Transport in a one-dimensional atomic ring: the role of grey solitons

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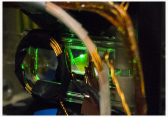
Atomtronics 2024

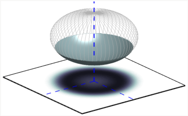
The BEC group at Villetaneuse

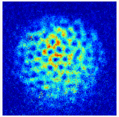
H. Perrin

http://bec.lpl.univ-paris13.fr

Bubble trap experiment — Rb

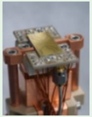


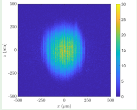


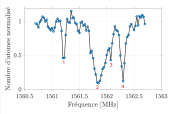


Thermal melting of a vortex lattice
arXiv:2404.05460

Atomchip experiment — Na







Fast manipulation of a quantum gas on an atom chip with a strong microwave field
arXiv:2405.07583

Main interest: study quantum gases dynamics in low dimensions

Motivations

From water waves to cold atoms

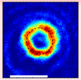
Transport in nonlinear systems

- ⇒ spectacular phenomena
- ⇒ in many systems — light in fibers, fluids, polaritons, quantum gases, ...
- ⇒ at all scales

Mascaret on Gironde, wikimedia

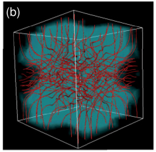
Why cold atoms ?

- ⇒ clean & tunable system
- ⇒ control of dimensionality
- ⇒ superfluid dynamics
- ⇒ easy measurements

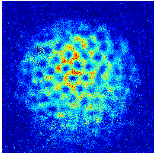


Different excitations:

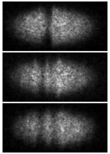
- ⇒ sound waves
- ⇒ vortex lines (3D)
- ⇒ point-like vortices (2D)
- ⇒ **solitons** (1D)



PRE **83** 066311 (2011)



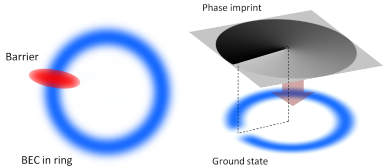
2D vortices @ LPL



PRL **83** 5198 (1999)

The link with atomtronics

The simplest atomtronics circuit:



[Kumar et al. PRA (2018)]

Main tool: nonlinear Schrödinger equation

$$i\frac{\partial\psi}{\partial t} = \left(-\frac{1}{2}\frac{\partial^2}{\partial z^2} + g|\psi|^2 + V(z,t) - \mu\right)\psi$$

- $\Rightarrow$ unwrap the ring $\quad z = \theta R$
- $\Rightarrow$ periodic boundary conditions $\quad \psi(z,t) = \psi(z+L,t)$
- $\Rightarrow$ normalization $\quad \int_0^L dz|\psi(z,t)|^2 = N$
- $\Rightarrow$ convenient units $\quad \hbar = M = 1$

P.Pedri, J. Polo, A. Minguzzi, M. Olshanii

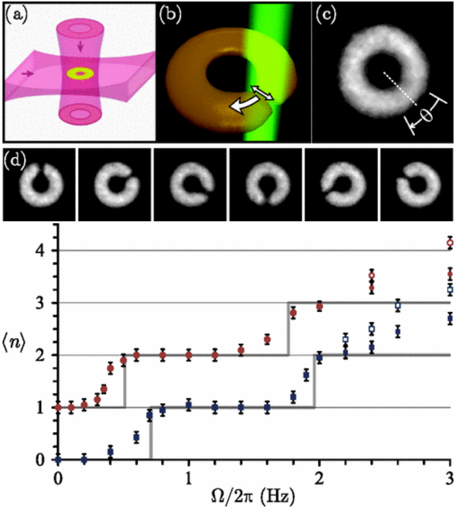
Investigate the decay mechanisms of a supercurrent.

Note: many related atomtronic topics, guided matter waves, bright solitons, ring lattices, ...

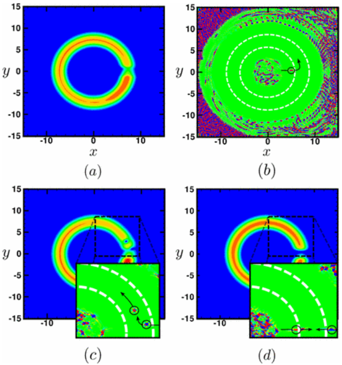
AVS Quantum Science **3** 039201 (2021)

What is known in higher dimensions

Key point: role of vortices



[PRL **110** 025302 (2013)]

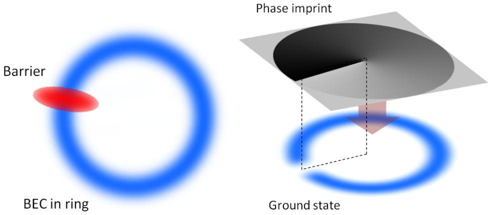


[PRA **80** 021601(R) 2009]

