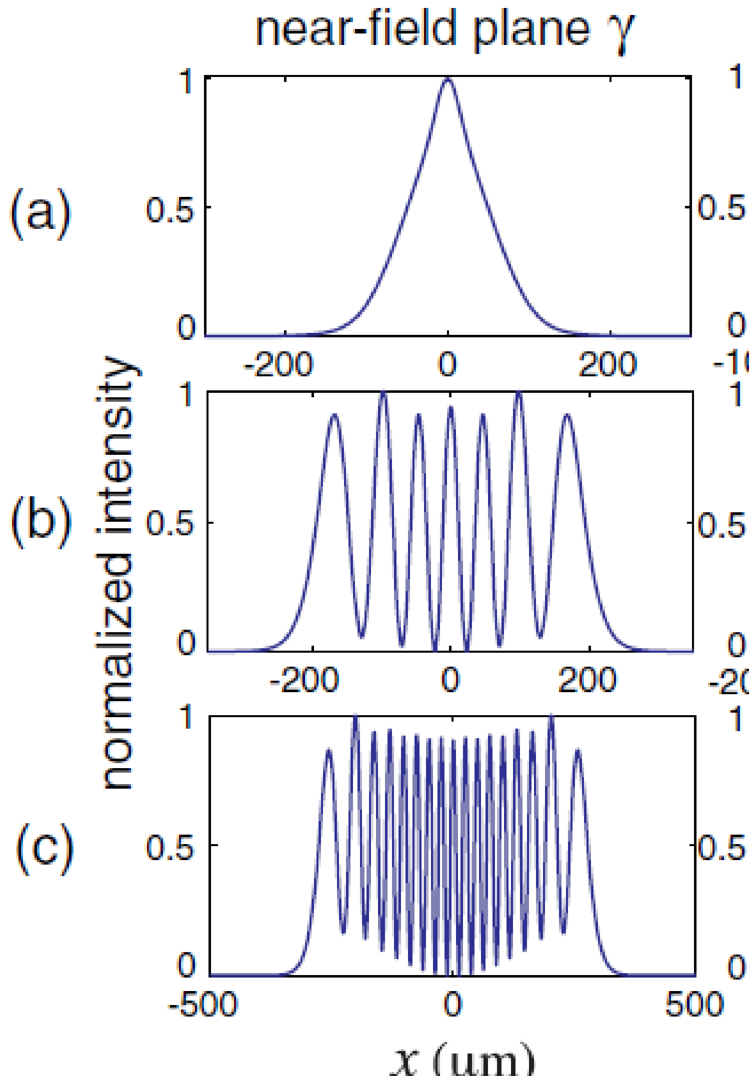
$$i\frac{\partial\psi}{\partial z}=\frac{1}{2}\left(-\frac{\partial^2}{\partial x^2}\right)^{\alpha/2}\psi+U(x)\psi,$$

where $U(x)$ is an effective potential, and z is the propagation distance instead of time in quantum mechanics.

The two-dimensional fractional Schroedinger equation may be realized in this setting as well, in the form of

$$i\frac{\partial\psi}{\partial z}=\frac{1}{2}\left(-\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}\right)^{\alpha/2}\psi+U(x,y)\psi.$$

Examples: the ground and excited **eigenstates** produced by the fractional one-dimensional Schrödinger equation with Lévy index ***α* = 1,** including the harmonic-oscillator trapping potential, ***U*(*x*) = const·*x*²**:



3. Experimental realization of the temporal fractional group-velocity dispersion (instead of the spatial diffraction) in fiber optics

The **cardinal problem** is the absence of any previously reported experimental realization of the fractional diffraction in linear or nonlinear optics (experimental realization of the fractional Schrödinger equation in quantum mechanics was not reported either).

Recently, an experimental realization of ***fractional dispersion*** (in the ***temporal domain***, rather than fractional diffraction in the spatial domain) has been reported, using a fiber-laser cavity.

nature communications

Article https://doi.org/10.1038/s41467-023-35892-8

Experimental realisations of the fractional Schrödinger equation in the temporal domain

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Published online: 14 January 2023

Shilong Liu[1,2], Yingwen Zhang[1,3], Boris A. Malomed[4,5] & Ebrahim Karimi[1,3]

The **main principle** is to split the temporal wave packet into its spectral components, and pass the light signal with the **spatially separated** spectral components through a **phase mask, realized as a hologram,** which imparts a particular phase shift to each component, so as to **emulate** the action of the **fractional GVD** (in the combination with the **regular GVD**) onto the original wave packet. With the Lévy index $\boldsymbol{\alpha}$, the phase shift emulating the action of the fractional GVD onto a spectral component with frequency $\boldsymbol{\omega}$ should be $\mathbf{const}\cdot|\boldsymbol{\omega}|^{\alpha}$.

The theoretical model

The propagation of wave packets in an optical fiber obeys the Schroedinger equation which is mathematically similar to the one in the spatial domain (planar waveguide), but its physical meaning is different, as it models the action of the group-velocity dispersions (**GVD**), rather than diffraction in the plane of the waveguide.

Thus, the propagation of light in the fiber laser may be affected by the action of both the **fractional** and usual (non-fractional) **GVD**:

$$i\frac{\partial\psi}{\partial z}=\frac{D}{2}\left(-\frac{\partial^2}{\partial\tau^2}\right)^{\alpha/2}\psi-\sum_{k=2,3,\ldots}\frac{\beta_k}{k!}\left(i\frac{\partial}{\partial\tau}\right)^k\psi,$$

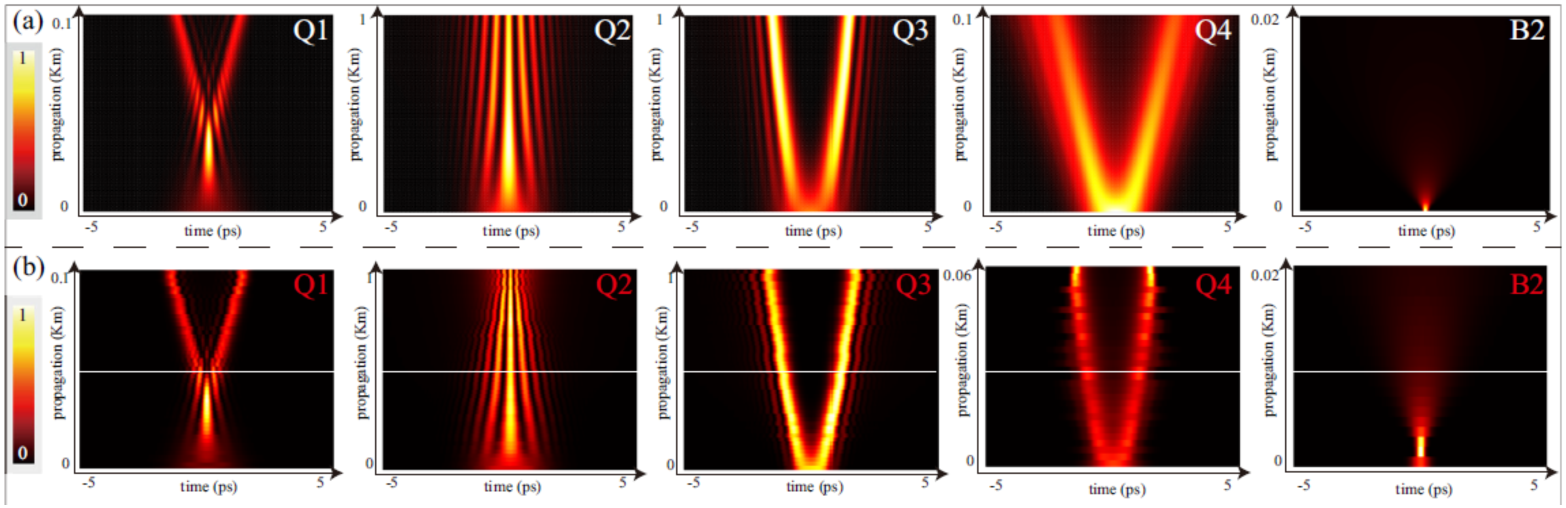
where z is again the propagation distance (along the fiber), while the *temporal coordinate* is $\tau=t-z/V_{gr}$. Further, D is an effective coefficient of the fractional dispersion with Levy index α, and β_k are coefficients of the regular (usual) dispersion of **integer** k-th orders.

