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Inertial effects in superfluid vortex dynamics

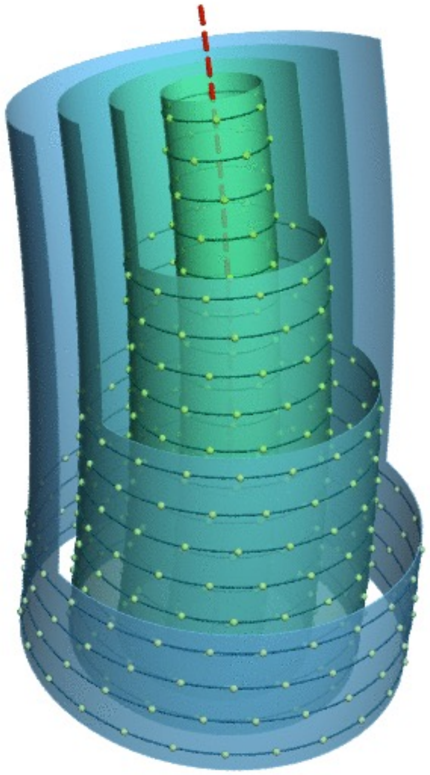
Benasque, May 20th, 2024

Outline

- Introduction to Superfluid Vortex Dynamics
- From Massless to Massive vortices
- Electromagnetic equivalence
- Active vortex cores
- Collective effects of massive vortices: superfluid Kelvin – Helmholtz instability
- Conclusions

INTRODUCTION TO SUPERFLUID VORTEX DYNAMICS

What is a vortex?

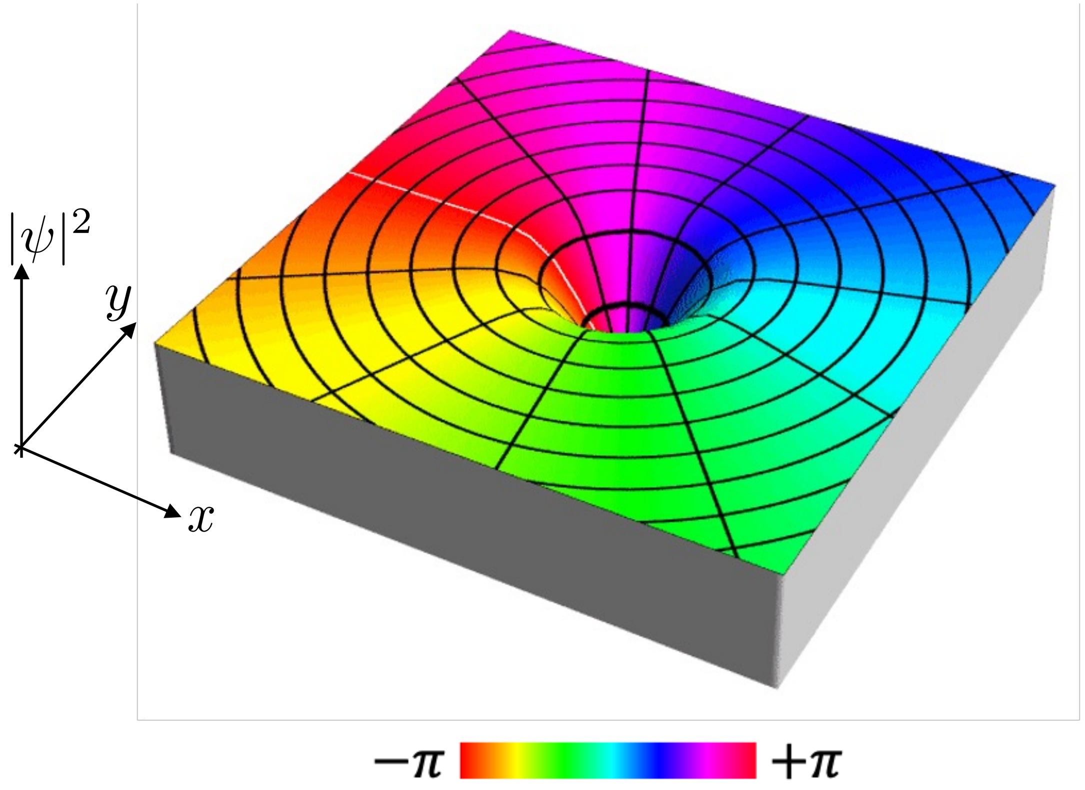


A vortex is a region in a fluid in which the flow revolves around an axis line.

Vortices as a manifestation of coherence

Dog puppies can be coherent too…

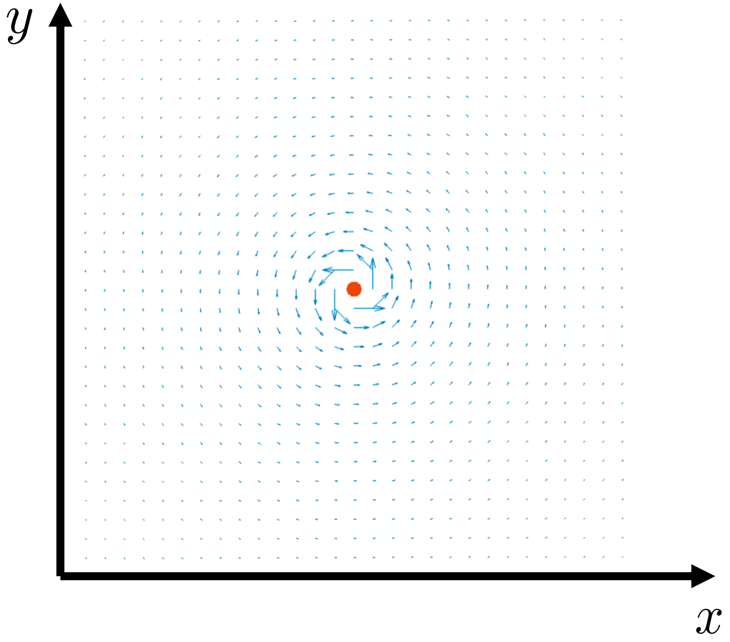
The ID of a quantum vortex



- The phase rolls up from 0 to 2π.
- The density goes to zero at the vortex centre.

Principle of superposition of the velocity fields

Each vortex generates a velocity vector field of the type:

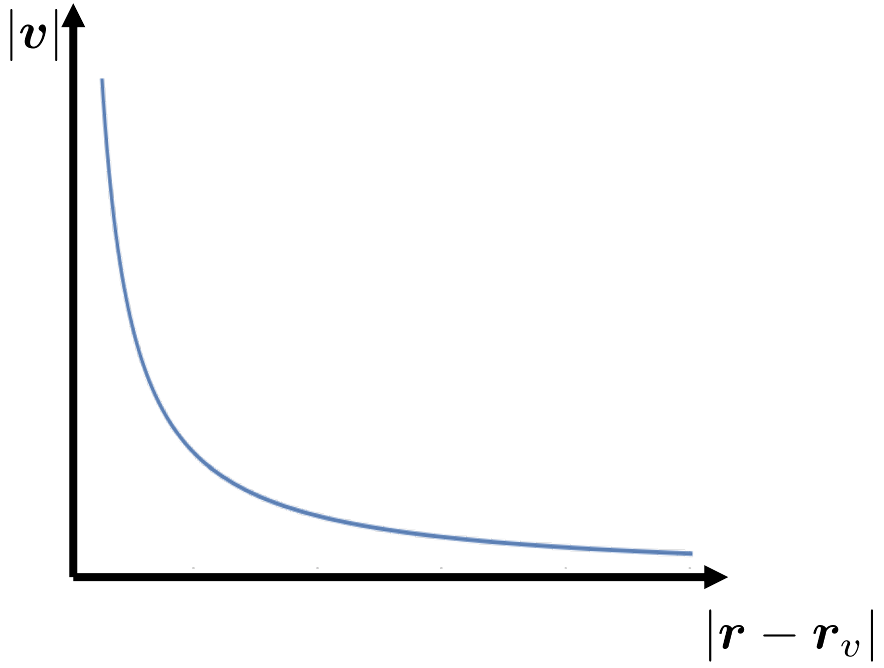


$$\boldsymbol{v}(\boldsymbol{r}) = \frac{\kappa}{2\pi}\hat{z} \times \frac{\boldsymbol{r} - \boldsymbol{r}_v}{|\boldsymbol{r} - \boldsymbol{r}_v|^2}$$

where

$$\kappa = \pm q\frac{h}{m}, \quad q \in \mathbb{N}$$

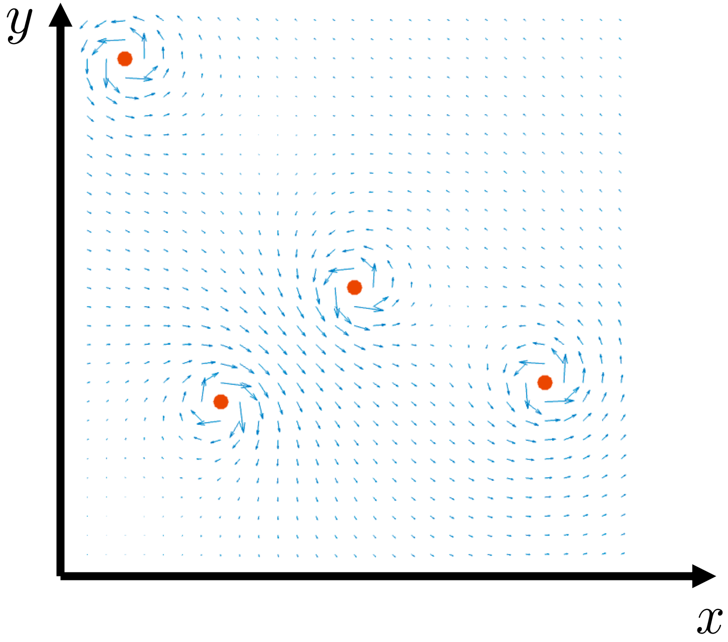
is the vortex strength.



Velocity diverges at the core!

Principle of superposition of the velocity fields

In a many-vortex system the total velocity field reads



$$\boldsymbol{v}(\boldsymbol{r}) = \sum_{i=1}^{N_v} \left[\frac{\kappa_i}{2\pi} \hat{z} \times \frac{\boldsymbol{r} - \boldsymbol{r}_i}{|\boldsymbol{r} - \boldsymbol{r}_i|^2} \right]$$

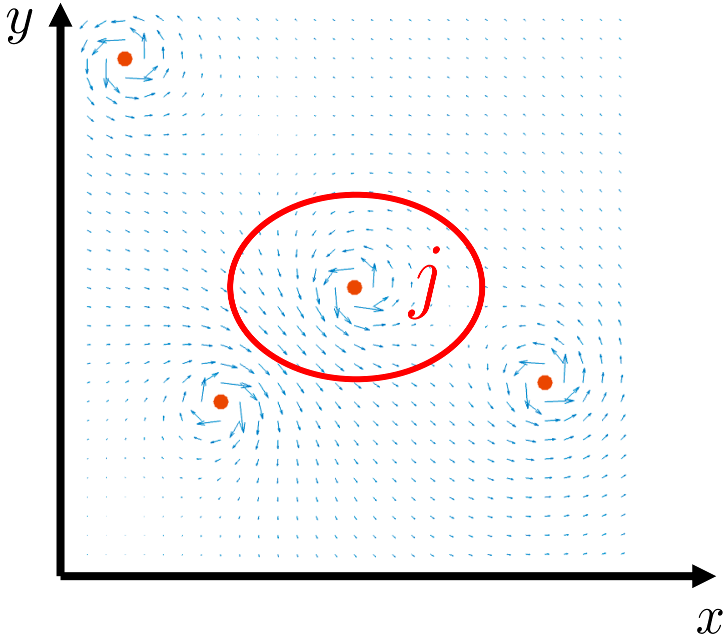
where

$$\kappa_i = \pm q \frac{h}{m}, \quad q \in \mathbb{N}$$

is the strength of the *i*-th vortex.

Principle of superposition of the velocity fields

The $\boldsymbol{i}$-th vortex moves under the influence of the remaining $\boldsymbol{N_v - 1}$ vortices



$$\boldsymbol{v}(\boldsymbol{r}_j) = \sum_{i \neq j} \left[\frac{\kappa_i}{2\pi} \hat{z} \times \frac{\boldsymbol{r}_j - \boldsymbol{r}_i}{|\boldsymbol{r}_j - \boldsymbol{r}_i|^2} \right]$$

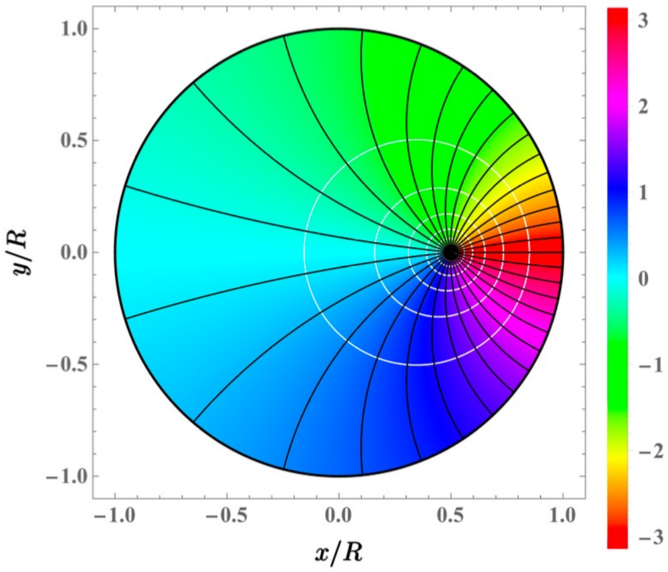
where the self-contribution $\boldsymbol{i} = \boldsymbol{j}$ has been removed from the summation.

The complex-potential framework

A convenient mathematical framework to study the dynamics of point vortices in quasi-2D superfluids is provided by the complex potential:

$$F(z) = \chi(\boldsymbol{r}) + i\,\theta(\boldsymbol{r})$$

Where $\boldsymbol{r} = (x,\, y)$ and $z = x + i\,y$

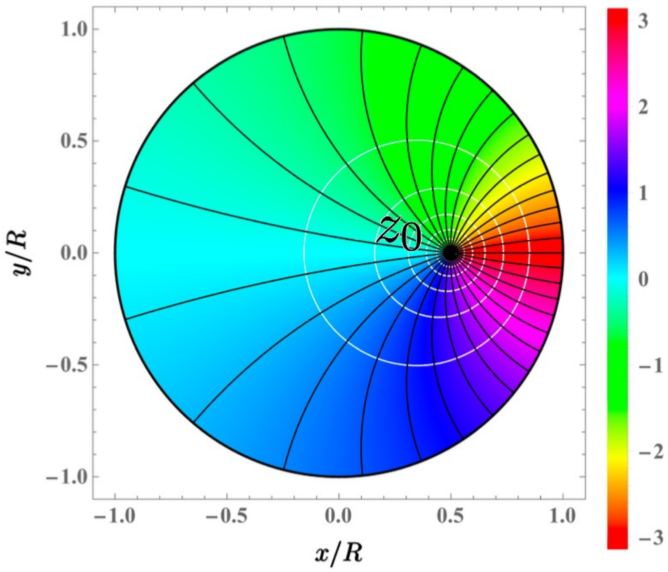


$\chi(\boldsymbol{r})$: **Stream function**
(streamlines represent the trajectories of fluid particles)

$\theta(\boldsymbol{r})$: **Velocity potential, or phase field**
(the gradient is the velocity field)

The complex-potential framework

The complex potential of a many-vortex system, where vortices are at $\{z_1,\ z_2,\ \ldots,\ z_{N_v}\}$ is



$$F(z) = \sum_{i=1}^{N_v} q_i \log(z - z_i) \qquad \text{with } q_i = \pm 1$$

In the case of a disk-like domain, if a vortex is present at

$$z_0 = x_0 + i y_0$$

an image vortex is automatically present at $z_0' = x_0' + i y_0'$

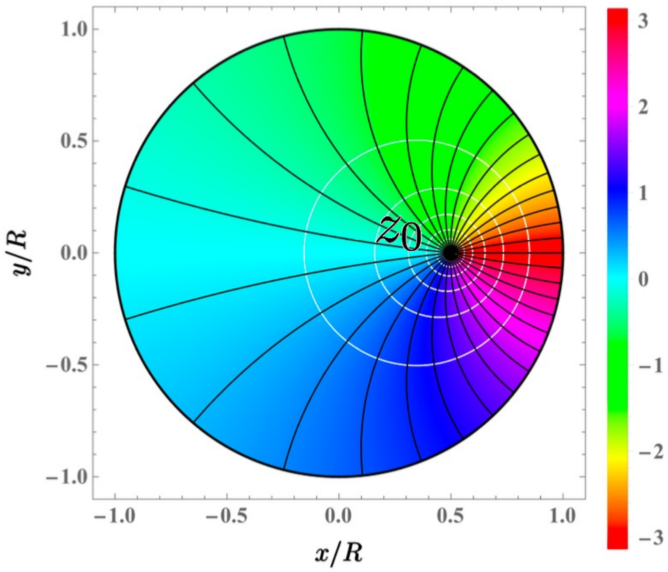
$$\begin{cases} x_0' = \dfrac{R^2}{x_0^2 + y_0^2} x_0, \qquad y_0' = \dfrac{R^2}{x_0^2 + y_0^2} y_0 \end{cases}$$

and has **opposite charge**.

