Hall Effect in Atomic Ladder Systems

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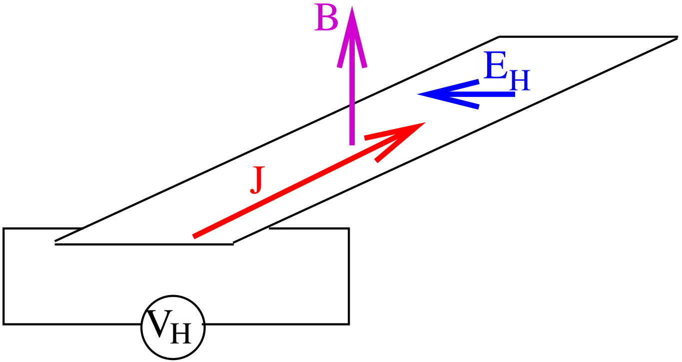
Hall effect



$$V_H = R_H B J$$

Non interacting 2D fermions

Single band, relaxation time approximation

$$\frac{d\vec{p}}{dt} = -e(\vec{E} + \vec{v} \times \vec{B}) - \gamma\vec{v} = \vec{0}$$

$$\vec{v} = \frac{d\epsilon}{d\vec{p}}$$

$$\vec{J} = -ne\vec{v}$$

$$\begin{pmatrix} E_x \\ E_y \end{pmatrix} = \begin{pmatrix} \frac{\gamma}{ne^2} & \frac{B}{ne} \\ -\frac{B}{ne} & \frac{\gamma}{ne^2} \end{pmatrix} \begin{pmatrix} J_x \\ J_y \end{pmatrix}$$

$$R_H = \frac{E_y}{J_x B} = -\frac{1}{ne}$$

Hall effect with interacting 2D fermions or bosons

Linear response theory

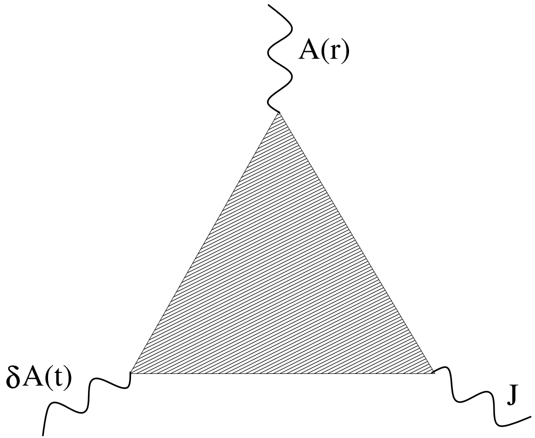
$$\vec{B} = \vec{\nabla} \times \vec{A}(\vec{r}), \quad \vec{A}(\vec{r},t) = \vec{A}(\vec{r}) + \delta\vec{A}(t)$$

$$H = H(\delta\vec{A}(t) = \vec{0}) - \vec{J}\cdot\delta\vec{A}(t) + \frac{1}{2}\sum_{\alpha} D_{\alpha}(\delta A_{\alpha})(t)^2 + o(\delta\vec{A}^2)$$

$$\sigma_{\alpha\beta} = \lim_{\omega\to 0}\int_0^{+\infty} e^{i(\omega+0_+)t}\,\frac{\langle[J_\alpha(t), J_\beta(0)]\rangle - \mathbb{D}_\alpha\delta_{\alpha\beta}}{i\omega}$$

$$\rho_{\alpha\beta} = (\sigma^{-1})_{\alpha\beta}, \quad R_H = \frac{\rho_{xy}}{B}$$

Low field expansion: triangle Feynman diagrams



H. Fukuyama et al. Prog. Theor. Phys. **42**, 494 (1969)

Lorentz transformation [S. M. Girvin, les Houches (1998)]

Frame moving with the current

$$
\begin{aligned}
v_x &= \frac{J_x}{ne} \\
A_y &= -B(x - v_x t) + O(v_x^2/c^2) \\
E_y &= -v_x B = -\frac{BJ_x}{ne} \\
&\Rightarrow R_H = -\frac{1}{ne}
\end{aligned}
$$

Hall effect and artificial gauge fields

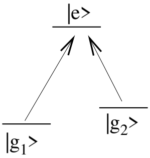
Geometric phases [Dalibard et al. Rev. Mod. Phys. 83, 1523 (2011)]

$$|\Psi\rangle = \psi(r)|D(\vec{r})\rangle = \begin{pmatrix} \cos\theta(\vec{r}) \\ \sin\theta(\vec{r})e^{i\varphi(\vec{r})} \end{pmatrix} \psi(\vec{r})$$