Note that **no nonlinearity** is included here.

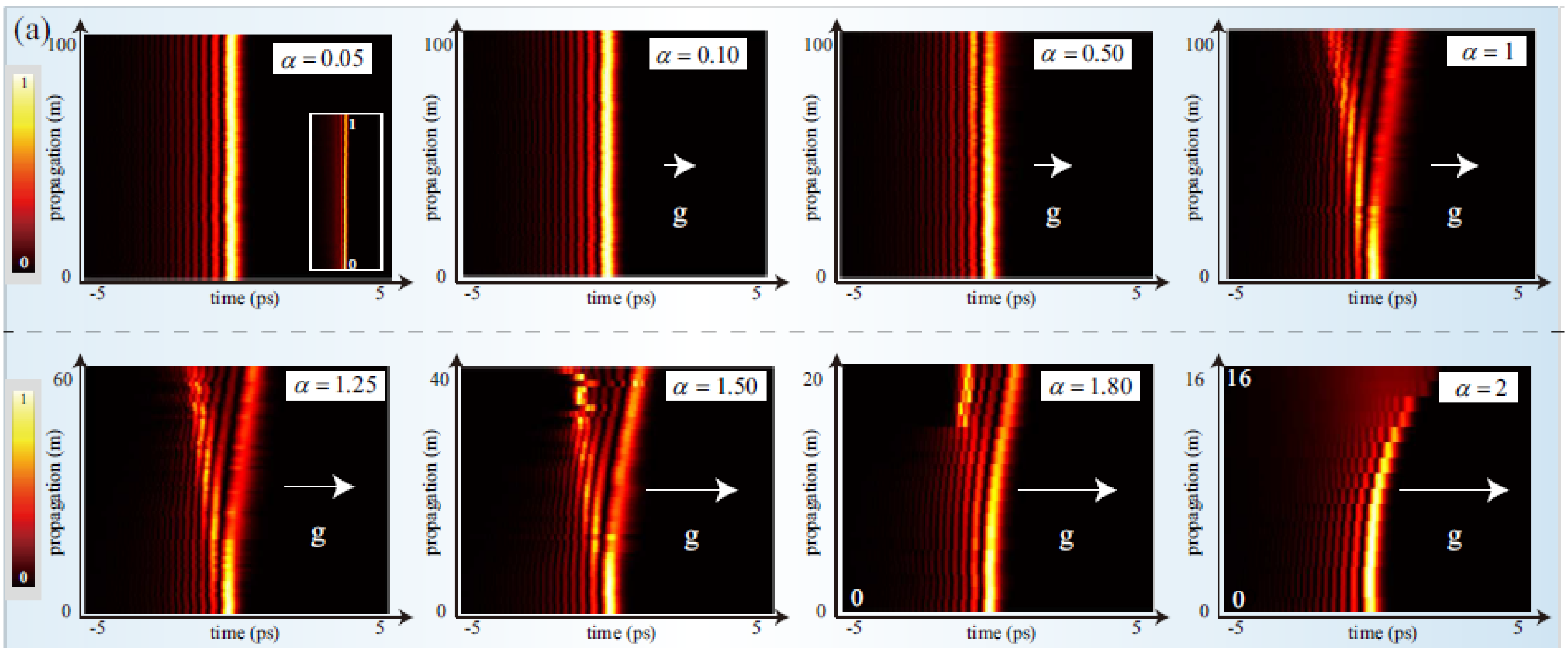
Basic experimental results and the corresponding simulations

Row (a): simulations; **row (b)**: experiment.

$\alpha = 1.25$	$\alpha = 0.25$	$\alpha = 0.25$	$\alpha = 1.25$	close to
$L_{GVD} = 5$	$L_{GVD} = 5$	$L_{GVD} = -5$	$L_{GVD} = -5$	$\alpha = 2$



Another set of experimental results: the evolution of **quasi-Airy** waves under the action of the fractional **GVD** with different values of the Lévy index, ***α***. The Airy wave is initiated by adding factor **exp(-*iCω*³)** to the Fourier transform of the input (in the experiment, it is generated by a fiber segment with the regular third-order **GVD**). White arrows indicate effective acceleration of the central lobe of the wave packet.



4. The interplay of the fractional diffraction and nonlinearity

Because the optical medium naturally includes the **Kerr nonlinearity (self-focusing),** the corresponding cubic term may be added to the fractional Schrödinger equation:

$$i\frac{\partial\psi}{\partial z}=\frac{1}{2}\left(-\frac{\partial^2}{\partial x^2}\right)^{\alpha/2}\psi+U(x)\psi-\gamma|\psi|^2\psi,$$

where γ is the nonlinearity coefficient. In the case of $\gamma>0$ (self-attraction), this 1D equation gives rise to the **critical collapse** (catastrophic self-compression of the wave field) at $\alpha=1$, and **supercritical collapse** at $\alpha<1$. **Stable solutions** are possible at $1<\alpha\leq 2$.

The **two-dimensional version** of the nonlinear fractional Schroedinger equation:

$$i\frac{\partial\psi}{\partial z}=\frac{1}{2}\left(-\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}\right)^{\alpha/2}\psi+U(x,y)\psi-\gamma|\psi|^2\psi.$$

The two-dimensional equation gives rise to the **supercritical collapse** at $\alpha<2$.

The same cubic term, with $\gamma>0$ or $\gamma<0$, may be added to the quantum-mechanical (**Laskin's**) one-, two-, and three-dimensional fractional Schroedinger equations (with time t instead of z). It is **an attempt** to introduce the **Gross-Pitaevskii** equation for a **Bose-Einstein condensate** of quantum *Levy-flighting* particles.

The nonlinear equations produce various one- and two-dimensional modes supported by the self-focusing (or defocusing) of light, such as bright and dark solitons, fronts, vortices, etc. Such modes were considered in many theoretical works. **A brief review**:
Photonics **8**, 353 (2021) (for the time being, cited **95 times**):

photonics

Review

Optical Solitons and Vortices in Fractional Media: A Mini-Review of Recent Results

Boris A. Malomed [1,2]

An updated recent review

(for the time being, cited only 5 times):

Chaos REVIEW pubs.aip.org/aip/cha

Basic fractional nonlinear-wave models and solitons

Cite as: Chaos **34**, 022102 (2024); doi: 10.1063/5.0190039
Submitted: 2 December 2023 · Accepted: 9 January 2024 ·
Published Online: 7 February 2024

View Online Export Citation CrossMark

Boris A. Malomed[a)]

5. A relatively simple example of theoretically elaborated nonlinear states in the fractional medium:
a one-dimensional **domain wall** separating two **immiscible** (mutually repelling) wave fields, produced by a system of coupled **fractional nonlinear Schrödinger** (**FNLS**) equations with the self- and cross-defocusing nonlinearities.