Phase-slips mediated by vortices crossing the weak link

What happens in 1D ? (no vortices)

Bosons on a 1D ring with a barrier: life & death of a super-current
PRL **123** 195301 (2019) & PRR **3** 013098 (2021)



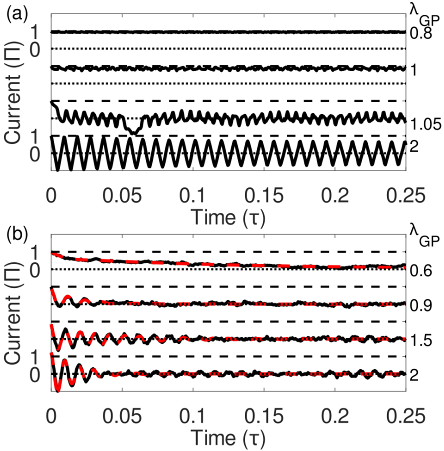
$$i\frac{\partial\psi}{\partial t} = \left(-\frac{1}{2}\frac{\partial^2}{\partial z^2} + g|\psi|^2 + V(z,t) - \mu\right)\psi$$

Decay of a super-current

Numerical solution of the 1D non-linear Schrödinger equation

LPL Laboratoire de Physique des Lasers

$$\Pi = -\frac{i}{N}\int_0^L dz\,\psi(z,t)^*\frac{\partial}{\partial z}\psi(z,t)$$



[Polo et al. PRL (2019)]

Zero-temperature

GPE simulation

- ⇒ weak barrier current is stable
- ⇒ critical behavior — dual of the Bose-Josephson model
- ⇒ large barrier: triangular oscillations

Finite-temperature $T \simeq \mu$

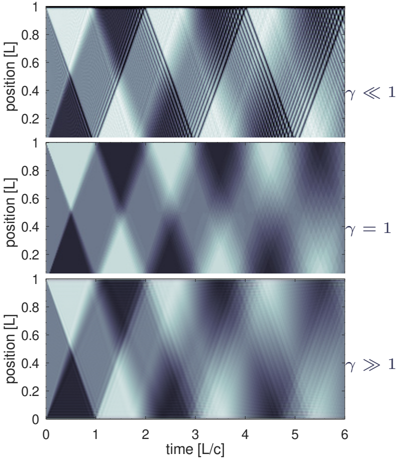
many realizations sampling thermal equilibrium

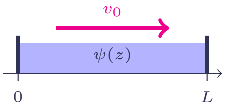
- ⇒ exponential damping
- ⇒ cross-over
- ⇒ damped oscillations

- ⇒ weak barrier: phase-slips mediated by solitons
- ⇒ large barrier: dispersive shock waves dynamics

A universal behavior for large barrier (impenetrable)

Simulation with **hard-wall boundaries**: gaz in a box





Density dynamics is **universal**

[Dubessy et al. PRR 2021]

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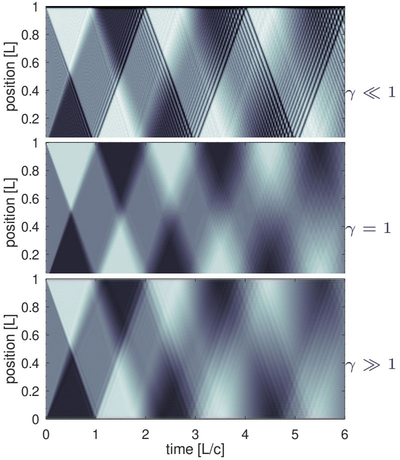

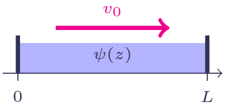

Density dynamics is **universal**, provided that:
times and velocities are scaled by the speed of sound.

$\Rightarrow c_{\mathrm{GP}} = \sqrt{\mu/M}$ $\quad \gamma \ll 1$

$\Rightarrow c_{\mathrm{GHD}} \to$ Lieb-Liniger sound $c(\gamma)$ $\quad$ all γ

$\Rightarrow c_{\mathrm{TG}} = \frac{\hbar \pi N}{ML}\sqrt{1 + \frac{3}{2N}}$ $\quad \gamma \gg 1$

initial velocity: $v_0 = 0.1 \times c(\gamma)$

[Dubessy et al. PRR 2021]