This ensures that **streamlines are tangent** to the circular boundary and **constant-phase lines are perpendicular** to it.

The complex-potential framework

The complex potential of a many-vortex system, where vortices are at $\{z_1, z_2, \ldots, z_{N_v}\}$ is



$$F(z) = \sum_{i=1}^{N_v} q_i \log(z - z_i) \qquad \text{with } q_i = \pm 1$$

In the case of a disk-like domain, the complex potential reads:

$$F(z) = \log(z - z_0) - \log(z - z_0')$$

Hence:

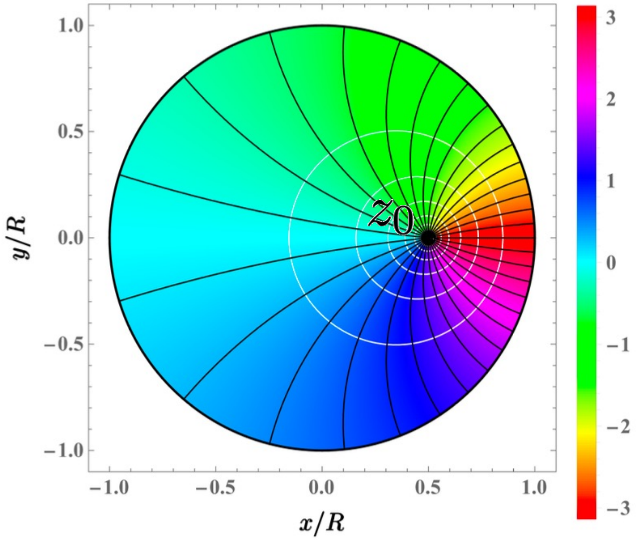
$$\chi(\boldsymbol{r}) = \mathrm{Re}(F) = \log \left| \frac{\boldsymbol{r} - \boldsymbol{r}_0}{\boldsymbol{r} - \boldsymbol{r}_0'} \right|$$

$$\theta(\boldsymbol{r}) = \mathrm{Im}(F) = \arctan\left(\frac{y - y_0}{x - x_0}\right) - \arctan\left(\frac{y - y_0'}{x - x_0'}\right)$$

The complex-potential framework

Once the complex potential is known, determining the vortex motion is straightforward:

$$i\dot{z}_0^* = \dot{y}_0 + i\dot{x}_0 = \frac{\hbar}{m} \lim_{z \to z_0} \left[F'(z) - \frac{1}{z - z_0} \right]$$



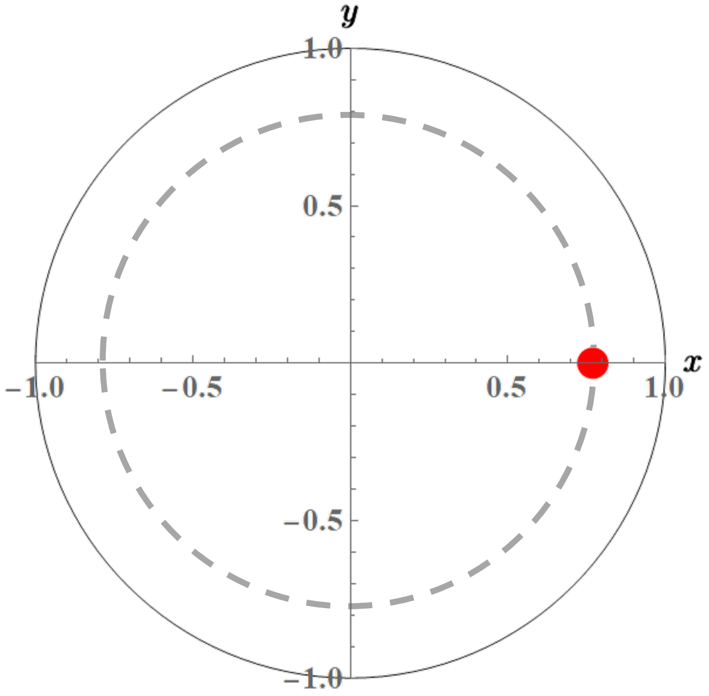
In the case of a vortex in a disk-like domain:

$$F(z) = \log(z - z_0) - \log(z - z_0')$$

$$\dot{\boldsymbol{r}}_0 = \hat{z} \times \frac{\hbar}{m} \frac{\boldsymbol{r}_0}{R^2 - r_0^2}$$

The complex-potential framework

$$\dot{\boldsymbol{r}}_0 = \hat{z} \times \frac{\hbar}{m} \frac{\boldsymbol{r}_0}{R^2 - r_0^2}$$



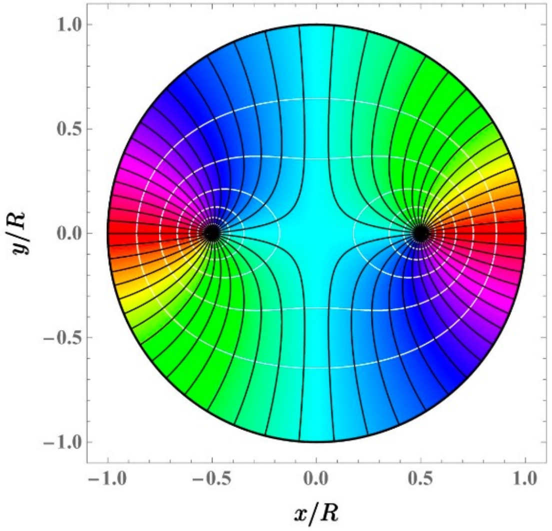
The equation of motion is a **first-orde**r differential equation, and its solutions are (trivial) **uniform circular orbits**.

More than one vortex

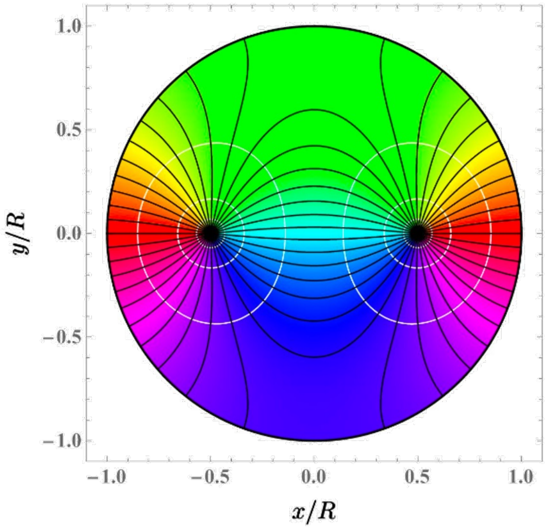
The complex-potential framework work well also with more than one vortex.

For the case of a two-vortex system in a disk, the complex potential reads

$$F(z) = q_1 \log\left(\frac{z - z_1}{z - z_1'}\right) + q_2 \log\left(\frac{z - z_2}{z - z_2'}\right)$$



$q_1 = q_2 = +1$



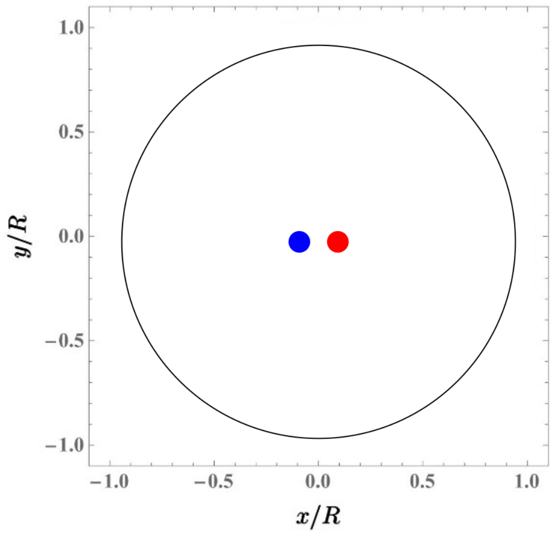
$q_1 = +1, \quad q_2 = -1$

More than one vortex

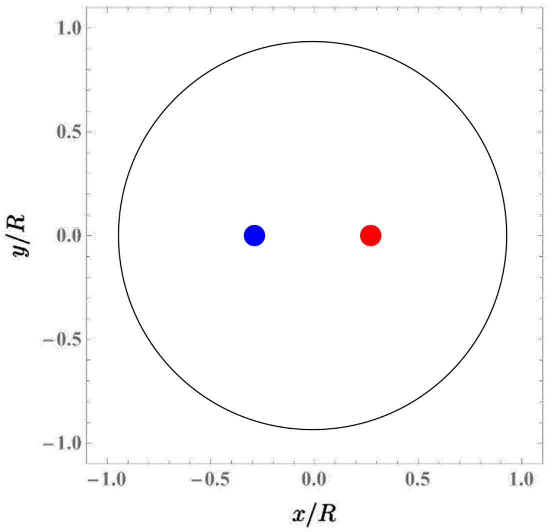
The complex-potential framework work well also with more than one vortex.

For the case of a two-vortex system in a disk, the complex potential reads

$$F(z) = q_1 \log\left(\frac{z - z_1}{z - z_1'}\right) + q_2 \log\left(\frac{z - z_2}{z - z_2'}\right)$$



$q_1 = q_2 = +1$

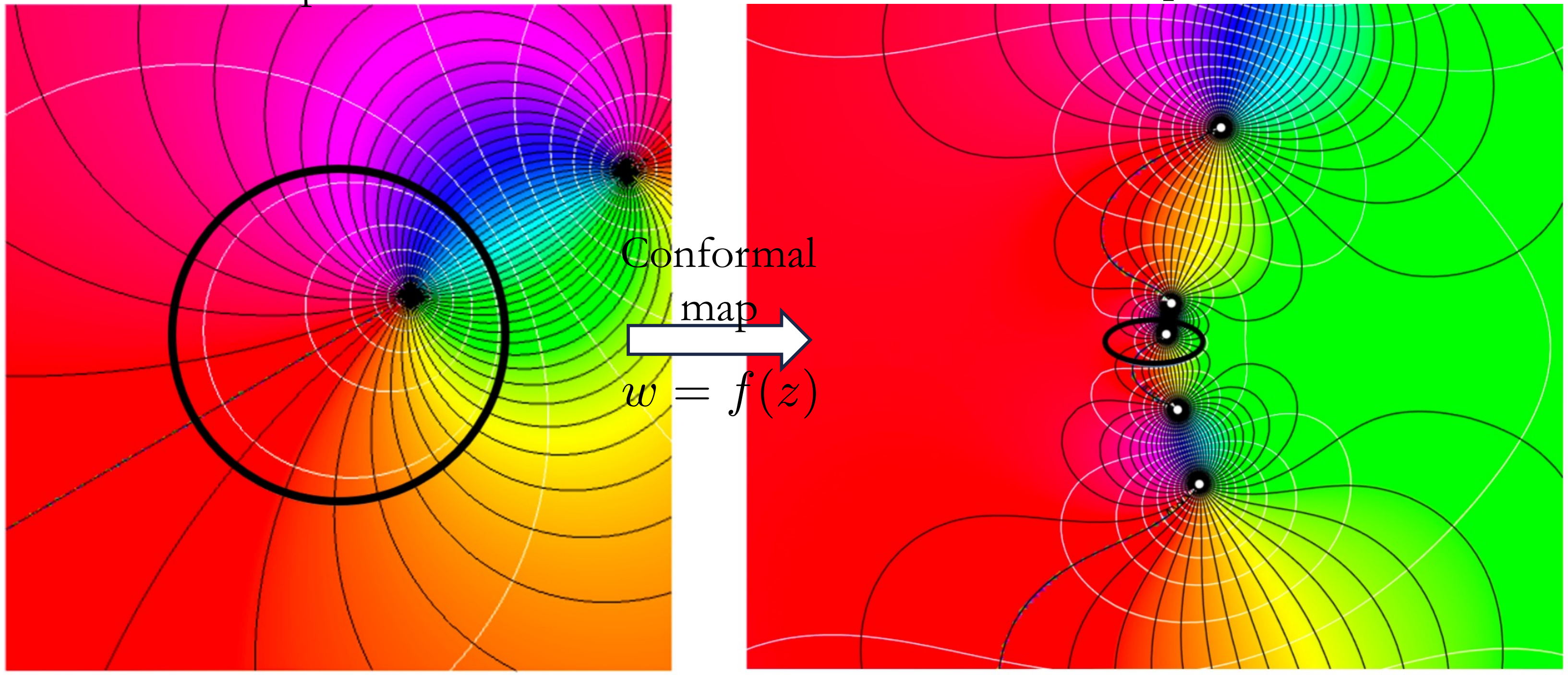


$q_1 = +1, \quad q_2 = -1$

More complex geometries

Suitable conformal transformations of the complex potential F(z) allow to fully solve the dynamics of vortices in different confining geometries, e.g. the ellipse:

w-plane   z-plane



$$F_{\text{disk}}(w) = \ln\left(\frac{w - w_0}{w - w_0'}\right)$$

$$F_{\text{ellipse}}(z) = F_{\text{disk}}(w = f(z)) = \ln\left(\frac{f(z) - w_0}{f(z) - w_0'}\right)$$

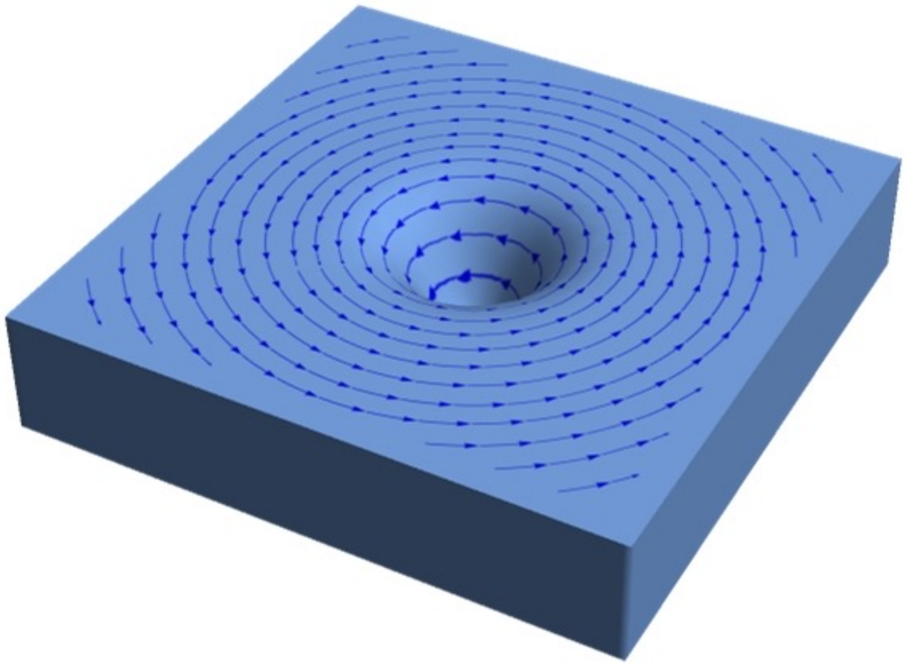
[M. Caldara, A. Richaud, P. Massignan, A. L. Fetter, arXiv:2311.13545]

FROM MASSLESS TO MASSIVE VORTICES

[A. Richaud, V. Penna, R. Mayol, M. Guilleumas, Phys. Rev. A **101**, 013630 (2020)]

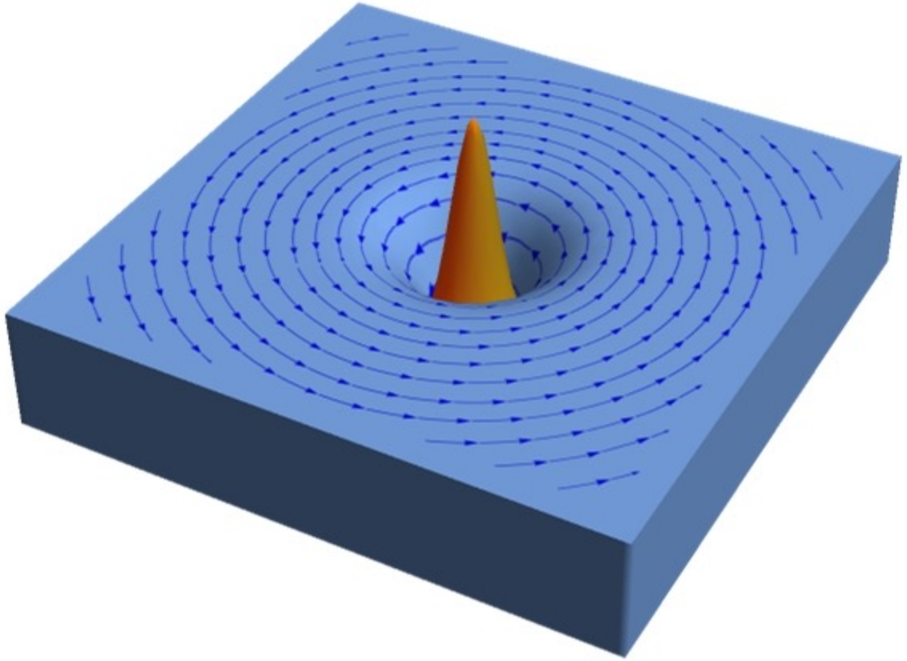
[A. Richaud, V. Penna, A. L. Fetter, Phys. Rev. A **103**, 023311 (2021)]

Vortices: just empty holes?