PHYSICAL REVIEW E **106**, 054207 (2022)

Domain walls in fractional media

Shatrughna Kumar[1] Pengfei Li[2,3] and Boris A. Malomed[1,4]
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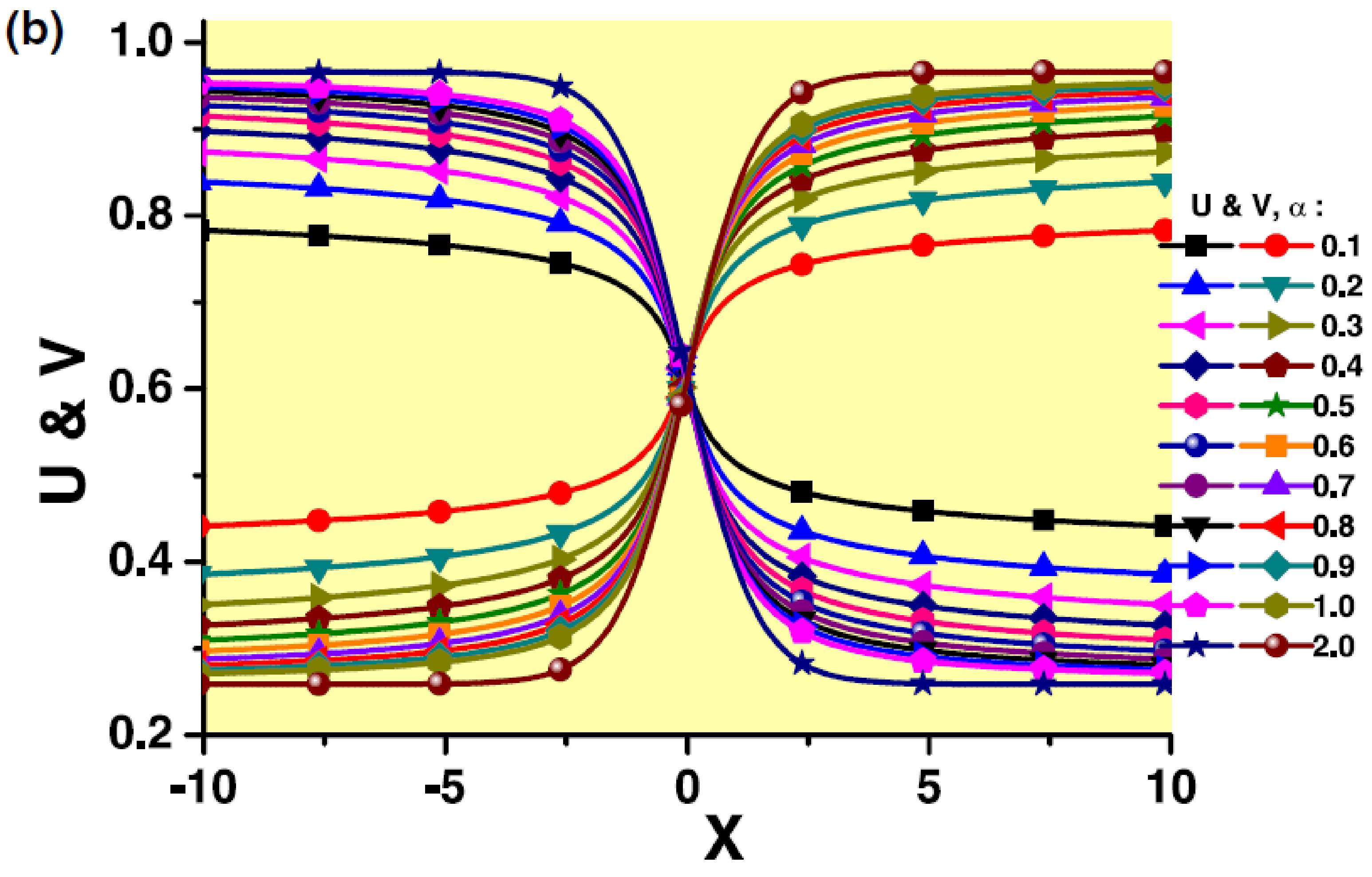
[4]*Instituto de Alta Investigación, Universidad de Tarapacá, Casilla 7D, Arica, Chile*

The system of coupled **FNLS** equations [***α*** is again the Lévy index, the **immiscibility** condition is ***β* > 1** (***β*** is the relative **cross-phase-modulation** (**XPM**) coefficient), and ***λ*** represents possible linear mixing between the fields]:

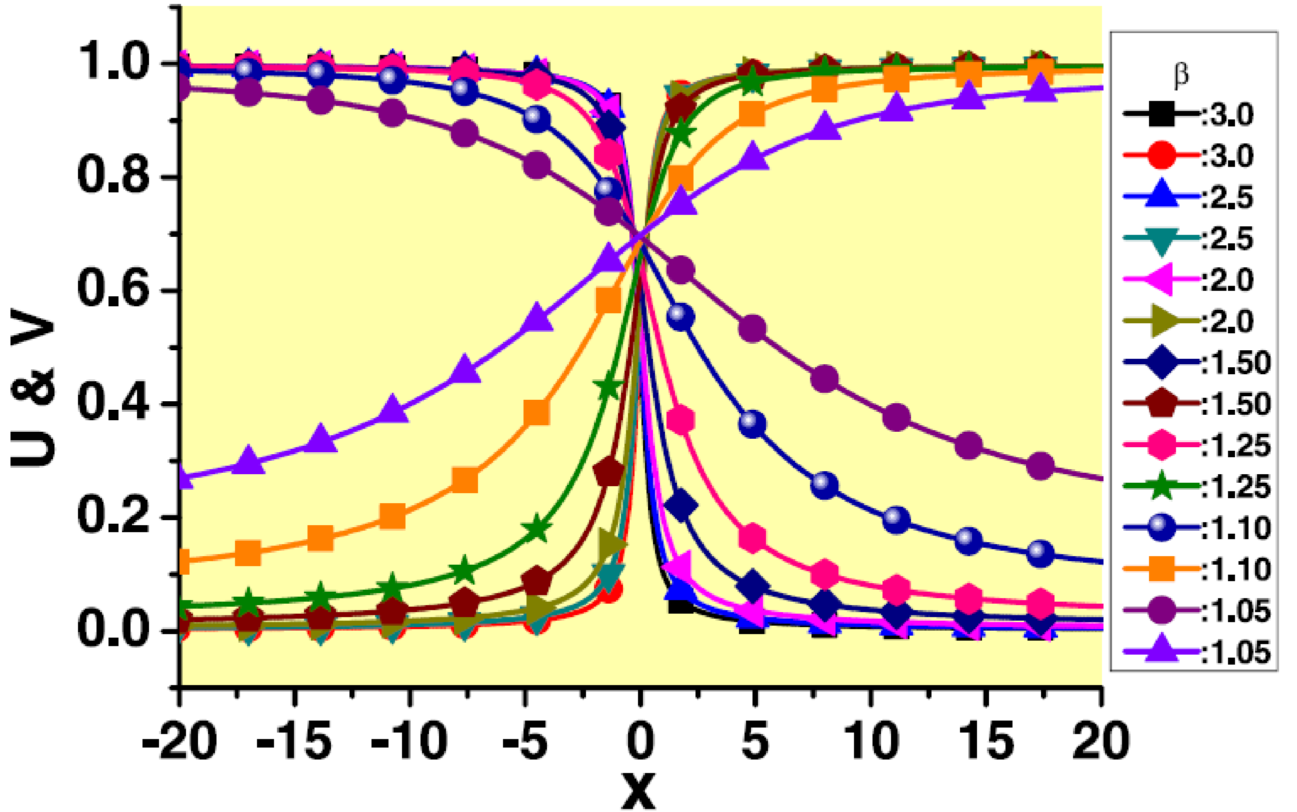
$$i\frac{\partial u}{\partial z} = \frac{1}{2}\left(-\frac{\partial^2}{\partial x^2}\right)^{\alpha/2} u + (|u|^2 + \beta|v|^2)u - \lambda v,$$

$$i\frac{\partial v}{\partial z} = \frac{1}{2}\left(-\frac{\partial^2}{\partial x^2}\right)^{\alpha/2} v + (|v|^2 + \beta|u|^2)v - \lambda u,$$

Stable **domain-wall patterns** produced by the coupled **FNLS** equations for ***β* = 3**, ***λ* = 0.5**, and Lévy indices between ***α* = 0.1** and ***α* = 1:**



Stable **domain-wall patterns** produced by the coupled **FNLS** equations for $\boldsymbol{\alpha = 1}$, $\boldsymbol{\lambda = 0}$, and the **XPM** coefficient taking values between $\boldsymbol{\beta = 1.05}$ and $\boldsymbol{\beta = 3}$:



6. An attempt to constract 1D and 2D matter-wave solitons under the action of the fractional diffraction and spin-orbit coupling (SOC)