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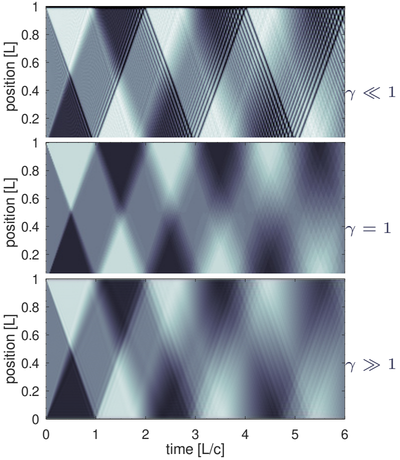
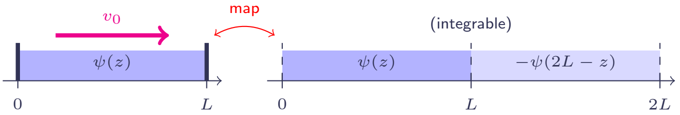


[Dubessy et al. PRR 2021]



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initial velocity: $v_0 = 0.1 \times c(\gamma)$

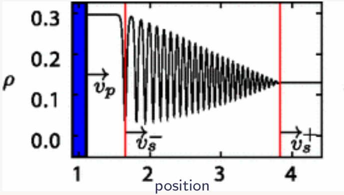
gas in a box $\Leftrightarrow$ anti-symmetric state on a ring
the barrier does not play a role in the damping !
dispersive shock waves for all γ

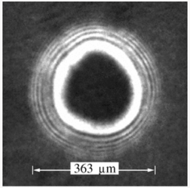
Connexion with other scenarii

For a complete review: *Dispersive shock waves and modulation theory*,

El and Hoefer Physica D **333** (2016)

The piston problem



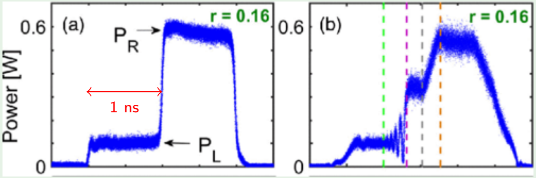


BEC experiment @ JILA

PRL **100** 084504 (2008)

PRA **74** 023623 (2006)

The dam break problem



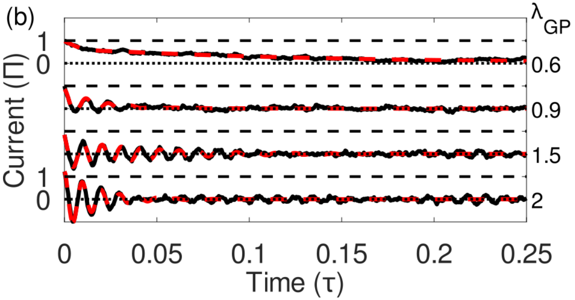
Light pulse in nonlinear fiber

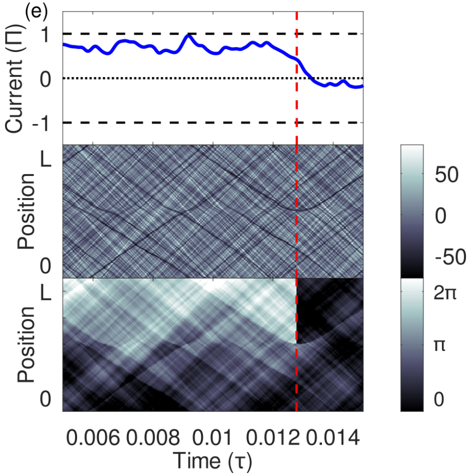
PRL **118** 254101 (2017)

Ring & barrier: observe dispersive shock waves for long times

Weak barrier regime

Evidencing the phase-slip mechanism





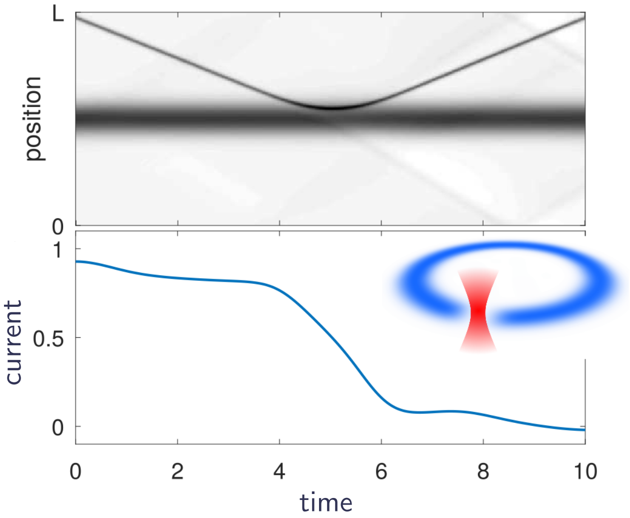
Exponential decay

- ⇒ average of many individual phase-slips
- ⇒ phase-slip ⇔ soliton reflection on the barrier
- ⇒ barrier slows down solitons

and enables nucleation of new solitons

[Polo et al. PRL (2019)]

Deterministic phase slip



Initial state with a phase winding of $+1$ and a single soliton.