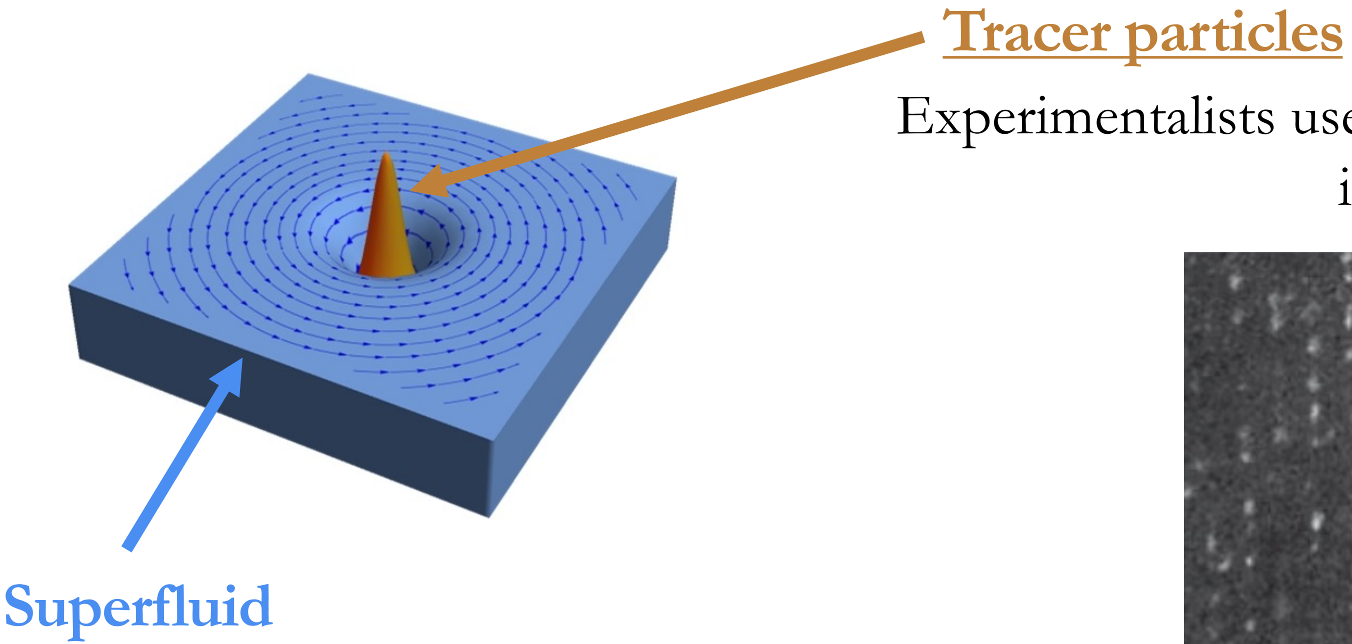
Traditionally, the core is represented as a funnel-like **hole** around which the superfluid exhibits a swirling flow, a sort of *tornado* in the corresponding wavefunction.

Vortices with filled massive cores

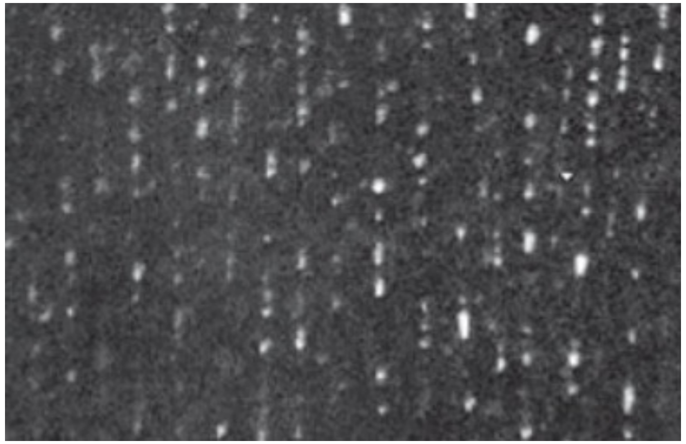


Actually, the vortex core turns out to be commonly filled by particles!

Vortices with filled massive cores



Experimentalists use particles as "vorticity tracers", e.g. in liquid helium.

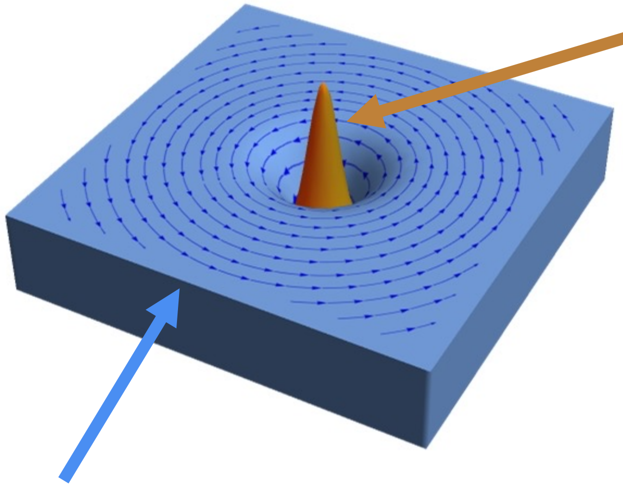


G. P. Bewley et al., Nature **441**, 588 (2006)

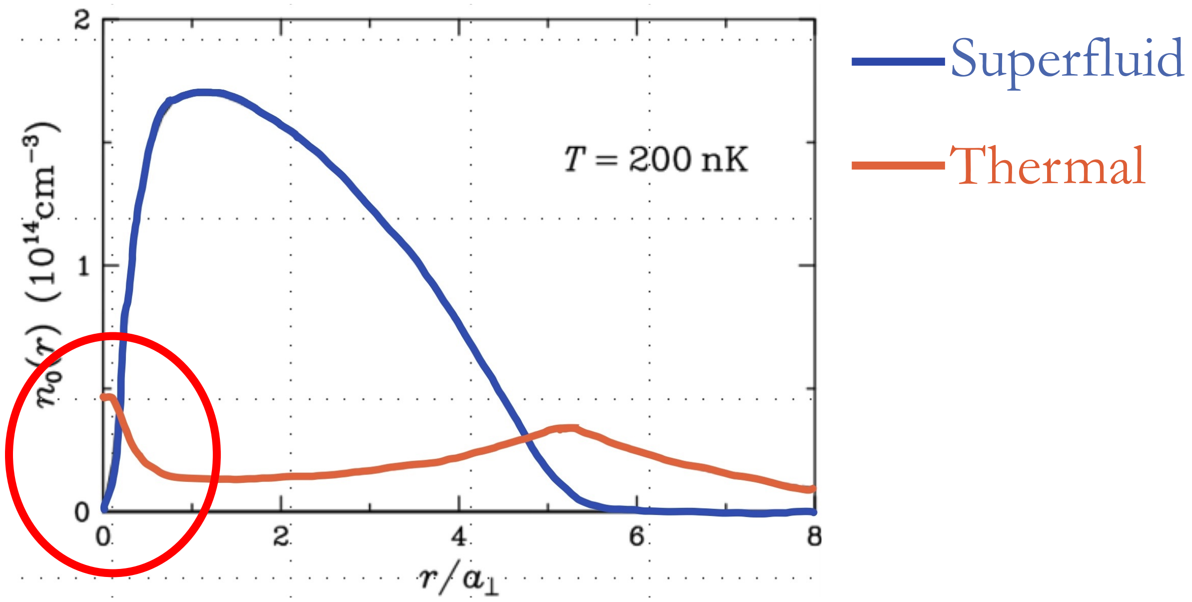
Vortices with filled massive cores

Thermal atoms

Atoms which do not belong to the superfluid fraction.

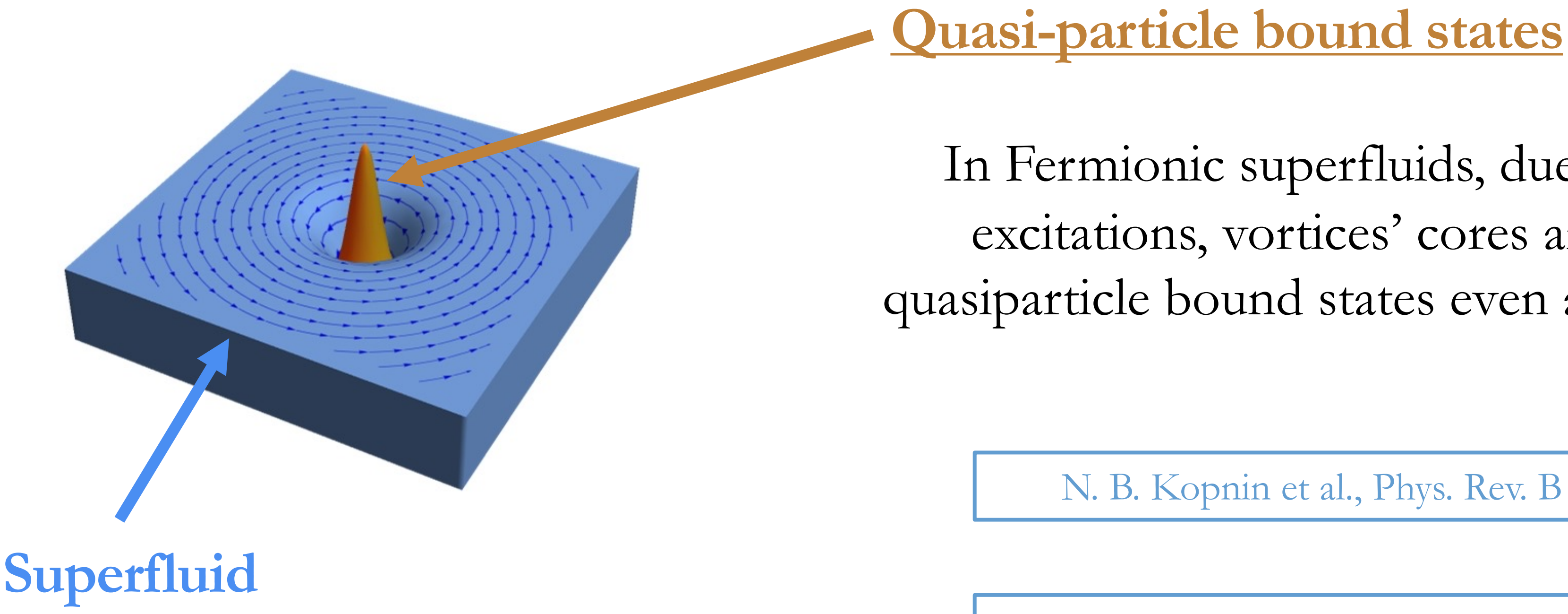


Superfluid



A. Griffin, T. Nikuni, E. Zaremba, *Bose-Condensed Gases at Finite Temperature*, Chap. 9, Cambridge University Press (2009)

Vortices with filled massive cores

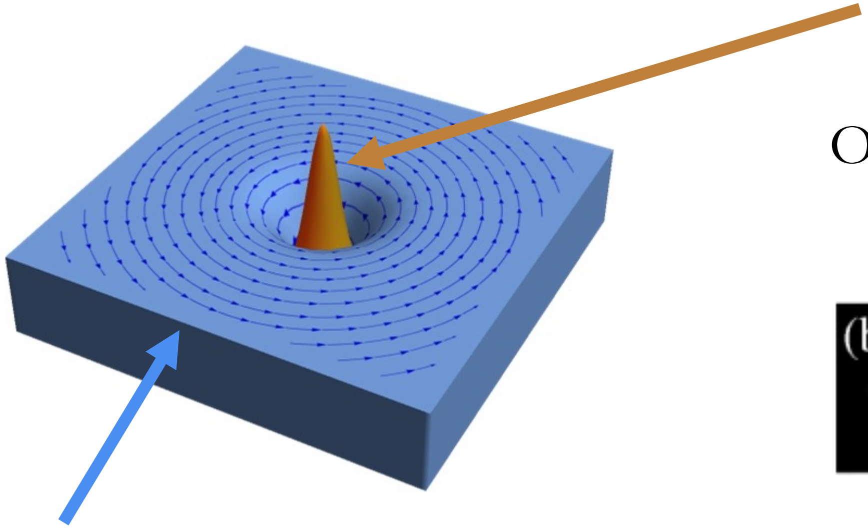


In Fermionic superfluids, due to pair-breaking excitations, vortices' cores are filled up with quasiparticle bound states even at zero temperature.

N. B. Kopnin et al., Phys. Rev. B **44**, 9667 (1991)

W. J. Kwon et al., Nature **600**, 64 (2021)

Vortices with filled massive cores

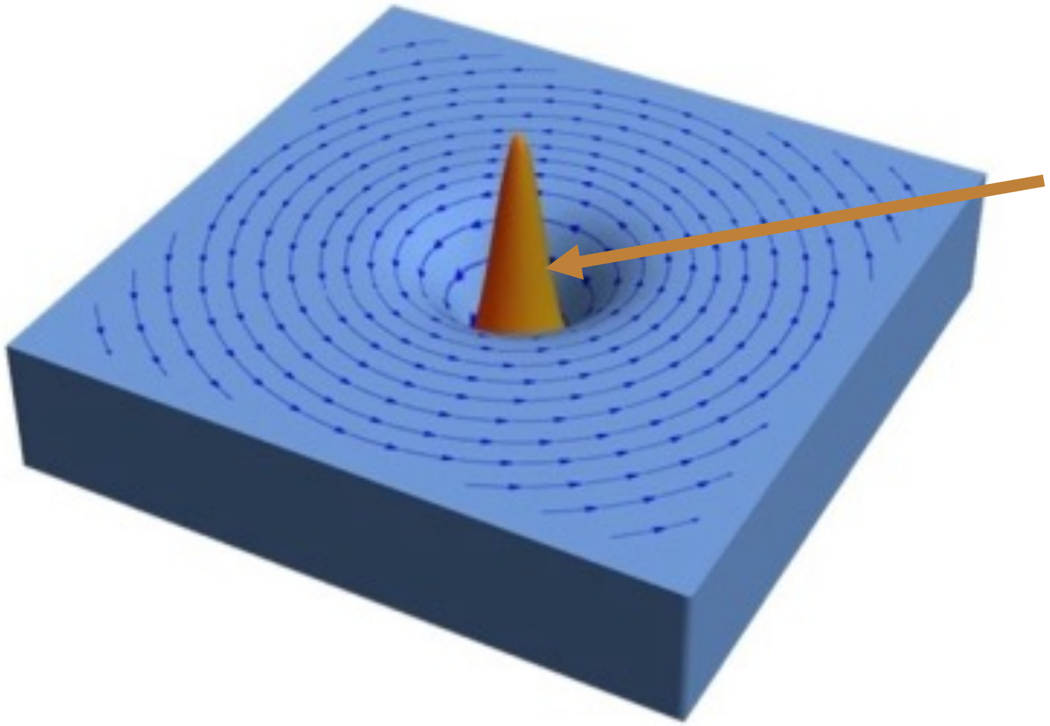


One of the first vortices ever observed in a BEC, had a core filled by another component!