IOP Publishing Journal of Physics B: Atomic, Molecular and Optical Physics

J. Phys. B: At. Mol. Opt. Phys. 55 (2022) 155301 (9pp) https://doi.org/10.1088/1361-6455/ac7685

One- and two-dimensional solitons in spin–orbit-coupled Bose–Einstein condensates with fractional kinetic energy

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[2] Department of Physical Electronics, School of Electrical Engineering, Faculty of Engineering, and Center for Light-Matter Interaction, Tel Aviv University, PO Box 39040 Tel Aviv, Israel
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The **2D** system of fractional Gross-Pitaevskii equations with **SOC** of the **Rashba type** and attractive interatomic interactions:

$$i\frac{\partial\phi_+}{\partial t} = \frac{1}{2}\left(-\nabla^2\right)^{\alpha/2}\phi_+ - \left(|\phi_+|^2 + \gamma|\phi_-|^2\right)\phi_+ + \lambda\left(\frac{\partial\phi_-}{\partial x} - i\frac{\partial\phi_-}{\partial y}\right),$$

$$i\frac{\partial\phi_-}{\partial t} = \frac{1}{2}\left(-\nabla^2\right)^{\alpha/2}\phi_- - \left(|\phi_-|^2 + \gamma|\phi_+|^2\right)\phi_- - \lambda\left(\frac{\partial\phi_+}{\partial x} + i\frac{\partial\phi_+}{\partial y}\right),$$

Stationary solutions with chemical potential μ:

$$\phi_\pm(x,y,t) = e^{-i\mu t} u_\pm(x,y),$$

Equations for the stationary wave functions:

$$\mu u_+ = \frac{1}{2}(-\nabla^2)^{\alpha/2} u_+ - (|u_+|^2 + \gamma |u_-|^2) u_+$$

$$+ \lambda \left(\frac{\partial u_-}{\partial x} - i \frac{\partial u_-}{\partial y} \right),$$

$$\mu u_- = \frac{1}{2}(-\nabla^2)^{\alpha/2} u_- - (|u_-|^2 + \gamma |u_+|^2) u_-$$

$$- \lambda \left(\frac{\partial u_+}{\partial x} + i \frac{\partial u_+}{\partial y} \right),$$

These equations can be derived form the corresponding Lagrangian:

$$
\begin{aligned}
L = {} & \mu \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \left[|u_+(x,y)|^2 + |u_-(x,y)|^2 \right] \\
& - \frac{1}{2\pi^2} \int_0^{+\infty} dp \int_0^{+\infty} dq (p^2+q^2)^{\alpha/2} \int_{-\infty}^{+\infty} d\xi \int_{-\infty}^{+\infty} d\eta \\
& \times \int_{-\infty}^{+\infty} \mathrm{d}x \int_{-\infty}^{+\infty} dy \cos\left[p(x-\xi)+q(y-\eta)\right] \\
& \times \left[u_+^*(x,y)u_+(\xi,\eta) + u_-^*(x,y)u_-(\xi,\eta) \right] - \lambda \int_{-\infty}^{+\infty} dx \\
& \times \int_{-\infty}^{+\infty} dy \left[u_+^* \frac{\partial u_-}{\partial x} + u_+ \frac{\partial u_-^*}{\partial x} \right. \\
& \left. - i \left(u_+^* \frac{\partial u_-}{\partial y} - u_+ \frac{\partial u_-^*}{\partial y} \right) \right] + \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} \\
& \times dy \left[\frac{1}{2} \left(|u_+|^4 + |u_-|^4 \right) + \gamma |u_+|^2 |u_-|^2 \right], \qquad (26)
\end{aligned}
$$

The objective is to construct stable **2D** solitons of the **semi-vortex** (**SV**) type.

In the case of the normal (non-fractional) diffraction, with $\boldsymbol{\alpha = 2}$, **2D** solitons of the **SV** type were first introduced in the paper

PHYSICAL REVIEW E **89**, 032920 (2014)

Creation of two-dimensional composite solitons in spin-orbit-coupled self-attractive Bose-Einstein condensates in free space

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(Received 12 December 2013; published 26 March 2014)

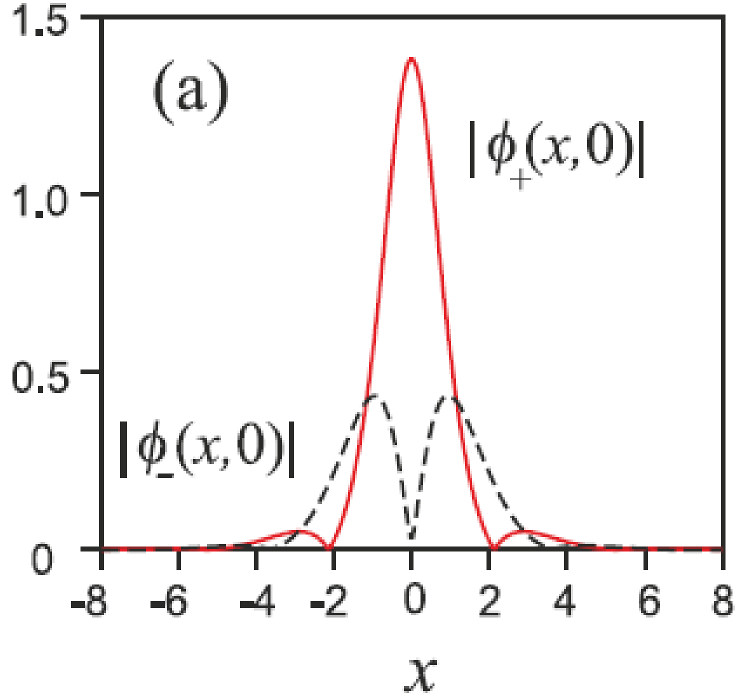
(**thus far, cited 215 times**). The ***ansatz*** for the **SV soliton**, written in the polar coordinates:

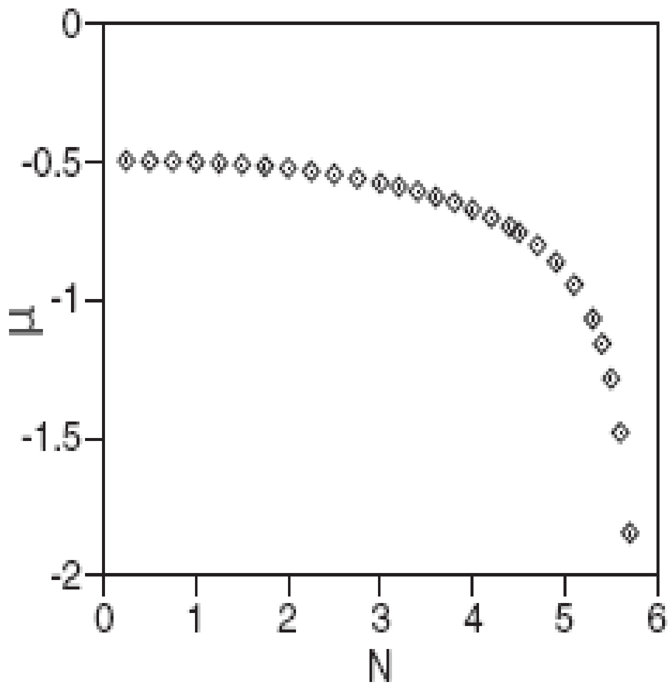
$$u_+ = f_1(r^2),\ u_- = \exp(i\theta) r f_2(r^2).$$

(a) An example of profiles **|$\phi_+(x)$, $\phi_-(x)$|** of the cross sections of the zero-vorticity and vortical components of a **stable SV soliton**.

(b) The family of the **SV solitons** in the plane of **(total norm, chemical potential)** is ***completely stable*** (as the ***ground state***) in spite of the possibility of the ***critical collapse*** in the same system.