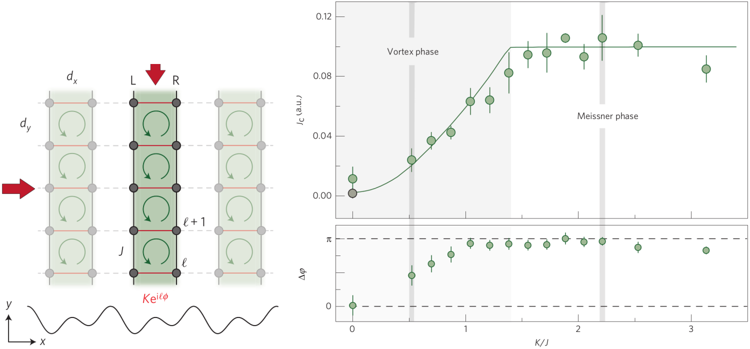
Berry connection:

$$\begin{aligned} \vec{A}(\vec{r}) &= \langle D(\vec{r})|\frac{\hbar}{i}\vec{\nabla}[|D(\vec{r})\rangle] \\ &= \hbar \sin^2\theta(\vec{r})\vec{\nabla}\varphi \end{aligned}$$

$|D\rangle$ is a dark state.

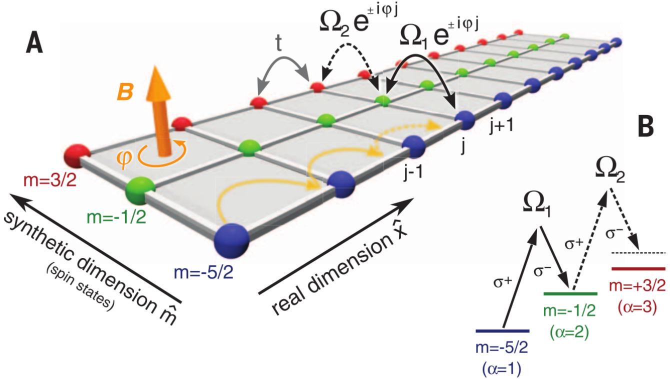


Bosonic ladder in a flux



From Atala et al. Nat. Phys. **10**, 588 (2014)

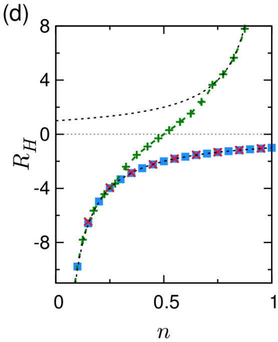
Fermionic ladder in artificial dimension



Mancini et al. Science **349**, 1510 (2015)

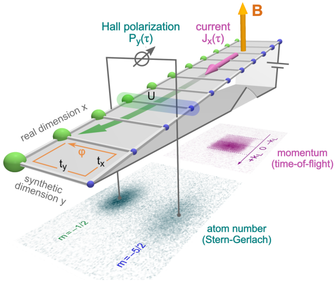
Hall effect in the two leg ladder geometry

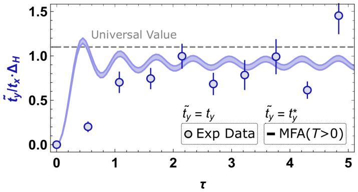
$R_H = -1/n$ dotted line
Fermions: green
Bosons: blue and purple



Greschner et al.
Phys. Rev. Lett. **122**, 083402 (2019)

Hall imbalance in a ladder system





$$\Delta_H = \frac{P_y}{J_x} = 2\frac{t_x}{t_y}\left|\tan\left(\frac{\varphi}{2}\right)\right| \quad (U \gg t_x)$$

Zhou et al. Science **381**, 427 (2023)

Tomonaga-Luttinger liquid theory (I)

Single chain of interacting bosons

$$
\begin{aligned}
H &= \sum_j -t(b_j^\dagger b_{j+1} + b_{j+1}^\dagger b_j) + \frac{U}{2} n_j(n_j - 1) \\
H_{TLL} &= \int \frac{dx}{2\pi} \left[uK(\nabla\theta)^2 + \frac{u}{K}(\nabla\phi)^2 \right] \\
\frac{n_j}{a} &= \rho_0 - \frac{\partial_x \phi}{\pi} + \sum_{m \neq 0} A_m e^{i2m(\phi(x) - \pi\rho_0 x)} \\
b_j &= e^{i\theta(x)} \sum_m B_m e^{i2m(\phi(x) - \pi\rho_0 x)}
\end{aligned}
$$

$[\phi(x), \nabla\theta(y)] = i\pi\delta(x - y)(x = ja)$

Tomonaga-Luttinger theory (II)

Non-perturbative definition of Tomonaga-Luttinger parameters

$$\mathcal{D} = uK$$
$$\chi = K/(\pi\rho_0^2 u)$$

$\mathcal{D}$ is the stiffness, $J_x = \mathcal{D}\varphi$ in the AB flux φ.
$\mathcal{D}$ interaction independent if Galilean invariant.
χ is the compressibility
F. D. M. Haldane Phys. Rev. Lett. **47**, 1840 (1981).