The two components were two different internal states of ^{87}Rb.

B. P. Anderson et al., Phys. Rev. Lett. **85**, 2857 (2000)

Vortices with filled massive cores

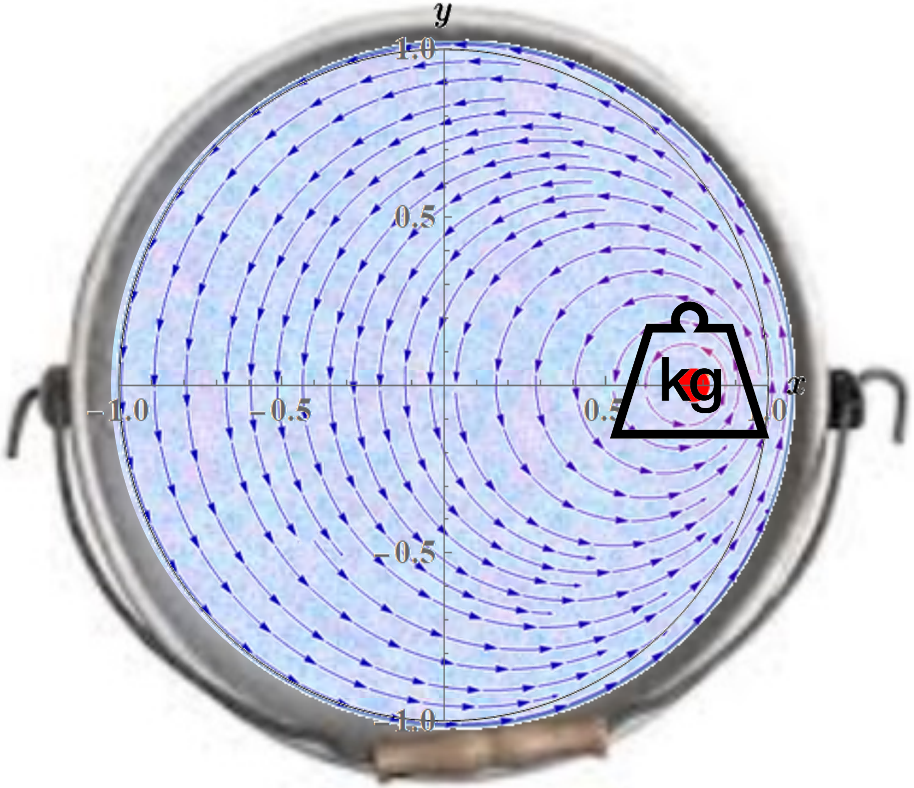


- **Tracer particles**
- **Thermal atoms**
- **Quasiparticle bound states**
- **Another (minority) BEC**

Superfluid vortices are often filled by massive cores (deliberately or accidentally!)

Massive Point Vortex Model

The Lagrangian of massive vortex in a disk can be derived in a rigorous way:



Start from the Lagrangian of a massless vortex in a disk:

$$L_a = \hbar n_a \pi (\dot{\boldsymbol{r}}_0 \times \boldsymbol{r}_0 \cdot \hat{z}) \frac{r_0^2 - R^2}{r_0^2} - \frac{\hbar^2 n_a \pi}{m_a} \log\left(1 - \frac{r_0^2}{R^2}\right)$$

Write the Lagrangian ensuing from the inertial contribution of the core:

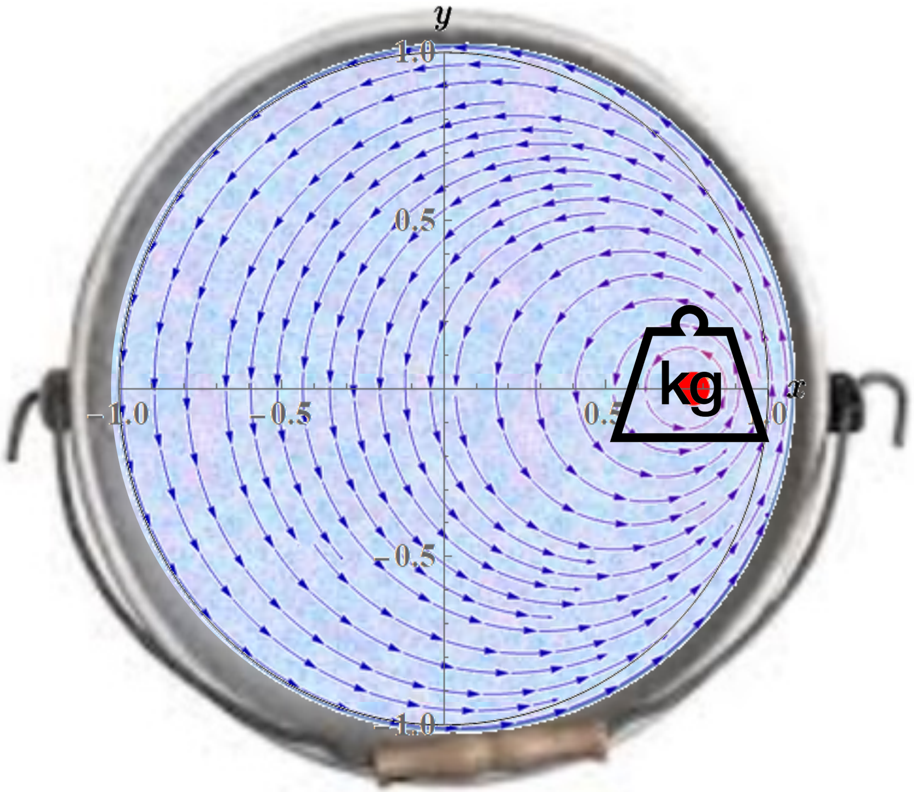
$$L_b = \frac{1}{2} M_b \dot{\boldsymbol{r}}_0^2$$

Recognize that the total Lagrangian of the system is:

$$L = L_a + L_b$$

[A. Richaud, V. Penna, A. L. Fetter, Phys. Rev. A **103**, 023311 (2021)]

Massive Point Vortex Model



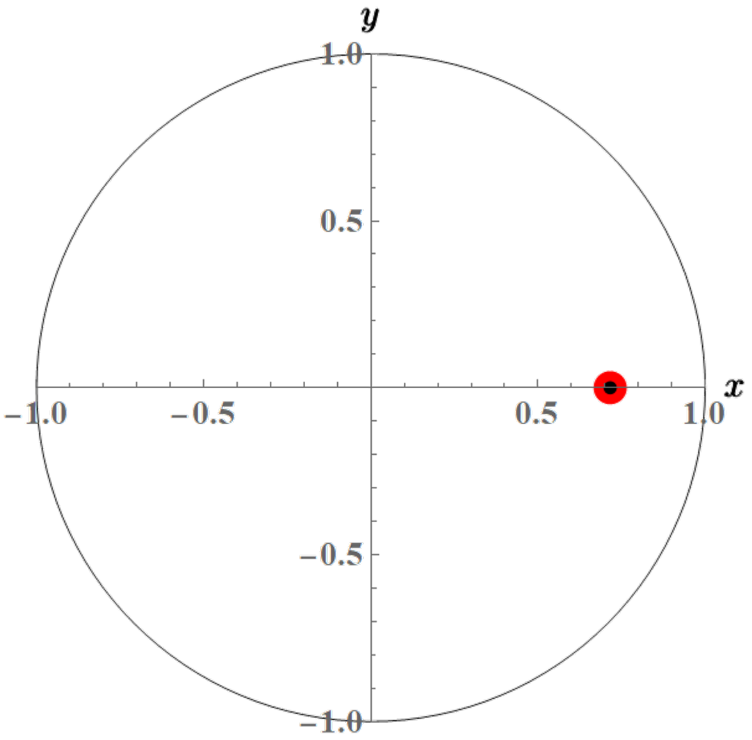
$$L = L_a + L_b$$

Compute the associated Euler-Lagrange equations:

$$M_b \ddot{\boldsymbol{r}}_0 = 2n_a \pi \hbar \left[\hat{z} \times \dot{\boldsymbol{r}}_0 + \frac{\hbar}{m_a} \frac{\boldsymbol{r}_0}{R^2 - r_0^2} \right]$$

- **This is a second-order equation of motion: the introduction of mass is a singular perturbation.**
- **The number of dynamical variables associated to each vortex doubles!**

Massive Point Vortex Model



$$L = L_a + L_b$$

Compute the associated Euler-Lagrange equations:

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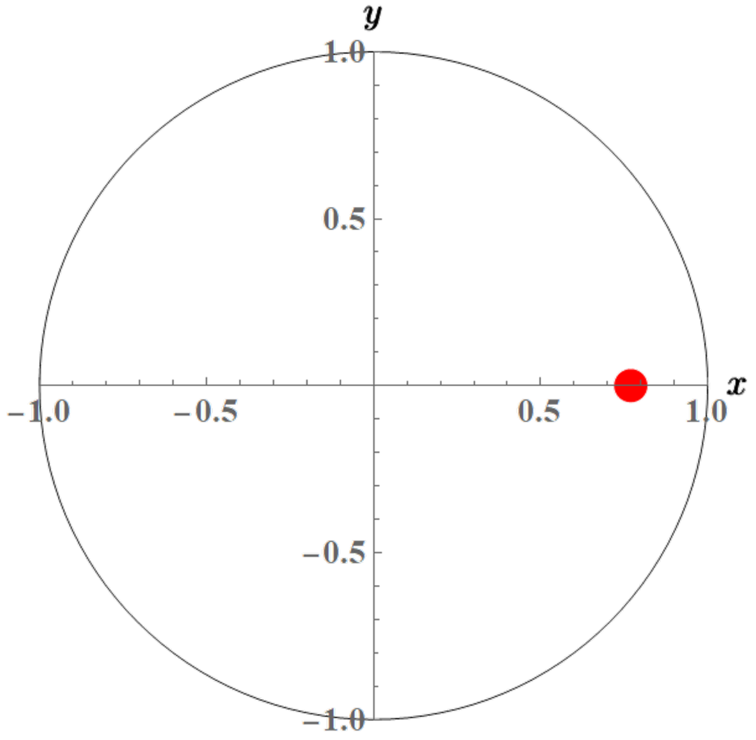
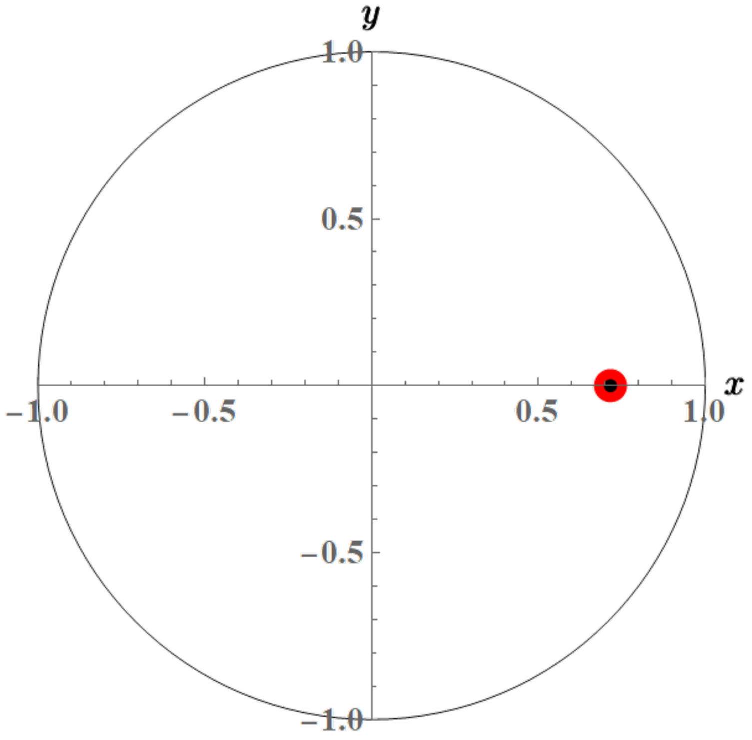
These equations tell us that the motion is not simply a uniform circular one!

Massless vs Massive Vortices



Massless → Only uniform circular orbits



Massive → **Radial oscillations** superimposed to circular orbits.

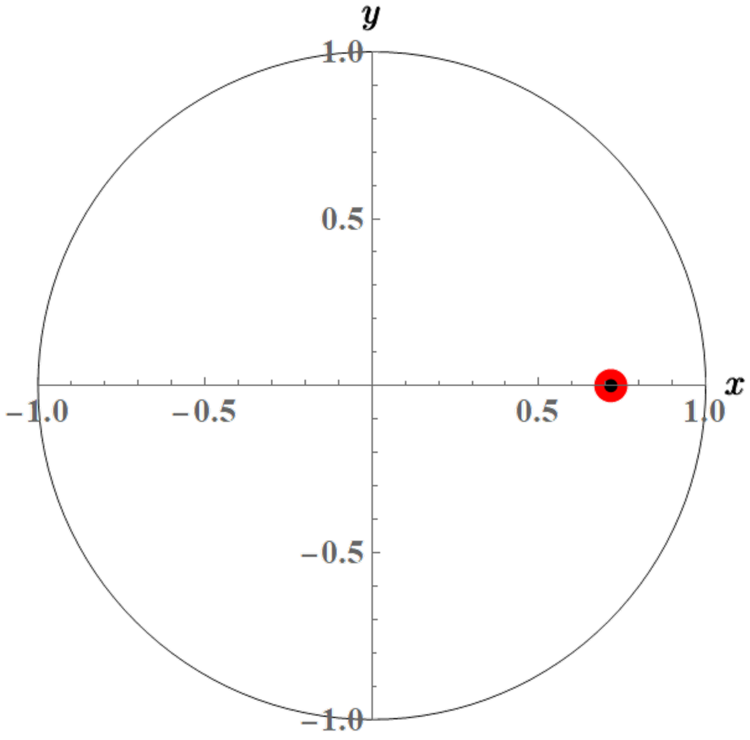
Transverse oscillation frequency as mass signature

The frequency ω of radial oscillations is **inversely proportional** to the core mass:

$$\omega = \frac{\hbar}{m_a R^2}\frac{2}{\mu}\sqrt{1-\mu\frac{2-\tilde{r}_0^2}{(1-\tilde{r}_0^2)^2}}.$$

where $\tilde{r}_0 = r_0/R$ and $\mu = M_b/M_a$.