(a) (b)





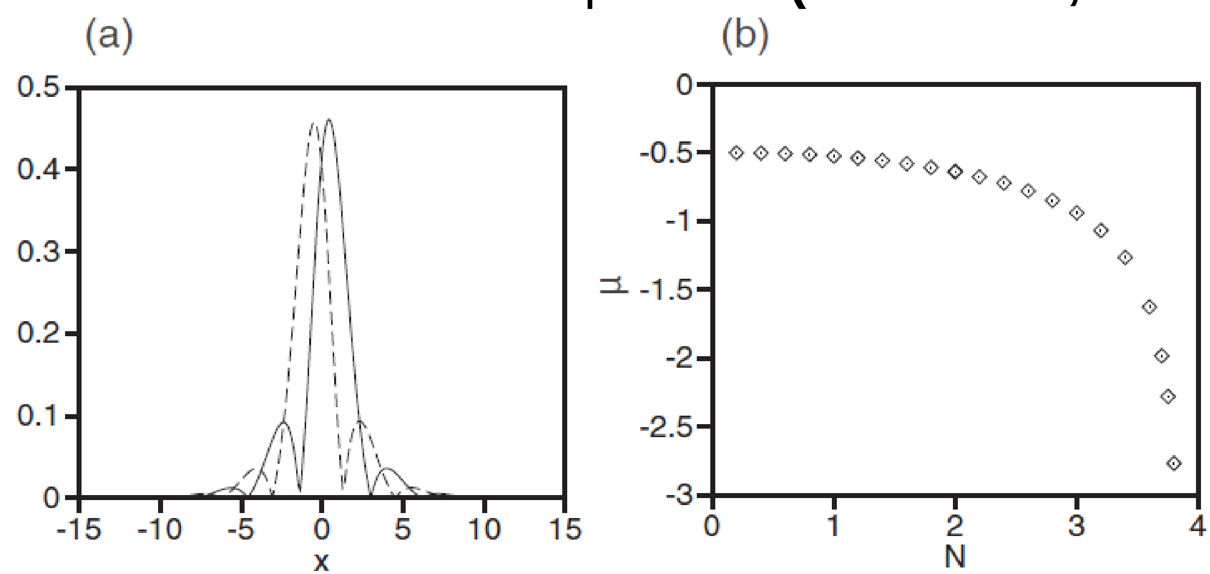
The **2D SV solitons** are ***stable*** in the case of **$\gamma < 1$** (the self-attraction of the components is stronger than the cross-attraction). At **$\gamma < 1$**, SV solitons are unstable, while stable ones (as the ***ground state***) are **mixed-mode** (**MM**) solitons, initiated by the ansatz

$$\phi_{+}^{(0)} = A_1 \exp(-\alpha_1 r^2) - A_2 r \exp(-i\theta - \alpha_2 r^2),$$

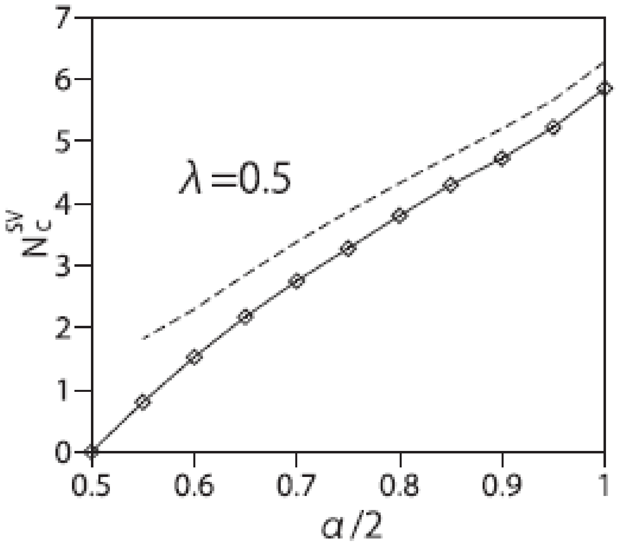
$$\phi_{-}^{(0)} = A_1 \exp(-\alpha_1 r^2) + A_2 r \exp(i\theta - \alpha_2 r^2).$$

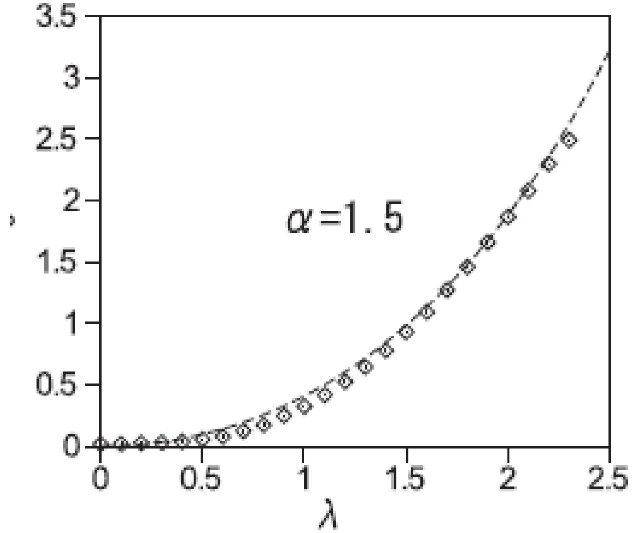
An example of profiles **$|u_+(x), u_-(x)|$** of the cross sections of the zero- (a) vorticity and vortical components of a **stable SV soliton**.

(b) The family of the ***stable*** **SV solitons** in the plane of **(total norm, chemical potential)** is