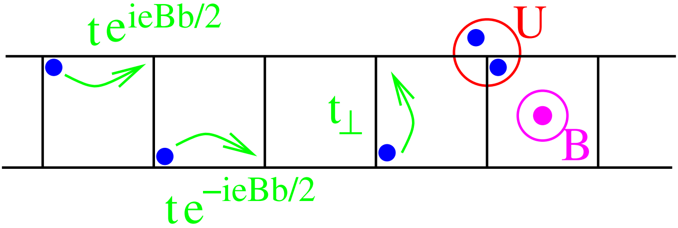
Tomonaga-Luttinger liquid theory (III)

Band curvature corrections [Matveev JETP 117, 508 (2013)]

$$
\begin{aligned}
H_{\mathrm{curv.}} &= \int dx \left[\alpha(\partial_x\phi)^3 + \gamma(\nabla\theta)^2\partial_x\phi\right] \\
\alpha &= -\frac{\partial}{\partial\rho_0}\left(\frac{u}{6\pi^2 K}\right) \\
\gamma &= -\frac{\partial}{\partial\rho_0}\left(\frac{uK}{2\pi^2}\right).
\end{aligned}
$$

$\Rightarrow$ irrelevant in the Renormalization group sense
But present unless there is particle-hole symmetry
$(\phi \to -\phi, \theta \to -\theta)$.
Galilean invariance $\Rightarrow \gamma$ interaction independent.

Two-leg bosonic ladder in flux



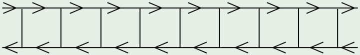
Bosonic ladder in flux

Bosonized Hamiltonian

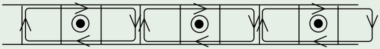
$$\begin{aligned}
H &= \int \frac{dx}{2\pi}\left[uK(\nabla\theta_s)^2 + \frac{u}{K}(\partial_x\phi_s)^2\right] \\
&+ \int \frac{dx}{2\pi}\left[uK\left(\nabla\theta_a - \frac{eBa}{\sqrt{2}}\right)^2 + \frac{u}{K}(\partial_x\phi_a)^2 - \frac{2t_\perp}{a_0}\cos(\sqrt{2}\theta_a)\right] \\
&+ \int \frac{dx}{\sqrt{2}}\left[\gamma\partial_x\phi_s(\nabla\theta_s)^2 + \alpha(\partial_x\phi_s)^3\right] \\
&+ \int \frac{dx}{\sqrt{2}}\left\{\partial_x\phi_s\left[\gamma\left(\nabla\theta_a - \frac{eBa}{\sqrt{2}}\right)^2 + 3\alpha(\partial_x\phi_a)^2\right]\right. \\
&\left. + 2\gamma\left(\nabla\theta_a - \frac{eBa}{\sqrt{2}}\right)(\nabla\theta_s)\partial_x\phi_a\right\}.
\end{aligned}$$

Phase diagram

Commensurate-incommensurate transition



(a)



(b)

(a) $B < B_c$ Meissner phase
$\langle \theta_a \rangle = 0$, gap
(b) $B > B_c$ Vortex phase
$\langle \theta_a \rangle = m(B)x$, gapless

Hall imbalance

First order perturbation theory in α, γ.

$$
\begin{aligned}
P_H &= -\frac{\sqrt{2}}{\pi}\int_{-\infty}^{+\infty} \partial_x \phi_a \\
&= \frac{2\gamma L \langle \nabla \theta_s \rangle}{\pi} \int d\tau dx \left\langle T_\tau \left[(\nabla \theta_a - eBa/\sqrt{2}) \partial_x \phi_a \right] (x, \tau) \right. \\
&\quad \times \partial_x \phi_a (0,0) \rangle + O(\gamma^2)
\end{aligned}
$$