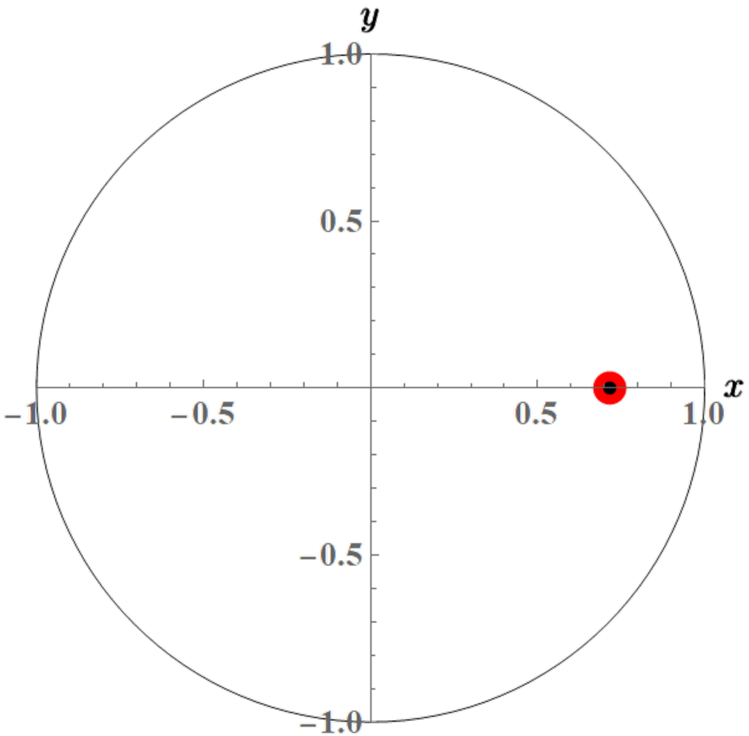
Typical signature of a **singular perturbation**.



Massive → Radial oscillations superimposed to circular orbits.

[A. Richaud, V. Penna, A. L. Fetter, Phys. Rev. A **103**, 023311 (2021)]

Magnus effect and Magnus force



[A. Richaud, P. Massignan, V. Penna, and A. L. Fetter, Phys. Rev. **A** 106, 063307 (2022)]

The equation of motion of a massive vortex

$$M_b \ddot{\boldsymbol{r}}_0 = 2 n_a \pi \hbar \left[\hat{z} \times \dot{\boldsymbol{r}}_0 + \frac{\hbar}{m_a} \frac{\boldsymbol{r}_0}{R^2 - r_0^2} \right]$$

can be rewritten as

$$M_b \ddot{\boldsymbol{r}}_0 = \boldsymbol{F}^M$$

where

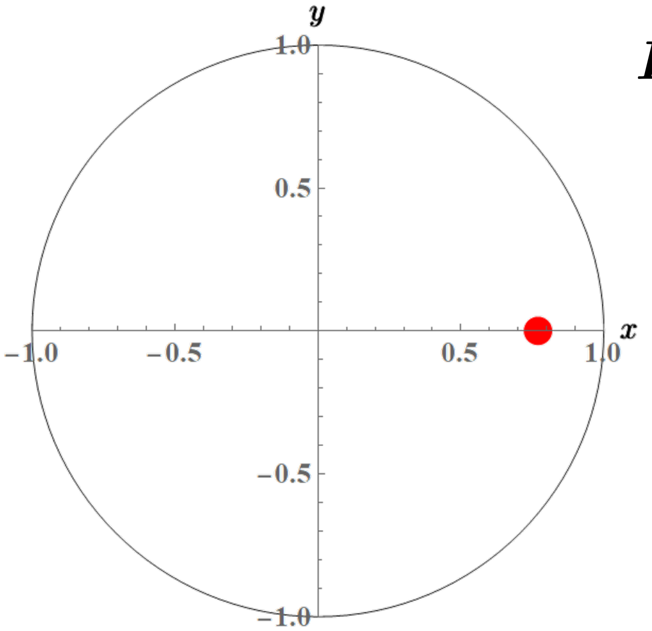
$$\boldsymbol{F}^M = 2 n_a \pi \hbar (\boldsymbol{v}_s - \dot{\boldsymbol{r}}_0) \times \hat{\boldsymbol{z}},$$

is the **Magnus force**, proportional to the difference between the actual vortex velocity, $\dot{\boldsymbol{r}}_0$ and the local superfluid velocity $\boldsymbol{v}_s$ (simply induced by the image vortex).

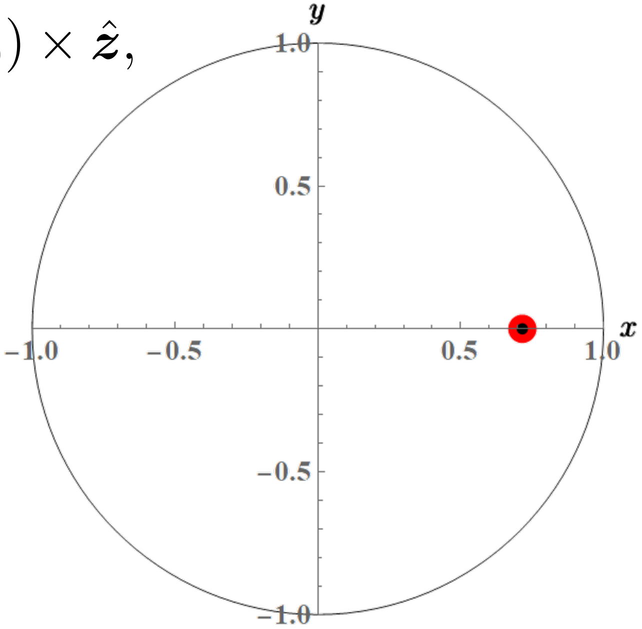
Magnus effect and Magnus force

$$M_b \ddot{\boldsymbol{r}}_0 = \boldsymbol{F}^M$$

$$\boldsymbol{F}^M = 2 n_a \pi \hbar (\boldsymbol{v}_s - \dot{\boldsymbol{r}}_0) \times \hat{\boldsymbol{z}},$$



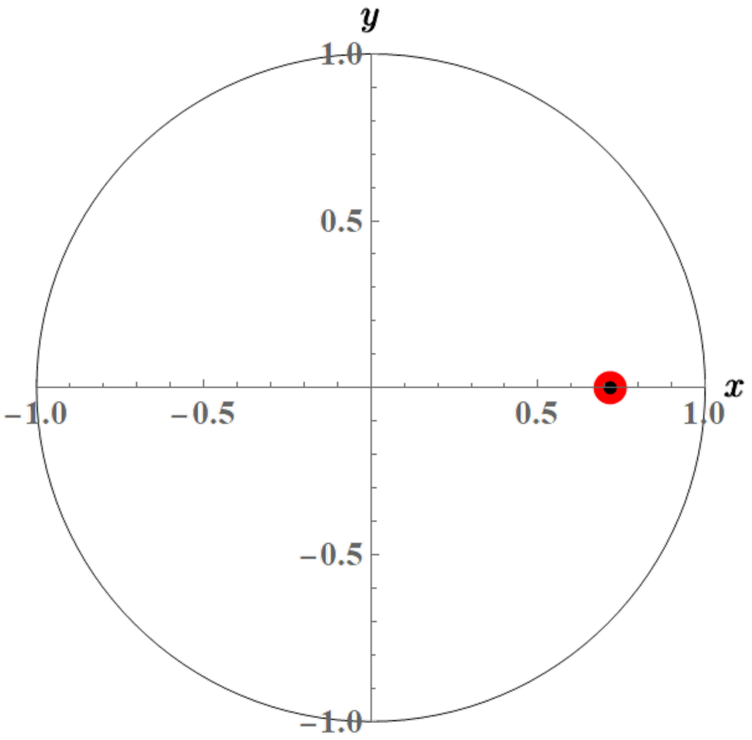
A **massless** vortex moves with the local superfluid velocity not to be subject to any net force. ← **Magnus effect.**



A **massive** vortex moves according to Newton's second law, where **F** is the Magnus force.

ELECTROMAGNETIC EQUIVALENCE

Vortices as interacting point charges subject to a transverse magnetic field



The equation of motion of a single massive vortex

$$M_b \ddot{\boldsymbol{r}}_0 = 2n_a \pi \hbar \left[\hat{z} \times \dot{\boldsymbol{r}}_0 + \frac{\hbar}{m_a} \frac{\boldsymbol{r}_0}{R^2 - r_0^2} \right]$$

can be also rewritten as

$$M_b \ddot{\boldsymbol{r}}_0 = \underbrace{\kappa \dot{\boldsymbol{r}}_0 \times (-m_a n_a \hat{z})}_{\text{Lorentz-like term}} + \underbrace{\frac{m_a n_a}{2\pi} \kappa \kappa' \frac{\boldsymbol{r}_0 - \boldsymbol{r}_0'}{|\boldsymbol{r}_0 - \boldsymbol{r}_0'|^2}}_{\text{Coulomb (2D) – like term}}$$

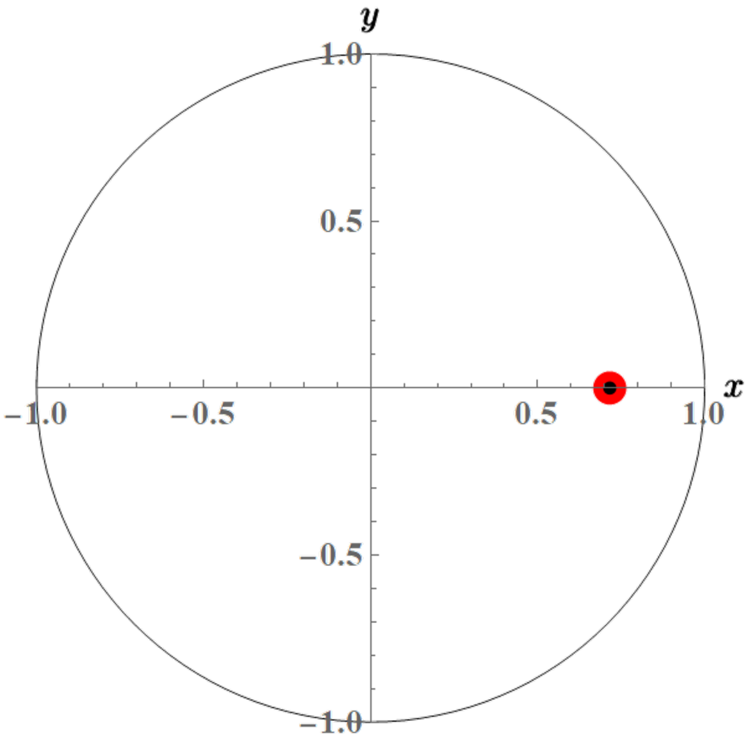
where $\boldsymbol{r}_0' = \frac{R^2}{r_0^2} \boldsymbol{r}_0$ and $\kappa' = -\kappa$

Lorentz-like term

Coulomb (2D) – like term

[A. Richaud, P. Massignan, V. Penna, and A. L. Fetter, Phys. Rev. **A** 106, 063307 (2022)]

Vortices as interacting point charges subject to a transverse magnetic field



[A. Richaud, P. Massignan, V. Penna, and A. L. Fetter, Phys. Rev. **A** 106, 063307 (2022)]

$$M_b \ddot{\boldsymbol{r}}_0 = \underbrace{\kappa \dot{\boldsymbol{r}}_0 \times (-m_a n_a \hat{z})}_{\text{Lorentz-like term}} + \underbrace{\frac{m_a n_a}{2\pi} \kappa\kappa' \frac{\boldsymbol{r}_0 - \boldsymbol{r}_0'}{|\boldsymbol{r}_0 - \boldsymbol{r}_0'|^2}}_{\text{Coulomb (2D) – like term}}$$

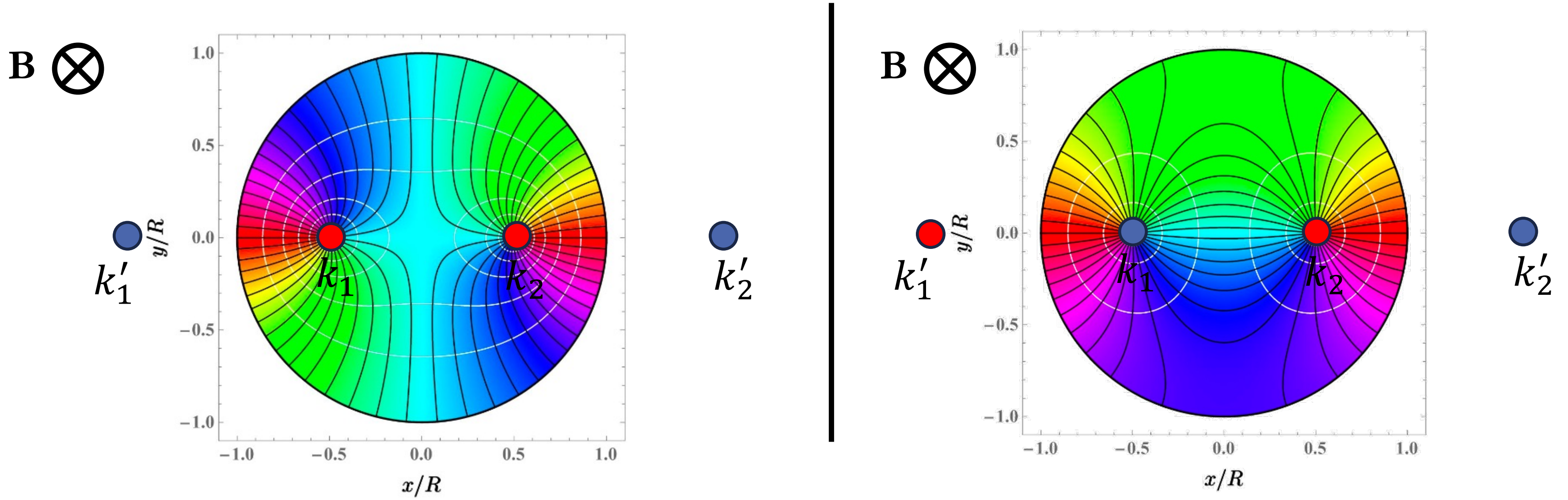
where $\boldsymbol{r}_0' = \frac{R^2}{r_0^2}\boldsymbol{r}_0$ and $\kappa' = -\kappa$

Lorentz-like term

Coulomb (2D) – like term

A massive vortex is formally equivalent to a massive particle of charge κ subject to an electric field (generated by all the other vortices, be them real or virtual) and a transverse magnetic field $\boldsymbol{B} = -m_a n_a \hat{z}$.