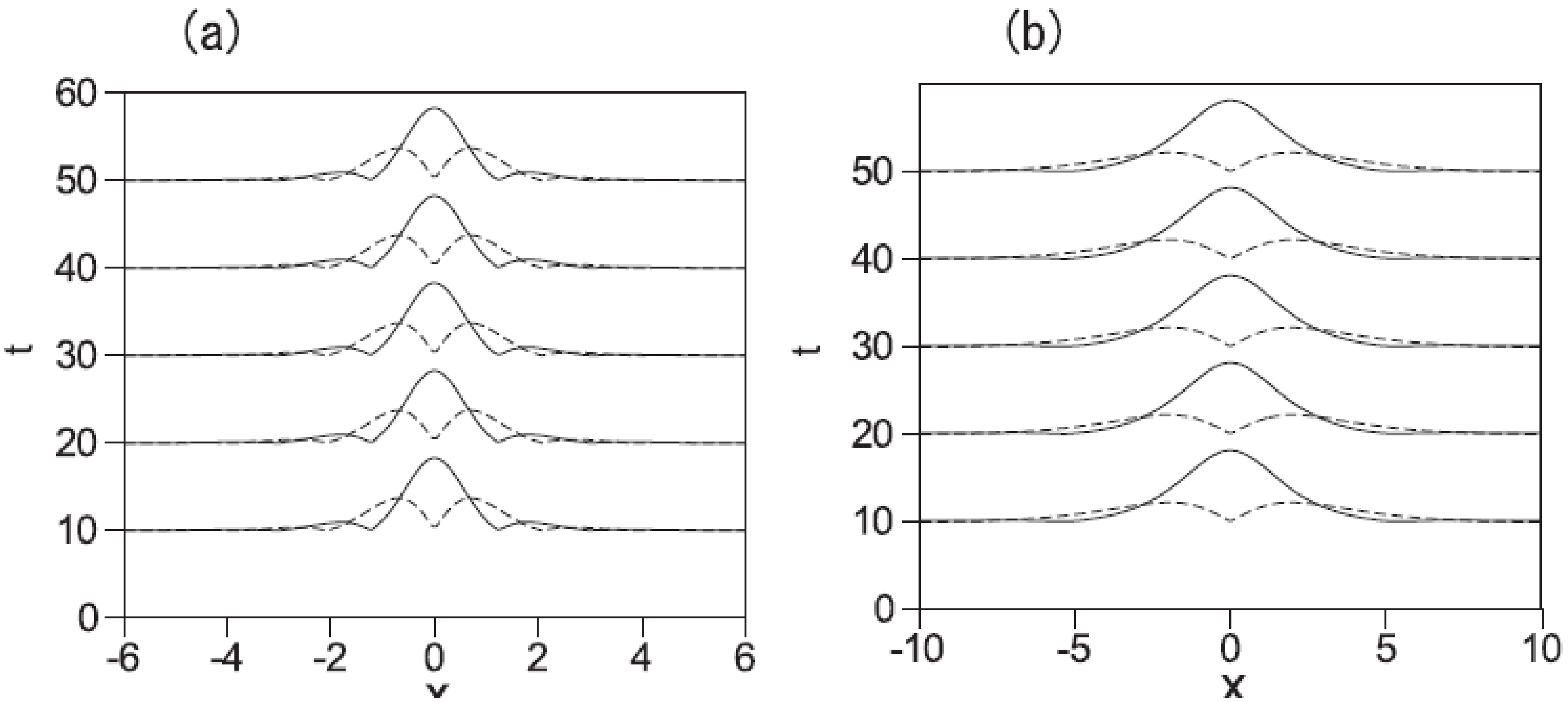


In spite of the ***supercritical collapse*** occurring in the **2D** nonlinear system in the interval of **1 < *α* < 2**, the linear **SOC-mediated** interaction between the two components makes the **SV solitons** ***stable*** at $\boldsymbol{N < N_{crit}(\alpha)}$ (and $\boldsymbol{\gamma < 1}$). Also shown is the dependence of the soliton's amplitude on the SOC strength, $\boldsymbol{\lambda}$, for a fixed Lévy index, $\boldsymbol{\alpha = 1.5}$:

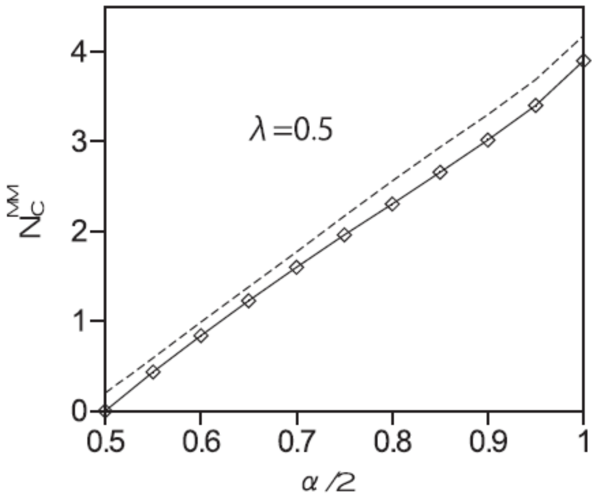


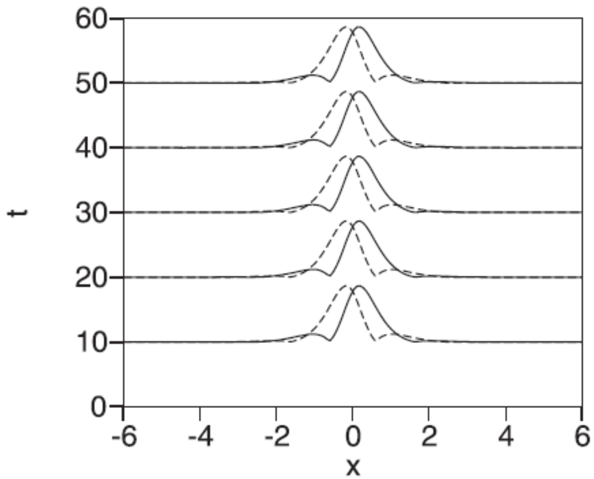


Typical examples of the evolution of **stable SV solitons** displayed by dint of their cross-sections:
(a) $\boldsymbol{\alpha = 1.5, \lambda = 1, N = 1}$; (b) $\boldsymbol{\alpha = 1.9, \lambda = 0.4, N = 5.15}$:



Next: in spite of the ***supercritical collapse*** occurring in the **2D** nonlinear system in the interval of **$1 < \alpha < 2$**, the linear **SOC-mediated** interaction between the two components makes the **MM solitons** ***stable*** at $\boldsymbol{N < N_{crit}(\alpha)}$ (and $\boldsymbol{\gamma > 1}$). Also shown is an example of the evolution of a ***stable*** **SV-soliton** for $\boldsymbol{\gamma = 2}$, **SOC** coupling strength $\boldsymbol{\lambda = 1}$, total norm $\boldsymbol{N = 0.8}$, and Lévy index $\boldsymbol{\alpha = 1.5}$:





7. Two-component solitons produced by the fractional second-harmonic-generation system

The **1D fractional** system for the amplitudes of the **fundamental-frequency** (**FF**) and **second-harmonic** (**SH**) fields with the fractional diffraction and quadratic nonlinearity (real ***Q*** is the **mismatch parameter, *** stands for the complex conjugation):

$$i\frac{\partial \Psi_1}{\partial z} - D_1\left(-\frac{\partial^2}{\partial x^2}\right)^{\alpha/2}\Psi_1 + \Psi_1^*\Psi_2 = 0,$$

$$2i\frac{\partial \Psi_2}{\partial z} - D_2\left(-\frac{\partial^2}{\partial x^2}\right)^{\alpha/2}\Psi_2 + Q\Psi_2 + \frac{1}{2}\Psi_1^2 = 0,$$

Stationary solutions to Eqs. (1) and (2) with FF and SH propagation constants β_1 and $\beta_2 \equiv 2\beta_1$ are looked for as

$$\Psi_1(x,z) = e^{i\beta_1 z}\psi_1(x), \Psi_2(x,z) = e^{2i\beta_1 z}\psi_2(x), \qquad (5)$$

The system was introduced and analyzed in

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Chaos, Solitons and Fractals

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Second-harmonic generation in the system with fractional diffraction