In Meissner phase

Hall imbalance

Using symmetry $(\theta_a, \phi_a) \to (-\theta_a, -\phi_a)$ gives

$$\langle P_H \rangle^{(1)} = \frac{\pi B a \gamma \langle j_s \rangle}{uK} \chi_{\rho_a \rho_a}(q = 0, \omega_n = 0)$$

$$\Rightarrow \frac{\langle P_H \rangle^{(1)}}{BL\langle j_s \rangle} = \frac{a\gamma}{u^2}$$

Hall voltage is the logarithmic derivative of stiffness

$$V_H = -\frac{\pi^2 \gamma \langle j_s \rangle}{euK} Ba = -\langle j_s \rangle \frac{Ba}{2e} \frac{1}{uK} \frac{\partial}{\partial \rho_0}(uK)$$

With Galilean invariance

Phase stiffness independent of interaction

$$uK = \frac{\pi \rho_0}{m}$$

$$R_H = -\frac{1}{e}\frac{\partial[\ln(uK)]}{\partial \rho_0} = -\frac{1}{e\rho_0}$$

$\Rightarrow$ The single band result is recovered.

General argument in Galilean invariant case

Galilean boost along the chain

$$U_v = e^{ivPt} e^{-imvX}$$
$$U_V^\dagger \psi(x) U_V = e^{imv(x+vt)} \psi(x+vt)$$
$$U_v^\dagger j_x U_v = U_v^\dagger \frac{P}{m} U_v = j_x + v$$
$$t_\perp \int dx e^{ieBax} U_v^\dagger \psi_1^\dagger(x) \psi_2(x) U_V = t_\perp \int dx e^{ieBa(x-vt)} \psi_1^\dagger(x) \psi_2(x)$$

$e^{-ieBavt}$ corresponds to a potential difference between the chains.

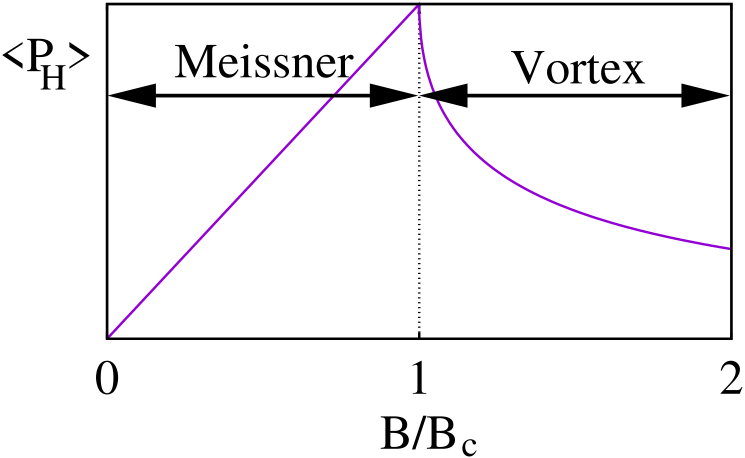
Hall imbalance in the vortex phase

Reduction of Hall imbalance

$$\begin{aligned}
\frac{\langle P_H \rangle^{(1)}}{L} &= \frac{\sqrt{2}\gamma \langle j_s \rangle}{eu^2}(\langle \partial_x \theta_a \rangle - eA_a). \\
\langle \partial_x \theta_a \rangle &= \sqrt{B^2 - B_c^2} \\
\lim_{B \to +\infty} \langle P_H \rangle^{(1)} &= 0,
\end{aligned}$$

High flux decouples the chains

Behavior of Hall imbalance



Hall effect in the two-leg fermion ladder

Hamiltonian

$$
\begin{aligned}
H &= \sum_{p=1,2} \int dx \frac{\psi_p^\dagger(x)}{2m} \left(\frac{\hbar}{i}\nabla - eA_p \right)^2 \psi_p(x) - \mu \psi_p^\dagger(x)\psi_p(x) \\
&\quad - t_\perp \int dx (\psi_1^\dagger \psi_2 + \psi_2^\dagger \psi_1) + U \int dx \rho_1(x)\rho_2(x) \\
&\quad + \frac{1}{2} \int dx V(x - x')[\rho_1(x)\rho_1(x') + \rho_2(x)\rho_2(x')]
\end{aligned}
$$