Vortex dipoles and pairs



$$M_{b,1}\ddot{\boldsymbol{r}}_1 = k_1\dot{\boldsymbol{r}}_1 \times (-m_a n_a \hat{z}) + \frac{m_a n_a}{2\pi}\left[k_1 k_1' \frac{\boldsymbol{r}_1 - \boldsymbol{r}_1'}{|\boldsymbol{r}_1 - \boldsymbol{r}_1'|^2} + k_1 k_2 \frac{\boldsymbol{r}_1 - \boldsymbol{r}_2}{|\boldsymbol{r}_1 - \boldsymbol{r}_2|^2} + k_1 k_2' \frac{\boldsymbol{r}_1 - \boldsymbol{r}_2'}{|\boldsymbol{r}_1 - \boldsymbol{r}_2'|^2}\right]$$

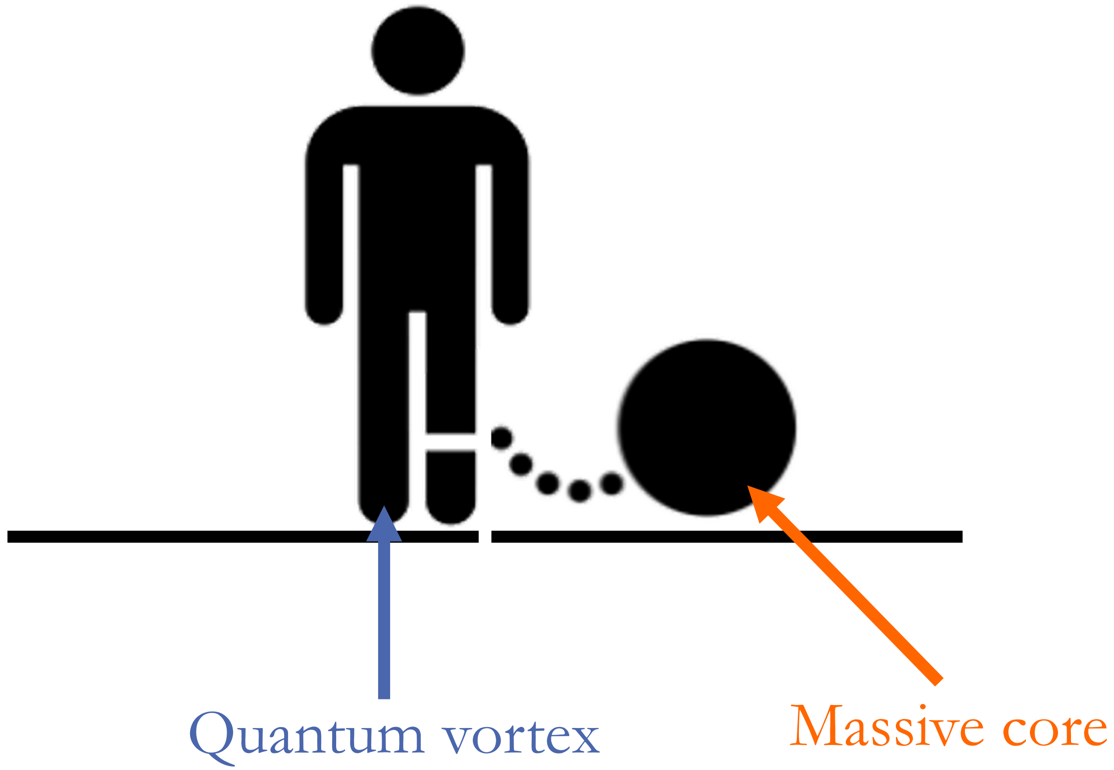
$$M_{b,2}\ddot{\boldsymbol{r}}_2 = k_2\dot{\boldsymbol{r}}_2 \times (-m_a n_a \hat{z}) + \frac{m_a n_a}{2\pi}\left[k_2 k_2' \frac{\boldsymbol{r}_2 - \boldsymbol{r}_2'}{|\boldsymbol{r}_2 - \boldsymbol{r}_2'|^2} + k_2 k_1 \frac{\boldsymbol{r}_2 - \boldsymbol{r}_1}{|\boldsymbol{r}_2 - \boldsymbol{r}_1|^2} + k_2 k_1' \frac{\boldsymbol{r}_2 - \boldsymbol{r}_1'}{|\boldsymbol{r}_2 - \boldsymbol{r}_1'|^2}\right]$$

ACTIVE

VORTEX CORES

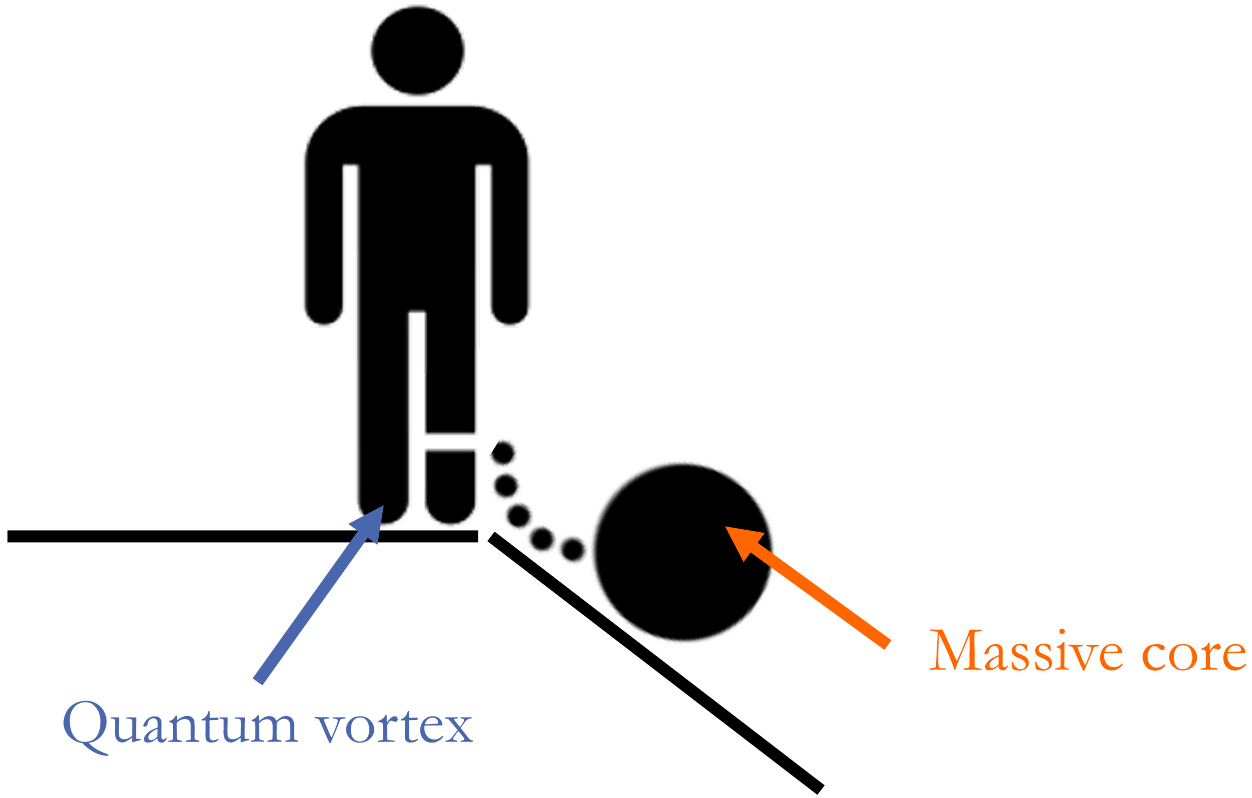
[A. Richaud, G. Lamporesi, M. Capone, A. Recati, Phys. Rev. **A** 107, 053317 (2023)]

A change of perspective



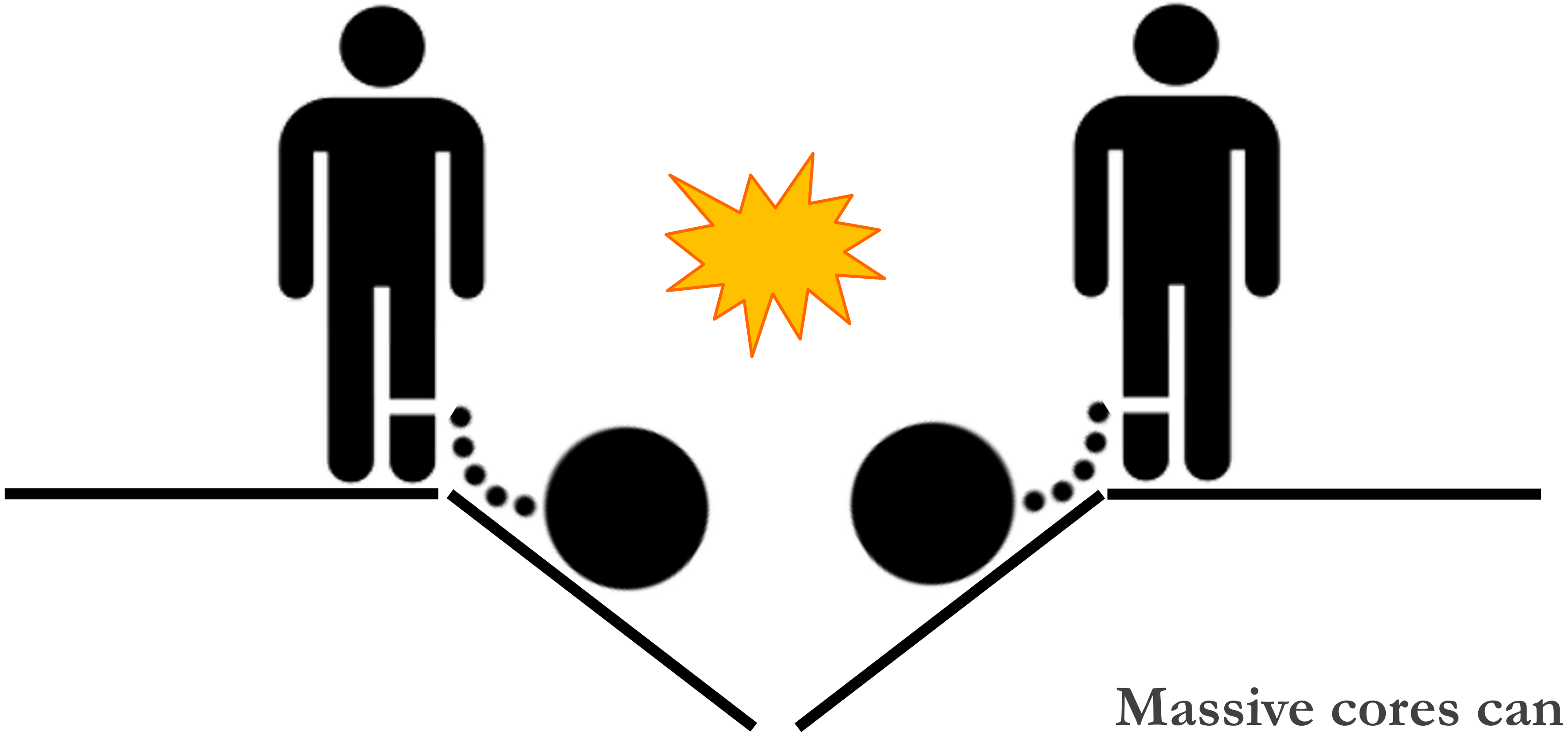
So far, the massive core has played a 'passive' role, meaning that it is like a **burden** which quantum vortices, deliberately or accidentally, have to live with.

A change of perspective



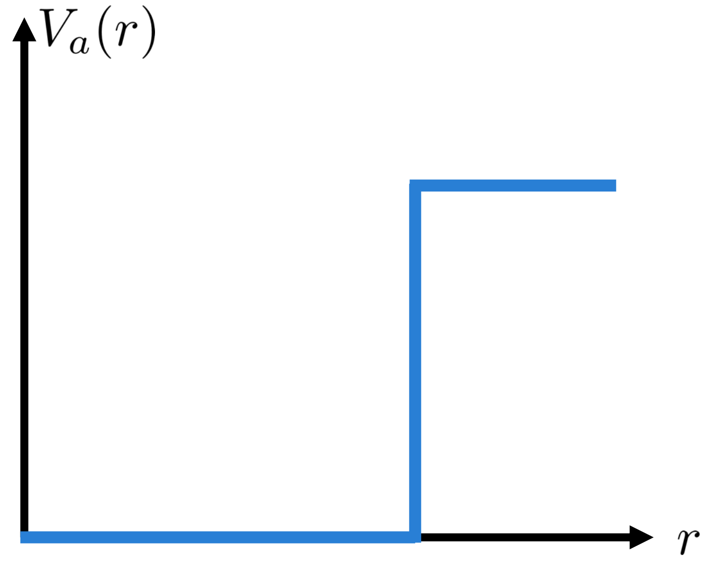
But the massive core can actually **drive** the hosting vortex!

A change of perspective

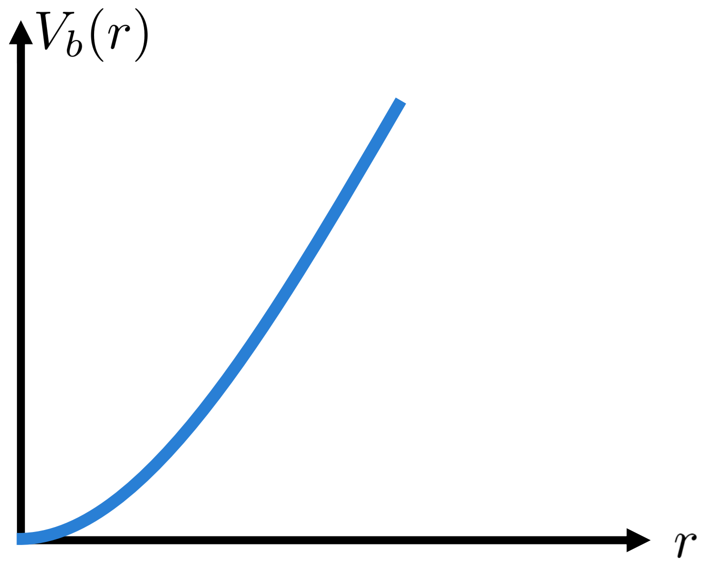


Massive cores can drive vortex collisions!

Species-selective traps

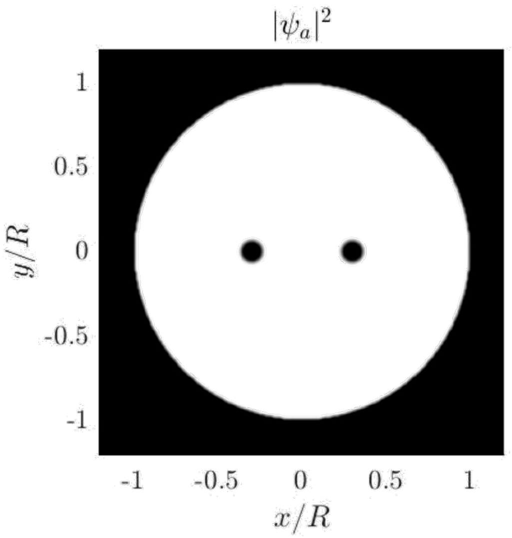


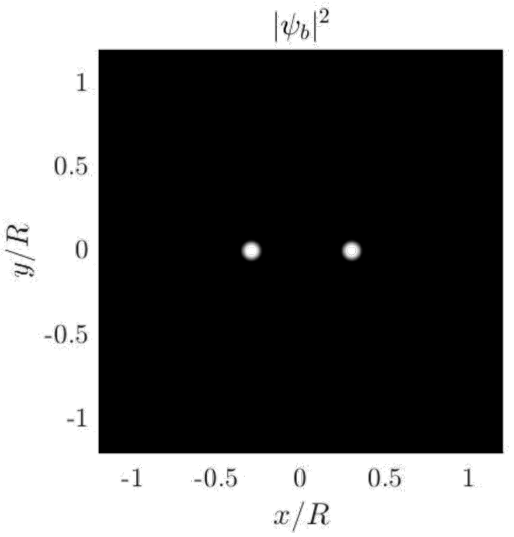
Species-a feels a **hard-wall** circular potential well



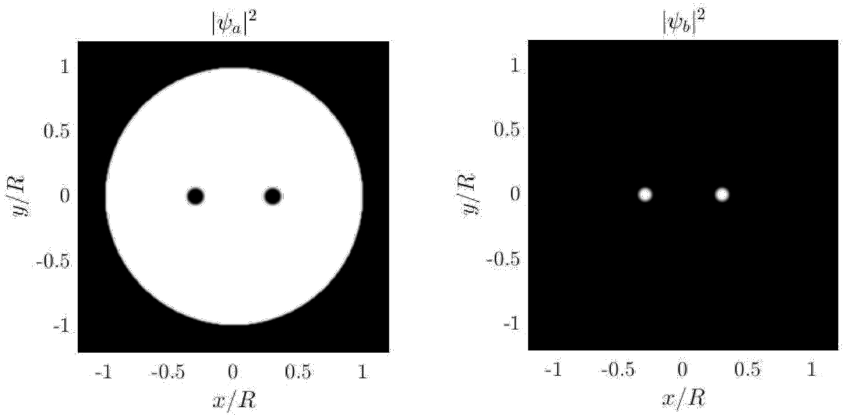
Species-b feels a **harmonic** potential

Massive Point-Vortex Model





Massive Point-Vortex Model



$$L = \sum_{j=1}^{2}\left[\frac{m_j}{2}\left(\dot{x}_j^2 + \dot{y}_j^2\right) + \frac{k_j\rho_*}{2}\left(y_j\dot{x}_j - x_j\dot{y}_j\right)\right] - \frac{\rho_*}{4\pi}\left\{k_1k_2\log\frac{|R^2 - z_1\bar{z}_2|^2}{|R(z_1 - z_2)|^2} + k_1^2\log\left(1 - \frac{|z_1|^2}{R^2}\right) + k_2^2\log\left(1 - \frac{|z_2|^2}{R^2}\right)\right\} + \sum_{j=1}^{2}\frac{1}{2}m_j\omega_b^2\left(x_j^2 + y_j^2\right)$$

- Blue: Usual Superfluid Vortex Dynamics in a circular box
- Green: Inertial term ensuing from the core mass
- Red: Harmonic trapping of species-b cores