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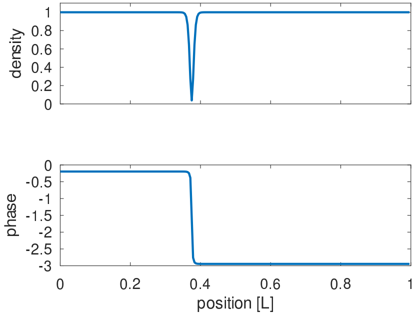
Grey solitons and the direct scattering transform in the nonlinear Schrödinger equation

arXiv:2210.09812v2

Reminder: what is a grey soliton ?

A special solution of a non-linear equation

$$i\frac{\partial\psi}{\partial t} = \left(-\frac{1}{2}\frac{\partial^2}{\partial z^2} + g|\psi|^2 - \mu\right)\psi$$



Energy balance

quantum pressure ⇔ interaction energy

or: diffraction ⇔ dispersion (waves)

Single soliton solution:

$$\psi(z,t) = \sqrt{n_0}\left(\cos\phi \tanh\left[\cos\phi\sqrt{gn_0}(z-\bar{z}(t))\right] + i\sin\phi\right)$$

traveling at a constant velocity: $v = \dot{\bar{z}}(t) = \sin\phi\sqrt{gn_0}$ recall: $c \equiv \sqrt{gn_0}$

[for a review see J. Phys. A **43** 213001 (2010), proper solution on a ring: PRA **62** 063610 (2000)]

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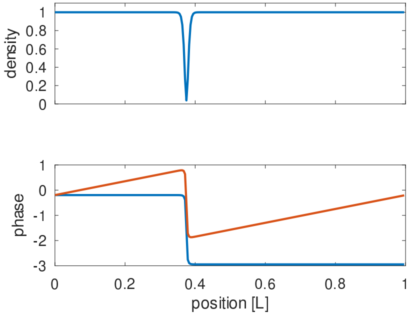

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$$\psi(z,t) = \sqrt{n_0}\left(\cos\phi\tanh\left[\cos\phi\sqrt{gn_0}(z-\bar{z}(t))\right] + i\sin\phi\right) \times e^{i(k_0 z - \omega t)}$$

traveling at a constant velocity: $v = \dot{\bar{z}}(t) = \sin\phi\sqrt{gn_0} + k_0$ recall: $c \equiv \sqrt{gn_0}$

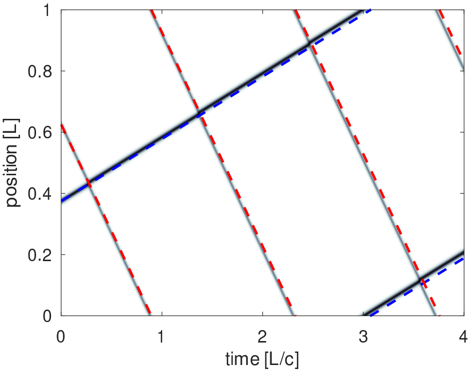
on a ring the background flow compensates the phase jump: $k_0 L = \pi - 2\phi\ [2\pi]$

$\omega = k_0^2/2$

[for a review see J. Phys. A **43** 213001 (2010), proper solution on a ring: PRA **62** 063610 (2000)]

Key properties

Solitons are independent



they cross each other

without changing shape.

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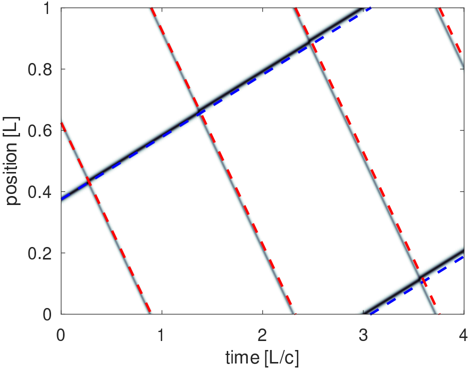


they cross each other

without changing shape.

Key properties:

- ⇒ on a flat background the soliton is completely determined by its velocity (or phase).
- ⇒ grey solitons interact by repelling each other

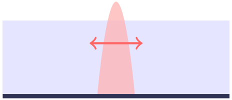
$$|\delta z_i| = \left| \frac{1}{2\cos\phi_i\sqrt{gn_0}} \ln \frac{1+\cos(\phi_j+\phi_j)}{1-\cos(\phi_i-\phi_j)} \right|$$

⇒ N-soliton interaction ⇔ pairwise interactions

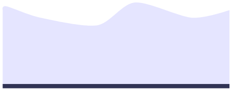
[Zakharov & Shabat 1973]

How to characterize a many-soliton state ?