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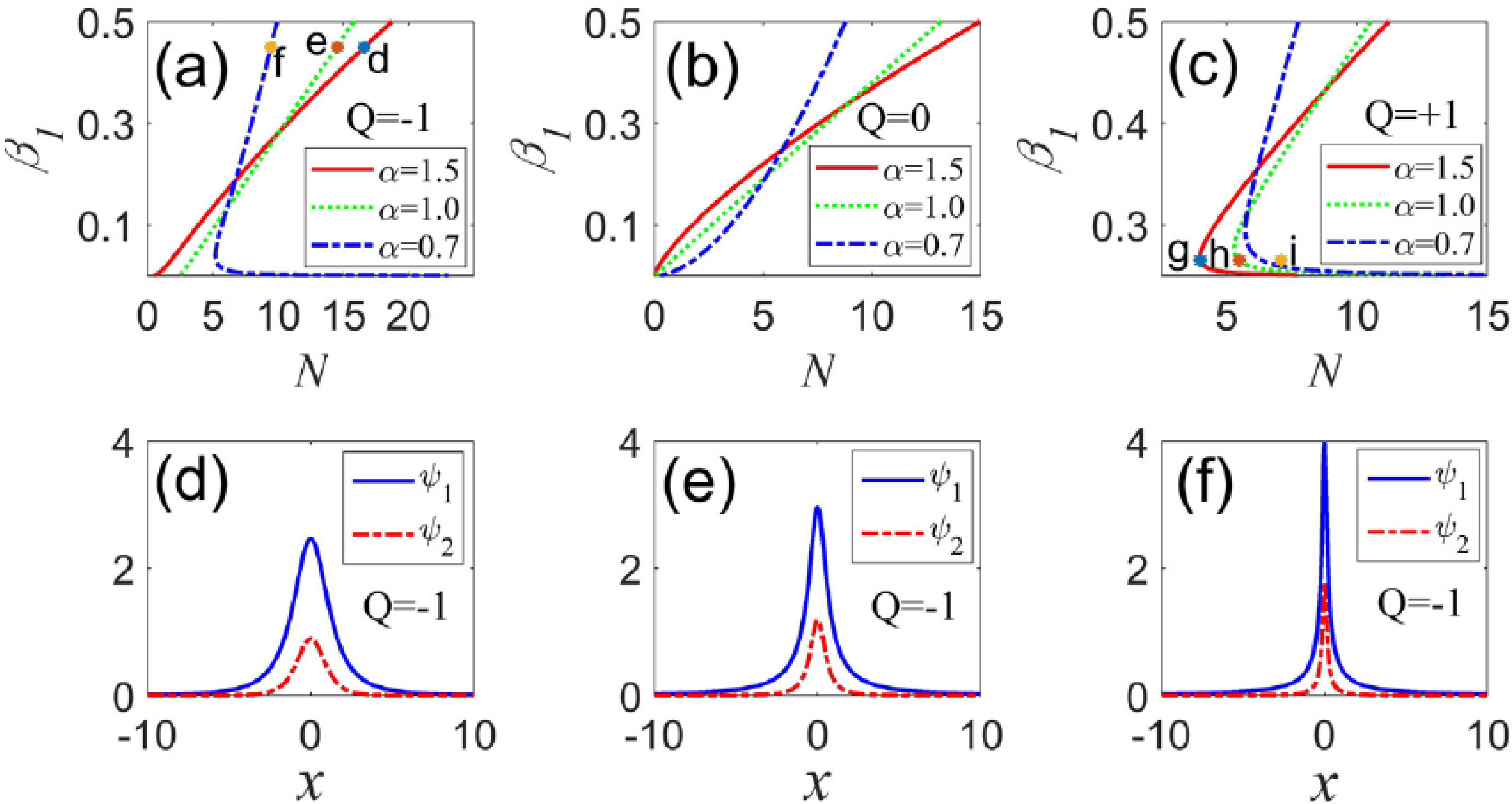
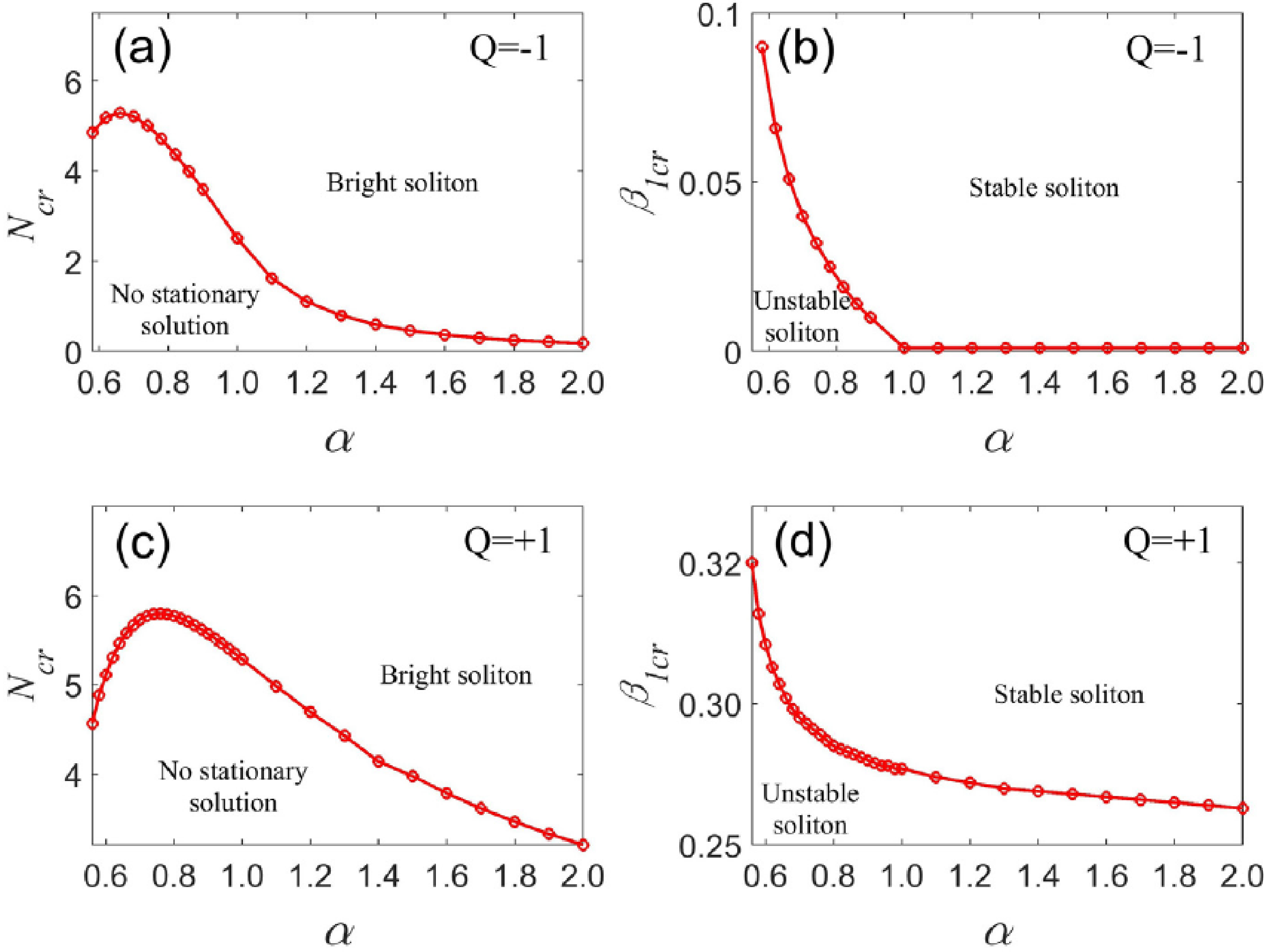
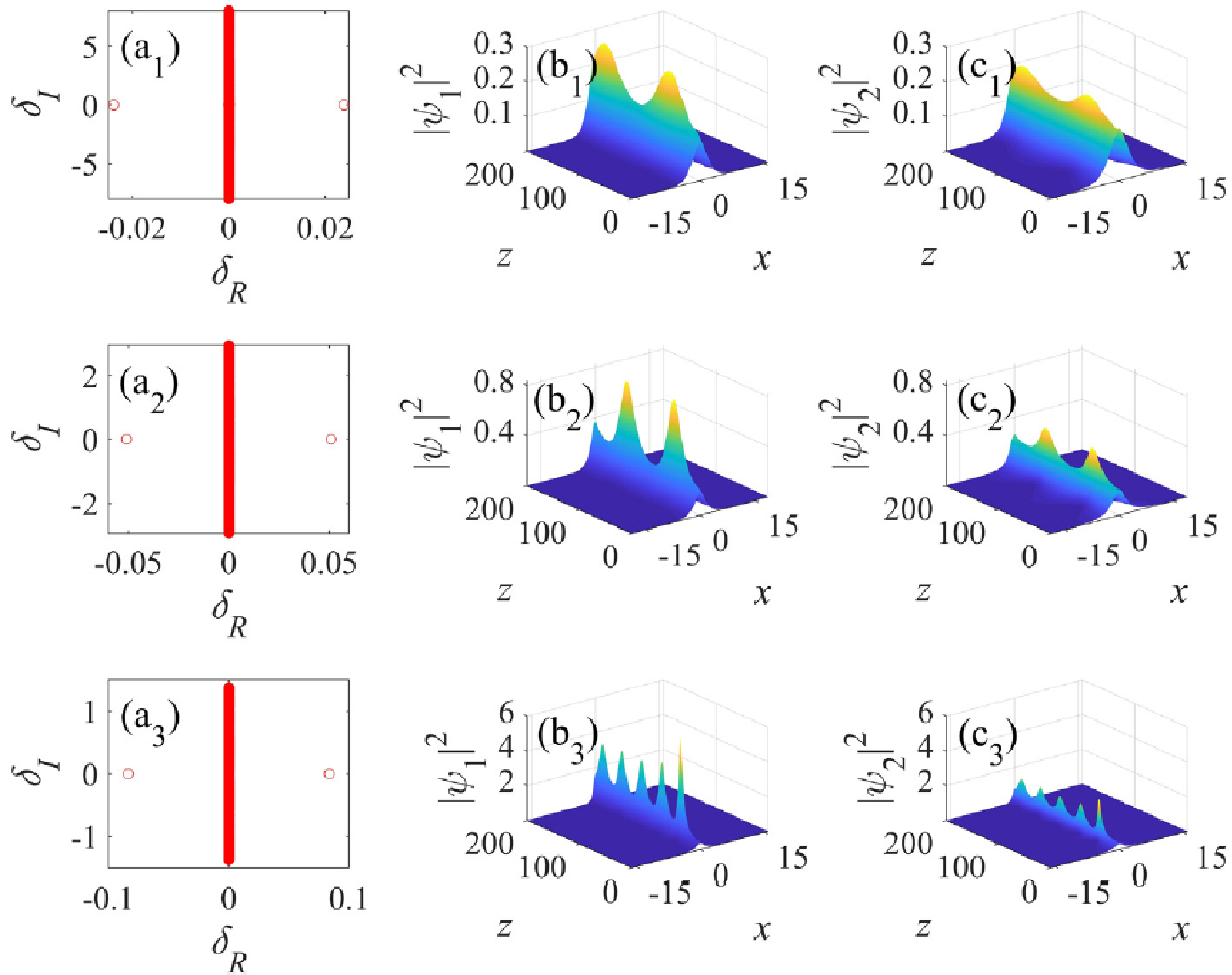
Families of soliton solutions for normalized values of the mismatch, ***Q* = -1, 0, +1**, and examples of stable solitons:



In the case of the **quadratic nonlinearity,** the **1D** system is ***free of the collapse*** in the interval of Lévy indices **0.5 < *α* ≤ 2**. ***Existence and stability*** areas for the solitons:



Examples of unstable solitons for mismatch ***Q* = 1**:



tra (a_1, a_2, a_3) and perturbed evolution of FF and SH components (b_1, b_2, b_3) and (c_1, c_2, c_3) for VK-unstable solitons with
$\beta_1 = 0.265$. Top, middle, and bottom rows correspond, respectively, to different LI values, *viz.*, $\alpha = 1.5$, $\alpha = 1.0$, and $\alpha = 0.7$.

8. Quasi-solitons in the fractional Lugiato-Lefever (LL) model

The fractional **LL** equation (the model of a ***passive driven laser cavity***), with the **Levi index LI**, **loss parameter *α*. mismatch *θ* ≡ 1**, and ***pump strength F***:

$$\frac{\partial E}{\partial t} = -\alpha E + F_{pump} + iE(|E|^2 - \theta) - \frac{i}{2}\left(-\frac{\partial^2}{\partial x^2}\right)^{LI/2} E.$$

The fractional LL model was introduced and investigated in

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Dynamics of solitons in Lugiato–Lefever cavities with fractional diffraction

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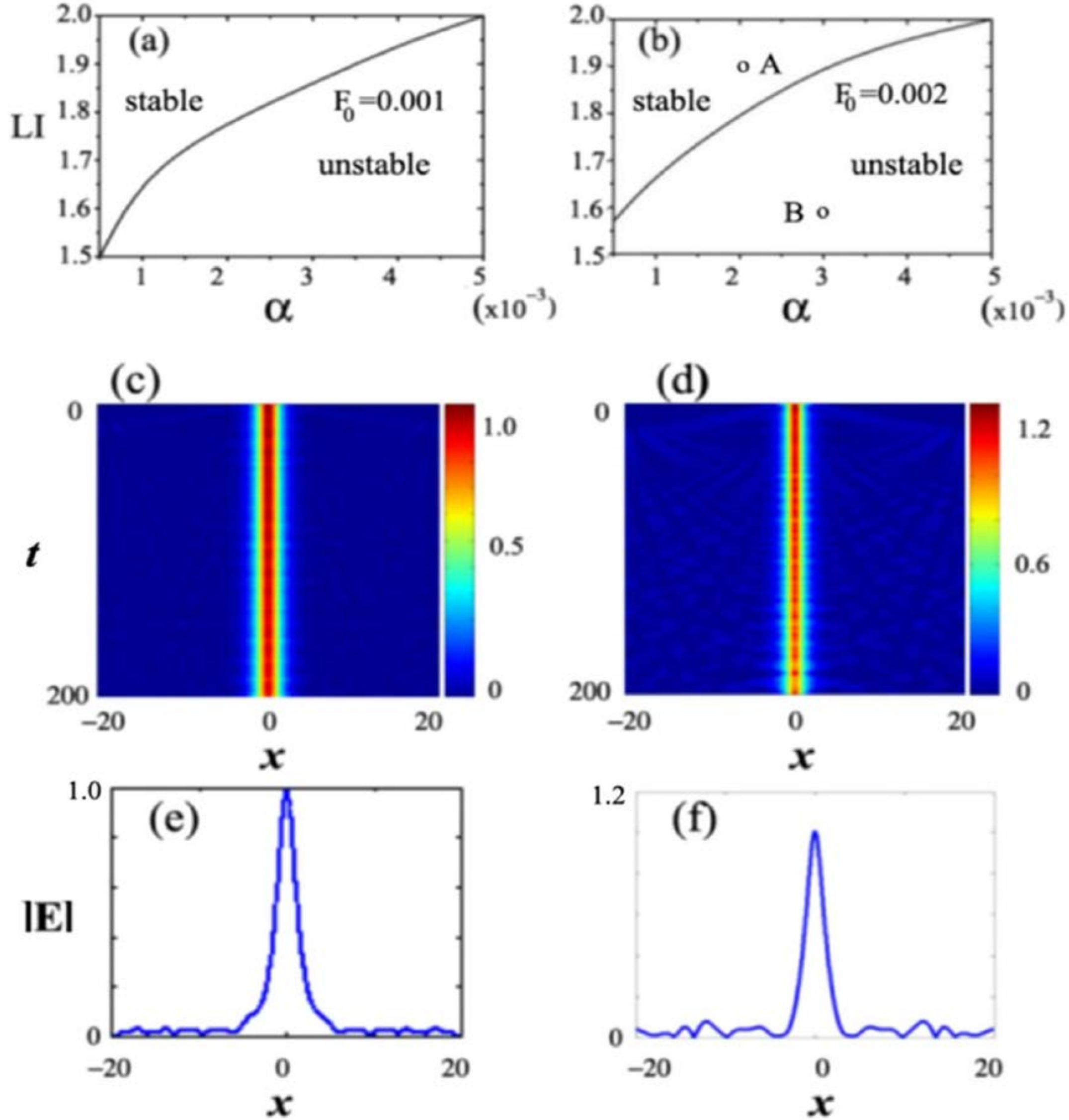
Stability regions for quasi-solitons, and examples of ***stable*** and ***weakly unstable*** ones (with the background uniform field $\boldsymbol{E \approx iF_0}$ **at** $\boldsymbol{|x| \rightarrow \infty}$:



9. Conclusion

The concept of fractional diffraction was introduced in physics by the Laskin's ***fractional quantum mechanics*** for particles which move, at the classical level, by ***Lévy flights***.

Experimental realization of fractional quantum mechanics was not reported as yet. It was proposed by Longhi to ***emulate*** the fractional quantum mechanics by the light propagation in an optical cavity, implementing the effect of the fractional diffraction by means of ***specific phase shifts*** imparted to separate spectral components of the optical beam.

A real experimental work, using a similar method – ***imparting specific phase shifts to spectral components*** of a temporal optical signal in a fiber cavity – has recently reported ***the first realization of the effective fractional group-velocity dispersion***.

Theoretically, many works have addressed ***dynamics of solitons*** and other self-trapped modes in the framework of the ***fractional nonlinear Schrödinger equation***. In particular, an attempt was made to introduce a nonlinear fractional Gross-Pitaevskii equation for a ***condensate of particles moving, at the classical level, by means of the Lévy flights***.

The remaining challenge to the experiment is realization of the effective ***fractional diffraction*** in the ***spatial domain***, i.e., for planar or bulk waveguides (linear or nonlinear), similar to the recently reported realization of the fractional group-velocity dispersion in optical fibers.

Finally, ***the most challenging objective*** may be the creation of a ***combination*** of fractional dispersion and diffraction for ***spatiotemporal optical pulses***.

Thank you for your interest!

Copies of this presentation, and/or of articles mentioned in it, can be requested from **malomed@tauex.tau.ac.il**