Massive Point-Vortex Model

$$L = \sum_{j=1}^{2}\left[\frac{m_j}{2}(\dot{x}_j^2 + \dot{y}_j^2) + \frac{k_j\rho_*}{2}(y_j\dot{x}_j - x_j\dot{y}_j)\right] -$$

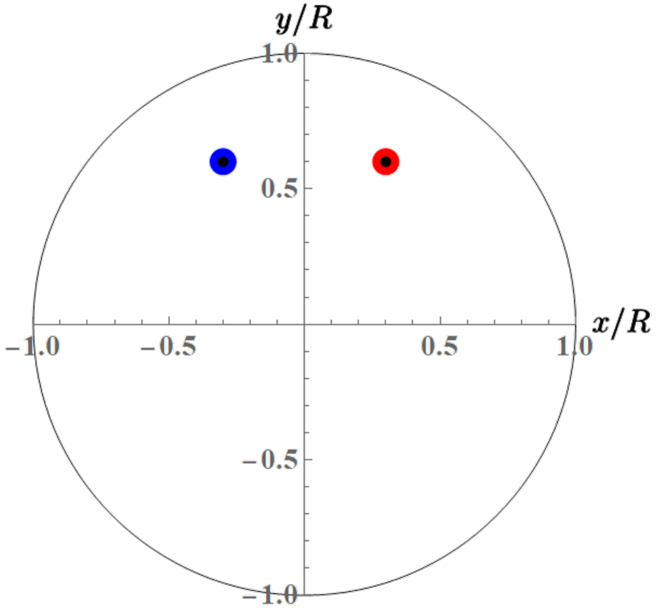
$$\frac{\rho_*}{4\pi}\left\{k_1k_2\log\frac{|R^2 - z_1\bar{z}_2|^2}{|R(z_1 - z_2)|^2} + k_1^2\log\left(1 - \frac{|z_1|^2}{R^2}\right) + k_2^2\log\left(1 - \frac{|z_2|^2}{R^2}\right)\right\} + \sum_{j=1}^{2}\frac{1}{2}m_j\omega_b^2(x_j^2 + y_j^2)$$

Euler-Lagrange Equations:

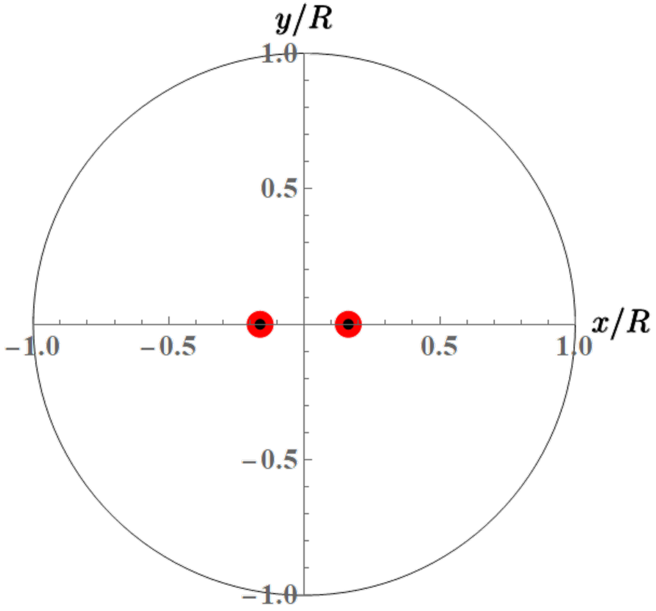
$$m_j\ddot{\vec{r}}_j = k_j\rho_*\vec{u}_3 \wedge \dot{\vec{r}}_j + \rho_*\frac{k_j}{2\pi}\left[k_i\frac{\vec{r}_j - \vec{r}_i}{|\vec{r}_j - \vec{r}_i|^2} + k_j\frac{\vec{r}_j}{R^2 - r_j^2} + k_i\frac{R^2\vec{r}_i - r_i^2\vec{r}_j}{R^4 - 2R^2\vec{r}_i\vec{r}_j + r_i^2r_j^2}\right] - m_j\omega_b^2\vec{r}_j,$$

Predictions of the Massive Point Vortex Model

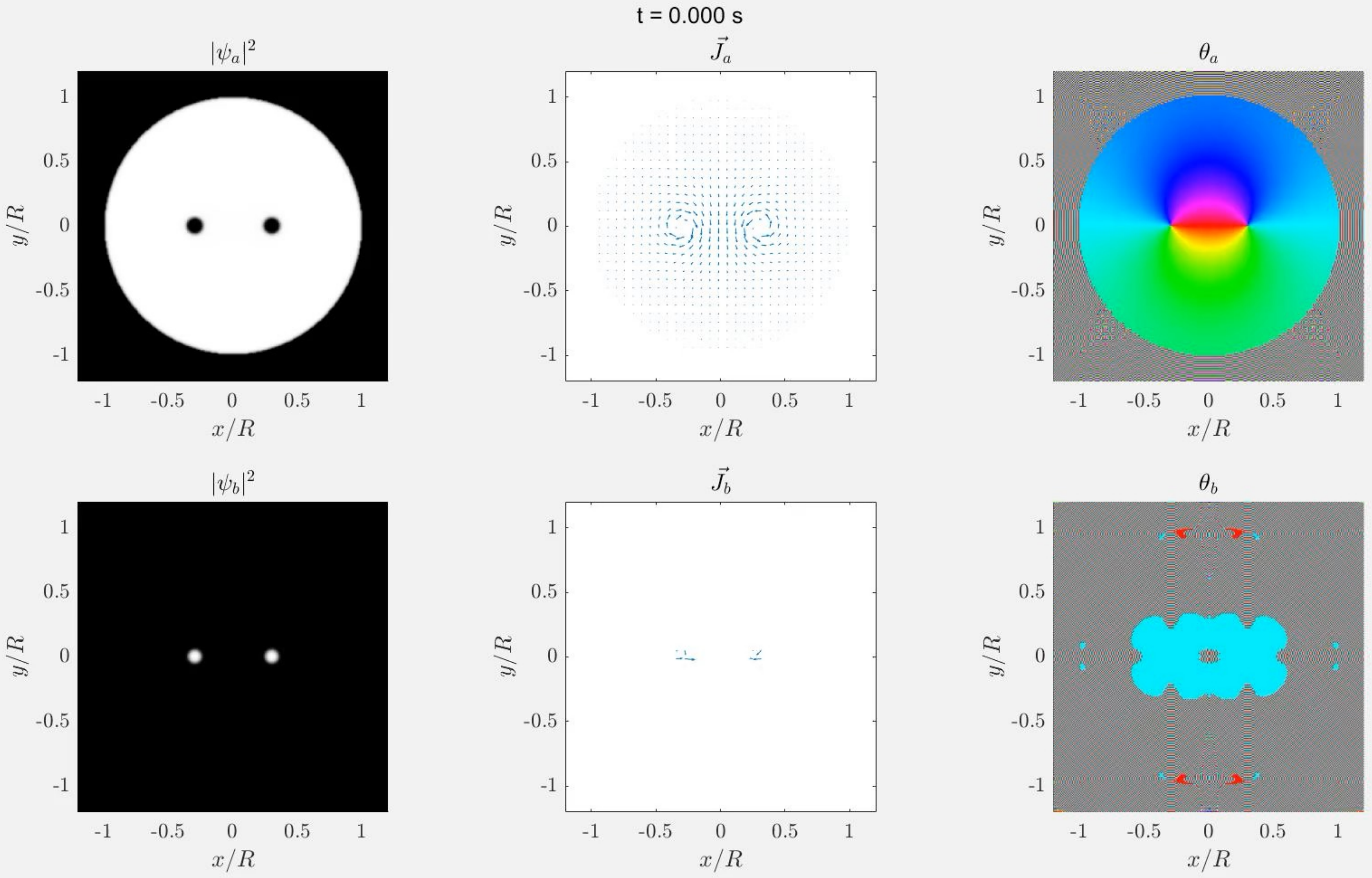
$$m_j \ddot{\vec{r}}_j = k_j \rho_* \vec{u}_3 \wedge \dot{\vec{r}}_j + \rho_* \frac{k_j}{2\pi} \left[k_i \frac{\vec{r}_j - \vec{r}_i}{|\vec{r}_j - \vec{r}_i|^2} + k_j \frac{\vec{r}_j}{R^2 - r_j^2} + k_i \frac{R^2 \vec{r}_i - r_i^2 \vec{r}_j}{R^4 - 2R^2 \vec{r}_i \vec{r}_j + r_i^2 r_j^2} \right] - m_j \omega_b^2 \vec{r}_j,$$



Pair of counter-rotating massive vortices.

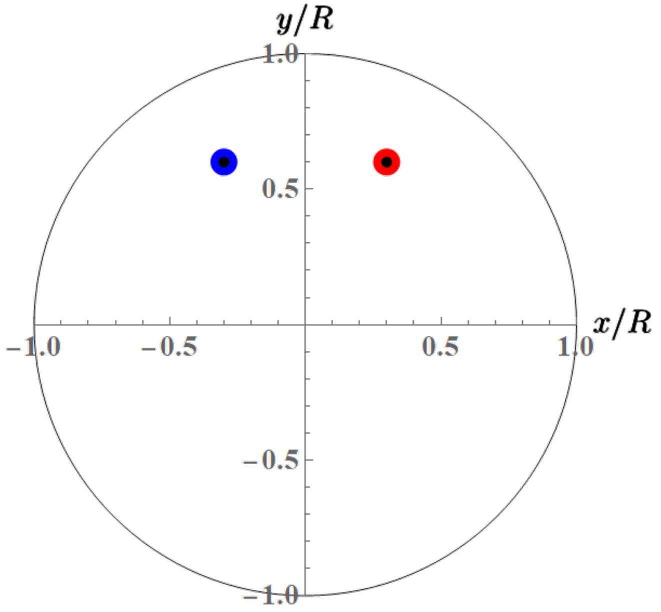


Pair of co-rotating massive vortices.



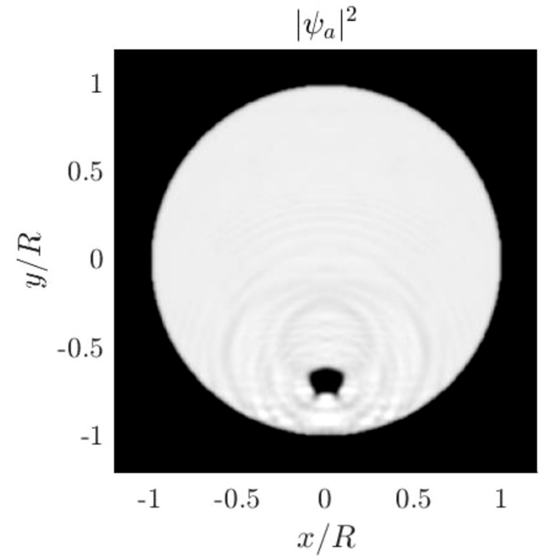
t = 0.000 s
$|\psi_a|^2$
$\vec{J}_a$
θ_a
$|\psi_b|^2$
$\vec{J}_b$
θ_b
x/R
y/R

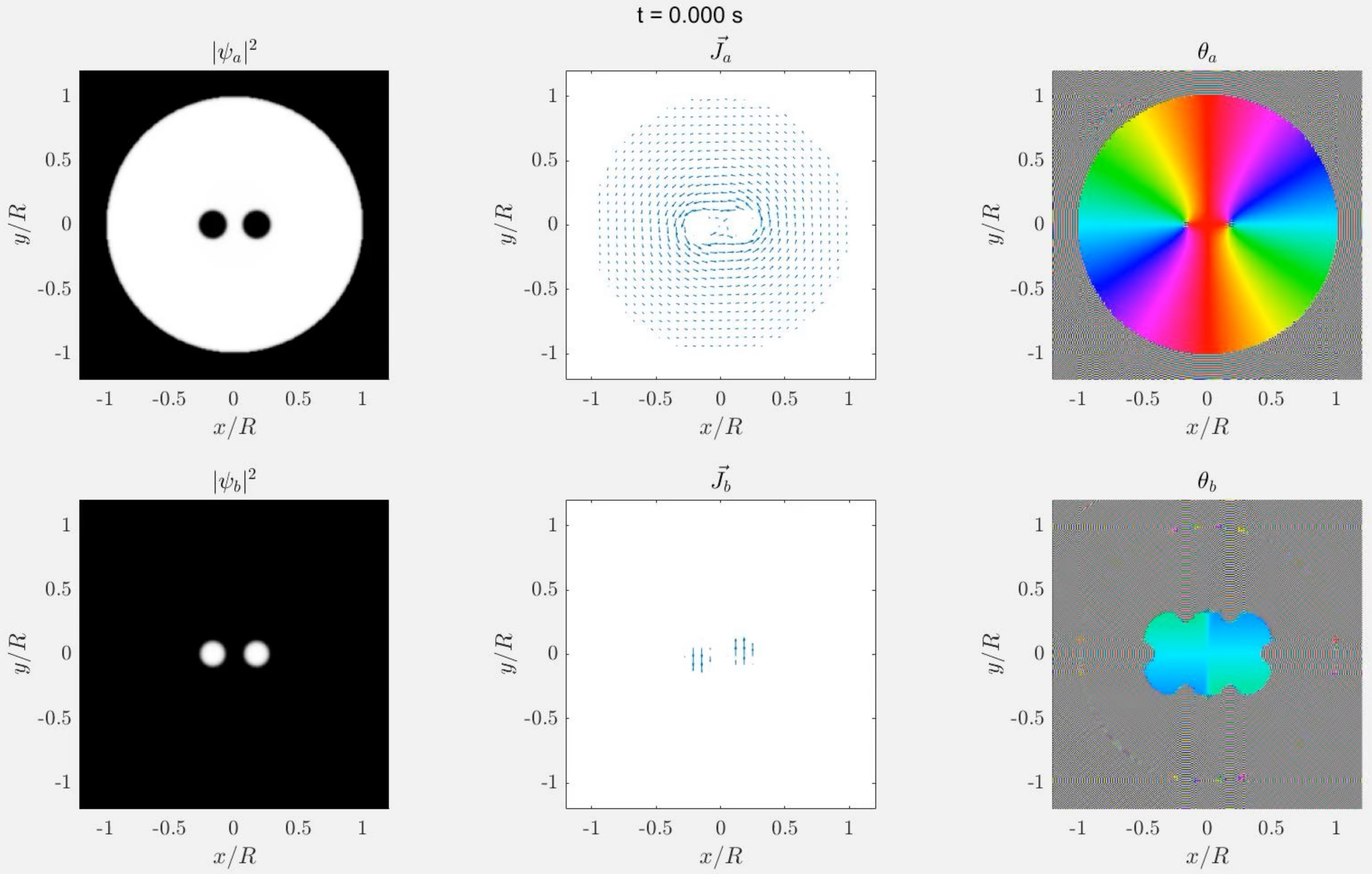
Collision of vortex/antivortex pair



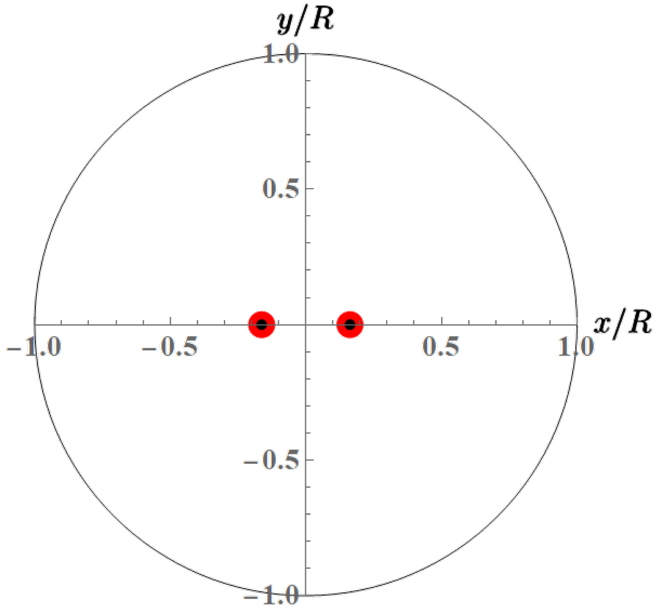
Mutual annihilation of the vortices

Sound-wave explosion!