groundstate

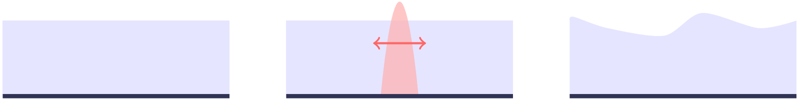


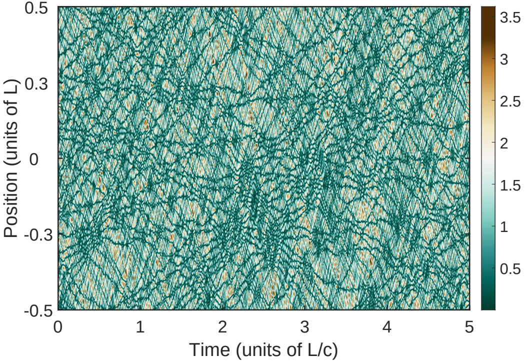
time-dependent perturbation



excited integrable system

How to characterize a many-soliton state ?





arXiv:2210.09812v2

Initial state far from equilibrium:

$\Rightarrow$ complex density dynamics

$\Rightarrow$ many propagating features

$\Rightarrow$ how to characterize this state ?

$\Rightarrow$ conserved quantities ?

(beyond E, N, P)

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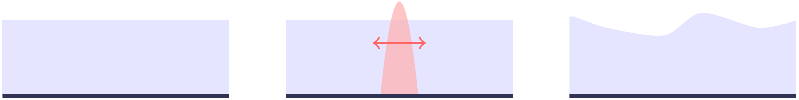
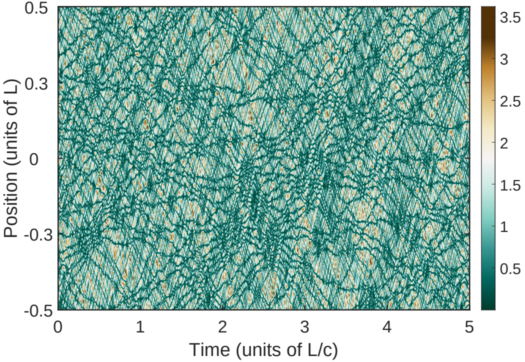


groundstate — time-dependent perturbation — excited integrable system



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Initial state far from equilibrium:

$\Rightarrow$ complex density dynamics

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$\Rightarrow$ how to characterize this state ?

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(beyond E, N, P)

Use the tools of

the inverse scattering transform

inspired by PRL **125** 264101 (2020)

The Lax spectrum

The homogeneous 1D non-linear Schrodinger equation is integrable:

a Lax pair of operators $\mathcal{L}$ and $\mathcal{A}$ exists such that

$$i\frac{\partial\psi}{\partial t} = \left(-\frac{1}{2}\frac{\partial^2}{\partial z^2} + g|\psi|^2\right)\psi \;\Leftrightarrow\; i\frac{\partial\mathcal{L}}{\partial t} = [\mathcal{A},\mathcal{L}]$$

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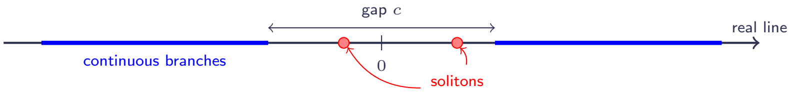
For the **repulsive** case $(g > 0)$, $\mathcal{L}$ is Hermitian:

$$\mathcal{L} = \frac{i}{2}\begin{pmatrix} \frac{\partial}{\partial z} & -\sqrt{g}\psi \\ \sqrt{g}\psi^* & -\frac{\partial}{\partial z} \end{pmatrix}$$

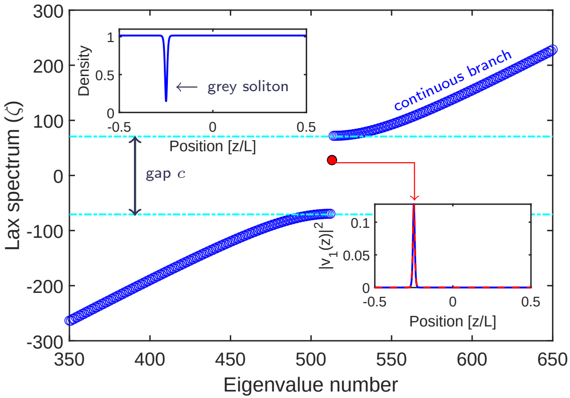
and its spectrum $(\mathcal{L}v = \zeta v)$ is time-independent.