Bonding/antibonding basis

$$
\psi_{0,pi} = \frac{1}{\sqrt{2}}(\psi_1 \pm \psi_2)
$$

Bosonized Hamiltonian

Non interacting bosonized Hamiltonian in bonding/antibonding basis

$$
\begin{aligned}
H_0 = {} & \frac{v_F}{2\pi}\int dx\,\left[(\nabla\theta_\sigma)^2 + (\partial_x\phi_\sigma)^2 + (\nabla\theta_\rho)^2 + (\partial_x\phi_\rho)^2\right] \\
& - \frac{2ev_F A_\sigma}{\pi a_0}\int dx \sin\sqrt{2}\theta_\sigma \sin\sqrt{2}\phi_\sigma \\
& - \frac{1}{6\pi m\sqrt{2}}\int dx\,\Big[3(\nabla\theta_\rho)^2\partial_x\phi_\rho + (\partial_x\phi_\rho)^3 \\
& + 3((\nabla\theta_\sigma)^2 + (\partial_x\phi_\sigma)^2)\partial_x\phi_\rho + 6\nabla\theta_\rho\nabla\theta_\sigma\partial_x\phi_\sigma\Big] \\
& - \int dx \frac{(eA_\sigma)^2}{\pi m\sqrt{2}}\partial_x\phi_\rho + \frac{\sqrt{2}t_\perp}{\pi}\int dx\partial_x\phi_\sigma \\
& + \frac{\sqrt{2}eA_\sigma}{\pi m a_0}\int dx\,\Big[\sin\sqrt{2}\theta_\sigma \sin\sqrt{2}\phi_\sigma\partial_x\phi_\rho \\
& - \cos\sqrt{2}\theta_\sigma \cos\sqrt{2}\phi_\sigma\nabla\theta_\rho\Big],
\end{aligned}
$$

Interactions

$$H_{int.} = \int dx \left[\frac{2U + 2\hat{V}(0) - \hat{V}(2k_F)}{4\pi^2} (\partial_x \phi_\rho)^2 + \frac{\hat{V}(2k_F)}{4\pi^2} (\Pi_\rho)^2 + \frac{\hat{V}(0)}{4\pi^2} (\nabla \theta_\sigma)^2 + \frac{\hat{V}(0) - 2U}{4\pi^2} (\partial_x \phi_\sigma)^2 \right] + \frac{V(0) - V(2k_F)}{(2\pi a_0)^2} \int dx \cos \sqrt{8}\theta_\sigma + \frac{2U + \hat{V}(0) - \hat{V}(2k_F)}{(2\pi a_0)^2} \int dx \cos \sqrt{8}\phi_\sigma$$

Hall polarization operator

$$P_H = 2 \int \frac{dx}{\pi a_0} \cos \sqrt{2}\theta_\sigma \cos \sqrt{2}\phi_\sigma .$$

Ground state in the absence of flux

[A. A. Nersesyan et al. Phys. Lett. A **176**, 363 (1993)]

1. θ_ρ, ϕ_ρ gapless
2. $\theta_\sigma, \phi_\sigma$ gapped
3. $\langle \theta_\sigma \rangle = \frac{\pi}{\sqrt{8}}$ with V repulsive
4. $\langle \theta_\sigma \rangle = 0$ with V repulsive

$\Rightarrow$ as in the bosonic case, a single gapless mode.

Perturbative result

Hall imbalance

$$\langle P_H \rangle = \frac{e^2 Ba\langle j_s \rangle}{m u_\rho K_\rho} \chi_{xx}\left(q = \frac{2t_\perp}{v_F}, \omega_n = 0\right),$$

Hall resistance

$$R_H = -\frac{a}{e}\frac{\partial}{\partial \rho_0}[\ln(u_\rho K_\rho)] = -\frac{a}{e\rho_0}.$$

Some consequences for the Hall coefficient

Near commensurate filling $1/n$: Commensurate-incommensurate transition

$$
\begin{aligned}
H_\rho &= \int \frac{dx}{2\pi}\left[u_\rho K_\rho (\nabla\theta_\rho)^2 + \frac{u_\rho}{K_\rho}(\partial_x\phi_\rho)^2\right] \\
&\quad -\frac{2g_U}{(2\pi\alpha)^2}\int dx \cos(n\sqrt{2}\phi_\rho) + \frac{\sqrt{2}}{\pi}\mu\int dx\partial_x\phi_\rho, \\
u_\rho^* &\sim (\rho-\rho_c) \quad \& \quad K_\rho^* \sim \frac{2}{n^2} \Rightarrow R_H \sim \frac{1}{e(\rho-\rho_c)}
\end{aligned}
$$

Conclusion

Summary [R. Citro, T. Giamarchi, E. Orignac arXiv:2404.16973]

- Calculation of Hall imbalance and Hall coefficient
- In the galilean invariant case, the Hall coefficient is inverse of density
- general relation between Hall coefficient and stiffness

Perspectives

- Beyond perturbation theory
- Systems of many coupled chains
- Quantum Hall phases
- Vortex lattice phases