See for example "Solitons and the IST" by Ablowitz and Segur

A simple example

The single soliton solution



Solve the direct scattering transform:

$$\mathcal{L}v = \zeta v, \ v \equiv \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

- $\Rightarrow$ method works — numerics vs analytical solution
- $\Rightarrow$ $\zeta_s = -\frac{v_s}{2} - \frac{k_0}{4}$ — soliton velocity
- $\Rightarrow$ discrete eigenvalue
  - $\Rightarrow$ localized state — *inside* the density dip

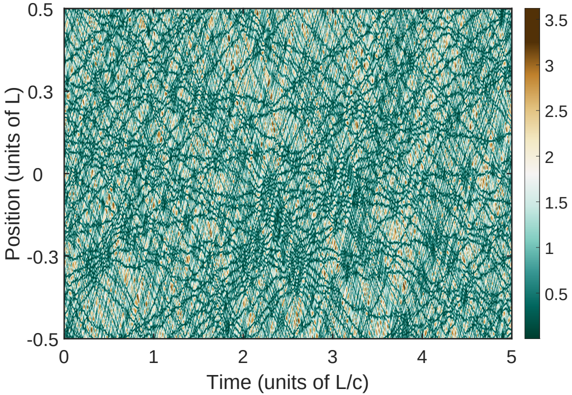
Remark: numerical solution on a grid $\Rightarrow$ $\mathcal{L}$ is a matrix operator, **discrete** spectrum

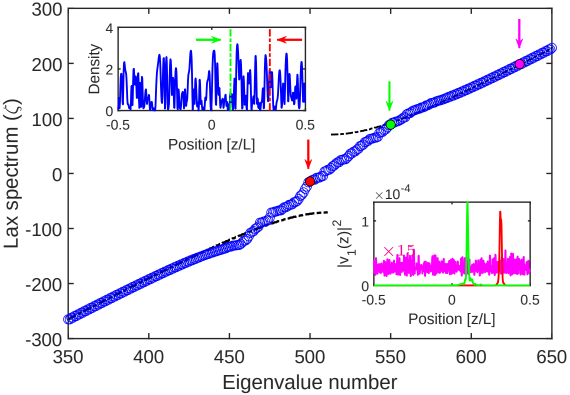
Challenge: how to identify the continuous spectrum ?

Note: simpler for the attractive case, for a recent application see [PRL **125** 264101 (2020)]

Application to a very excited state

Which eigenvalues are “discrete” ?





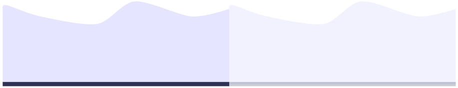
⇒ gap is *filled* with **discrete** eigenvalues

⇒ identification: use state *localization* ?

⇒ better criterion : study eigenvalue degeneracy

continuous transition from small soliton to sound wave

Our contribution
arXiv:2210.09812v2



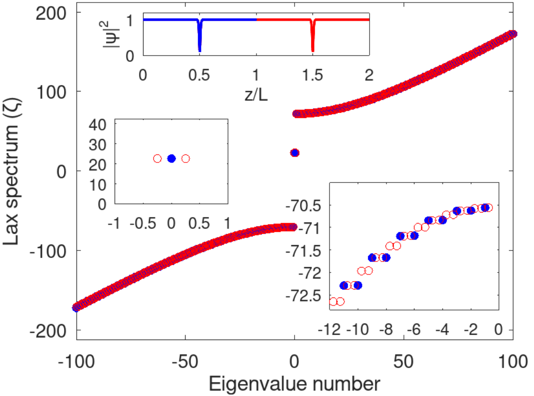
excited integrable system | system "copy"

Eigenvalue distribution

- ⇒ study the **Lax spectrum**
- ⇒ compare with a *double system*

Conjecture: each soliton will appear twice,
continuous spectrum is denser

A.K Saha



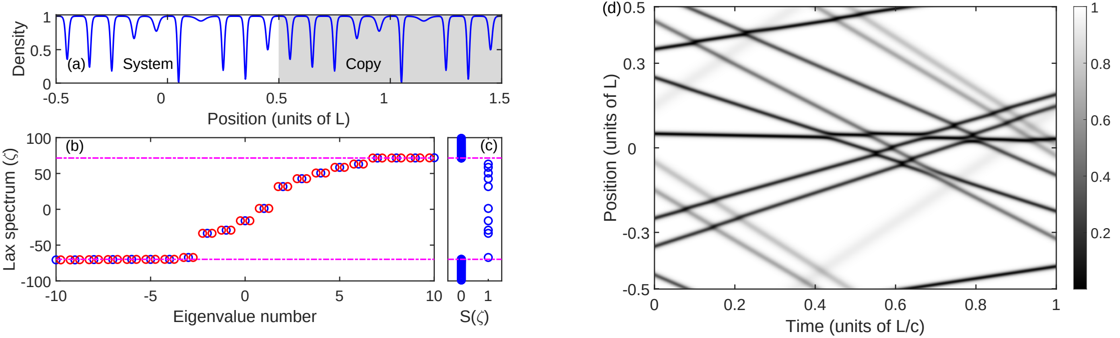
Define **solitons** as Lax eigenvalues with double degeneracy in the extended system

Systematic benchmark

Imprint a N-soliton state with random phases:

$$\psi_N(z,t=0) = \sqrt{n_0} e^{ik_{\text{tot}} z} \prod_{j=1}^{N} \left(\cos\phi_j \tanh\left[\sqrt{g n_0} \cos\phi_j (z - z_j)\right] + i \sin\phi_j\right)$$

(valid if solitons do not overlap)



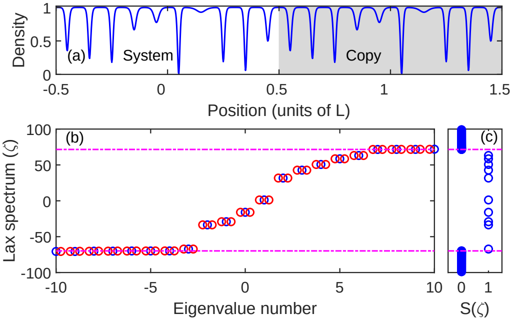
Define a soliton **indicator** $S(\zeta)$

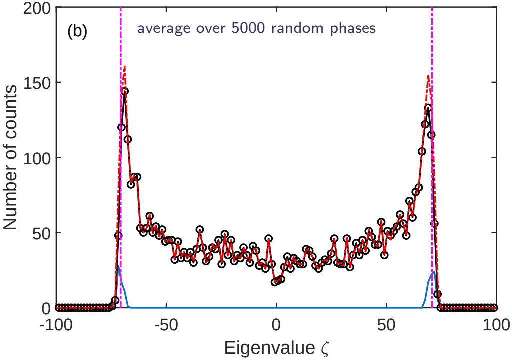
Systematic benchmark

Imprint a N-soliton state with random phases:

$$\psi_N(z,t=0) = \sqrt{n_0}e^{ik_{\rm tot}z}\prod_{j=1}^{N}\left(\cos\phi_j \tanh\left[\sqrt{gn_0}\cos\phi_j(z-z_j)\right] + i\sin\phi_j\right)$$

(valid if solitons do not overlap)





Define a soliton **indicator** $S(\zeta)$

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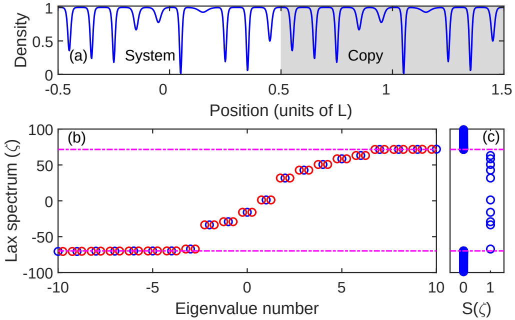
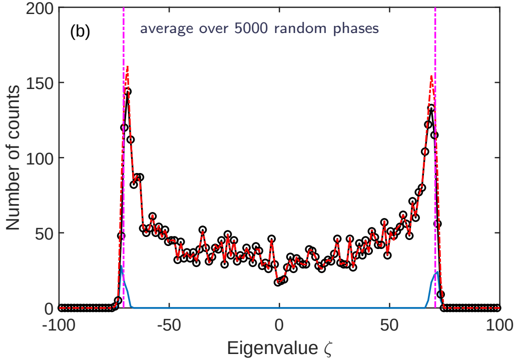
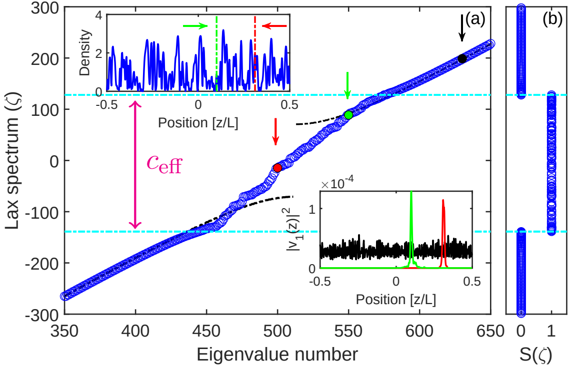




Define a soliton **indicator** $S(\zeta)$

evaluate $\bar{N}_{\text{sol}} \pm \delta N_{\text{sol}}$, $\bar{c}_{\text{eff}} \pm \delta c_{\text{eff}}$, ... in *single* realizations



Efficient method to study solitons

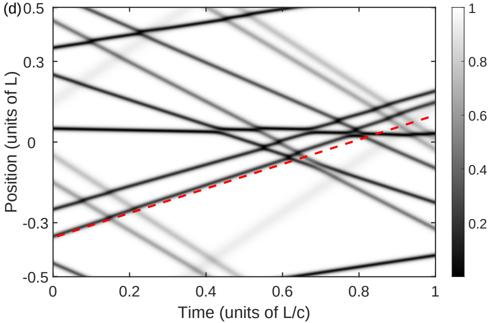
Main results

- ⇒ distinguish the discrete / continuous spectra
- ⇒ count the **soliton number**
- ⇒ the soliton's velocity distribution

Conjectures

- ⇒ the gap size gives the effective speed of sound c_{eff}
- ⇒ the highest peak in the eigenvector modulus gives the soliton position

Emerging hydrodynamics: the soliton gas picture



Simplified recipe

$\Rightarrow$ $\rho(v,z)dvdz$: number of solitons in $[z, z+dz] \times [v, v+dv]$

$\Rightarrow$ find the **effective** soliton velocity:

$$v_{\text{eff}} = v + \int dv' \Delta(v,v')\rho(v',z) \times [v_{\text{eff}}(v) - v_{\text{eff}}(v')]$$

$\Delta(v,v')$ collision induced shift

$\Rightarrow$ write a continuity equation:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v_{\text{eff}})}{\partial z} = 0$$

Lax spectrum $\{\zeta\} \Rightarrow \rho(v,z)$ at $t=0$.

Crucial point: integrable system ($\{v\}$ is conserved)

Classical mechanics J. Stat. Phys. **31** 577 (1983)
Lieb-Liniger model PRL **117** 207201 (2016) & PRX **6** 041065 (2016)
Bright soliton gases PRL **120** 045301 (2018)
Grey soliton gases PRE **103** 042201 (2021)

Conclusion

Take home messages

Box-trap to study 1D shock-waves

$\Rightarrow$ long wavelength $c(\gamma)$ controls propagation

$\Rightarrow$ even for non perturbative quenches

simple for periodic or hard-wall boundary conditions

Solitons are responsible for phase-slips in 1D

$\Rightarrow$ phase slips $\Leftrightarrow$ soliton reflection

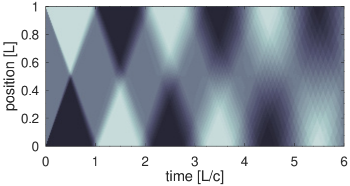
$\Rightarrow$ competition with dispersive effects

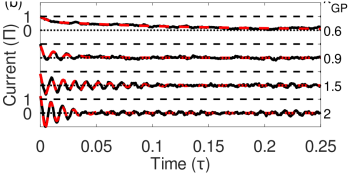
Direct scattering transform (Lax spectrum)

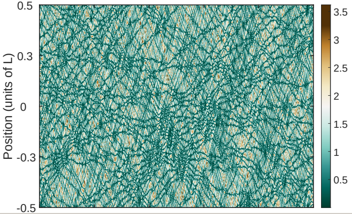
$\Rightarrow$ solitons $\Leftrightarrow$ localized eigenstates

$\Rightarrow$ quantitative measure of velocities

$\Rightarrow$ measure the speed of sound far from equilibrium





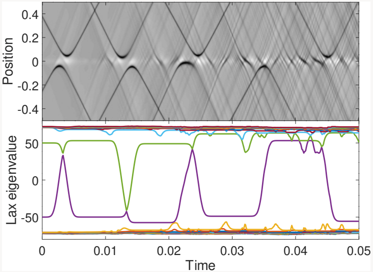


Perspectives

LPL
Laboratoire de Physique des Lasers

Integrability breaking

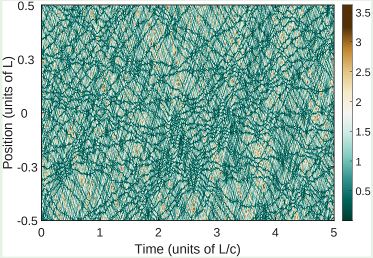
dynamics with a (weak) barrier



Use the Lax spectrum to:

- ⇒ evidence the phase-slips
- ⇒ detect soliton nucleation
- ⇒ study **relaxation**

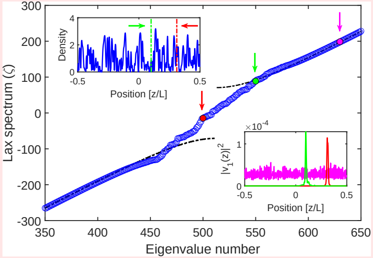
Classification of states



How far is a excited state from a thermal state ? (or a turbulent flow ?)

⇒ distribution of solitons (Lax eigenvalues)

Soliton gases



Soliton as a particle:

$$\frac{\partial \rho_s}{\partial t} + \frac{\partial(\rho_s v_{\rm eff}(\zeta))}{\partial z} = 0$$

hydrodynamic approach

Thank you for your attention !