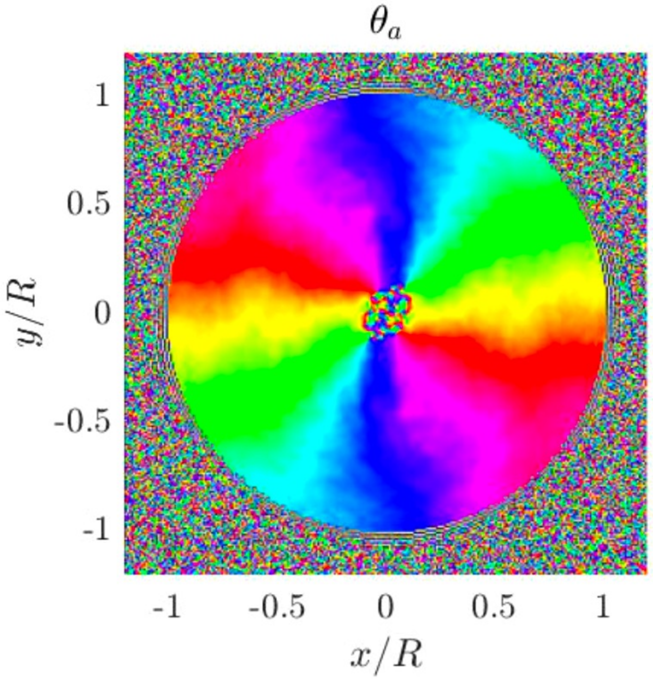



t = 0.000 s
$|\psi_a|^2$
$\vec{J}_a$
θ_a
$|\psi_b|^2$
$\vec{J}_b$
θ_b
x/R
y/R
-1
-0.5
0
0.5
1

Collision of vortex/vortex pair



Stabilization of a double-charge vortex with filled core!

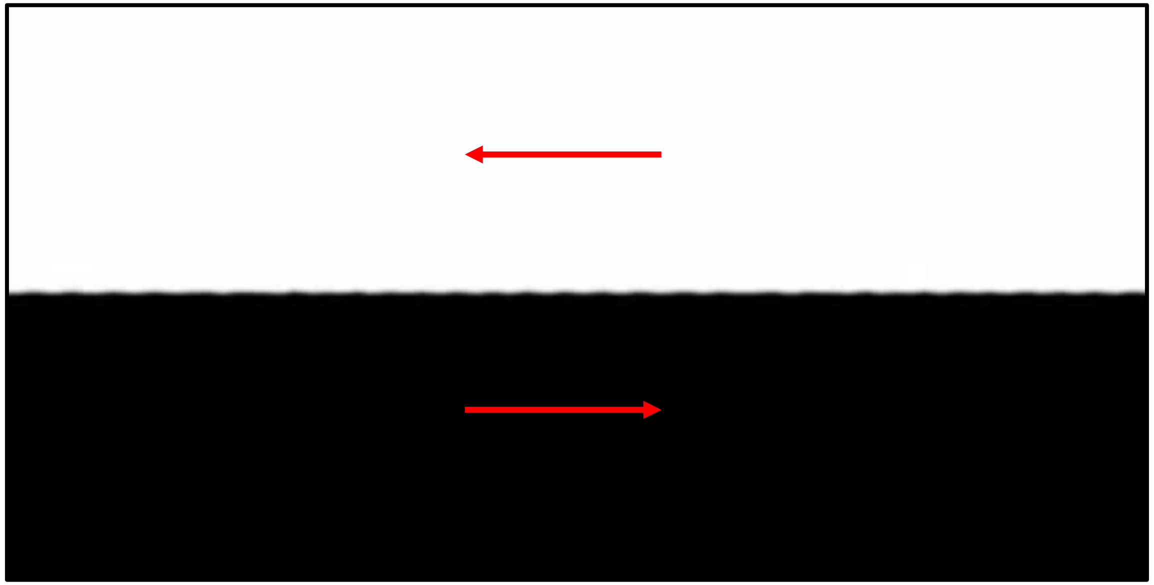


COLLECTIVE EFFECTS IN MASSIVE-VORTEX SYSTEMS:

SUPERFLUID KELVIN – HELMHOLTZ INSTABILITY

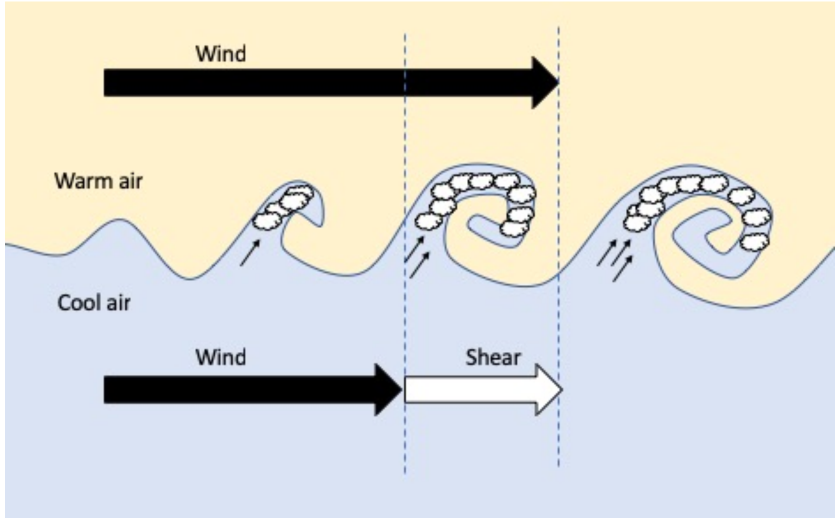
[M. Caldara, A. Richaud, M. Capone, P. Massignan arXiv:2403.11987]

Classical Kelvin-Helmholtz instability



At the interface between two fluid layers in relative motion, infinitesimal fluctuations can be exponentially amplified, inducing vorticity and the breakdown of the laminar flow.

Kelvin-Helmholtz instability in atmospheric phenomena



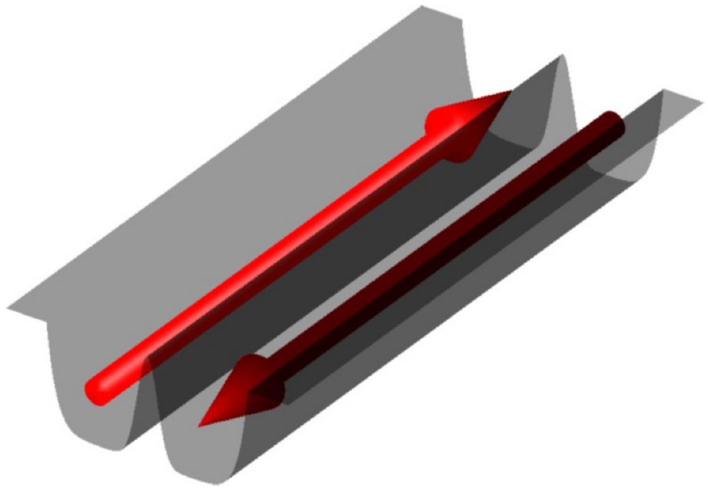
Kelvin-Helmholtz instability rendered visible by clouds, known as fluctus

Kelvin-Helmholtz instability in atmospheric phenomena

Kelvin-Helmholtz instability in the sky of Rivarolo Canavese (TO), 24th December 2023

Kelvin-Helmholtz instability in superfluids

First theoretical study of the Kelvin-Helmholtz instability in a single-component BEC
[A. W. Baggaley, N. G. Parker, Phys. Rev. A **97**, 053608 (2018)]



An atomic BEC confined in a channel is divided by a **central barrier**.

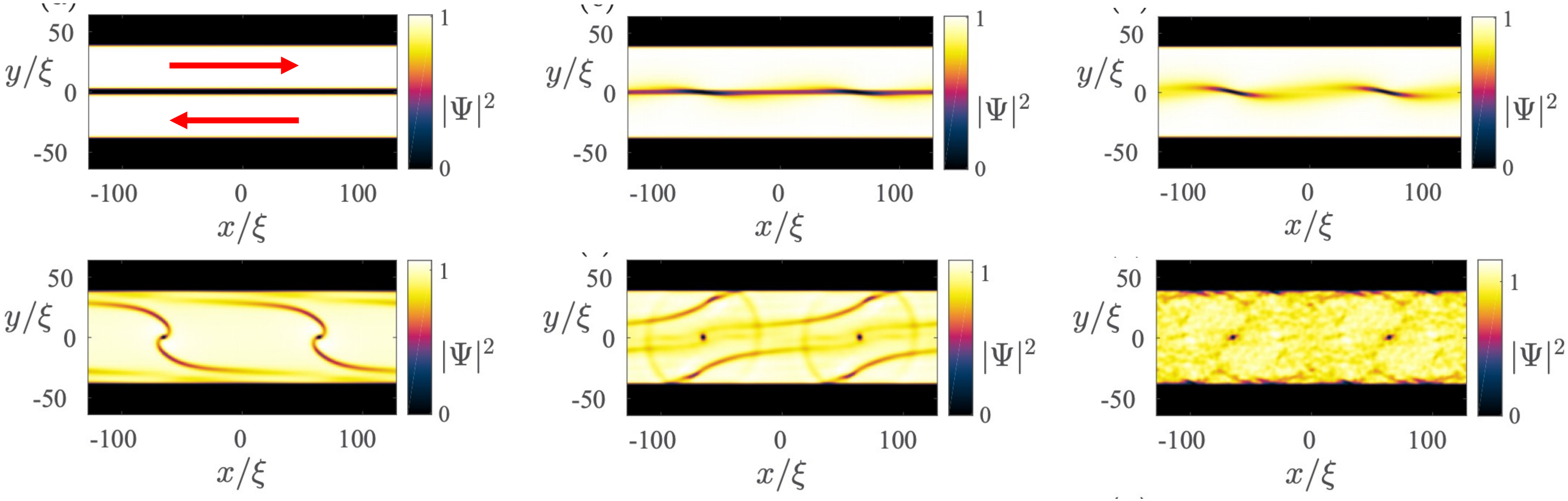
The superfluid on either side **flows in opposite directions**.

The central **barrier is then lowered** to create a region of high shear.

Kelvin-Helmholtz instability in superfluids

First theoretical study of the Kelvin-Helmholtz instability in a single-component BEC

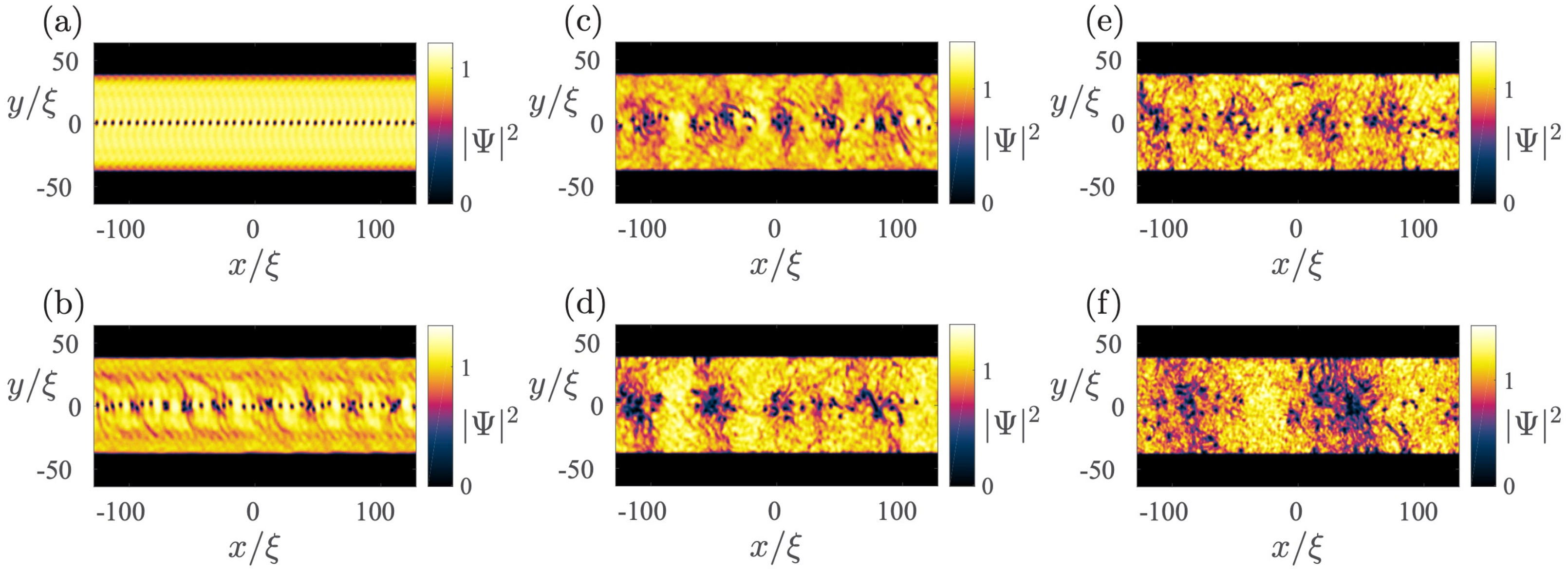
[A. W. Baggaley, N. G. Parker, Phys. Rev. A **97**, 053608 (2018)]



Kelvin-Helmholtz instability in superfluids

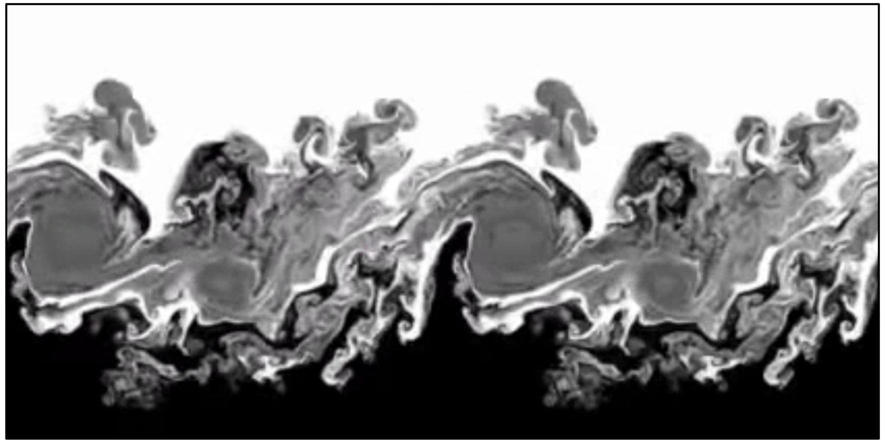
First theoretical study of the Kelvin-Helmholtz instability in a single-component BEC

[A. W. Baggaley, N. G. Parker, Phys. Rev. A **97**, 053608 (2018)]

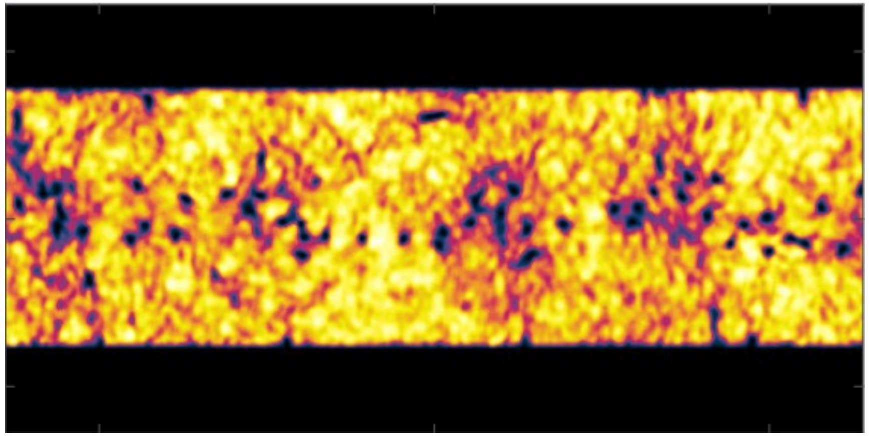


The **vortex chain** that forms at the interface is **unstable** and **rolls up** into small **clusters** of same-sign vortices. Over time, these clusters merge to create **larger clusters** (mimicking classical patches of vorticity).

Difference between classical KHI and superfluid KHI

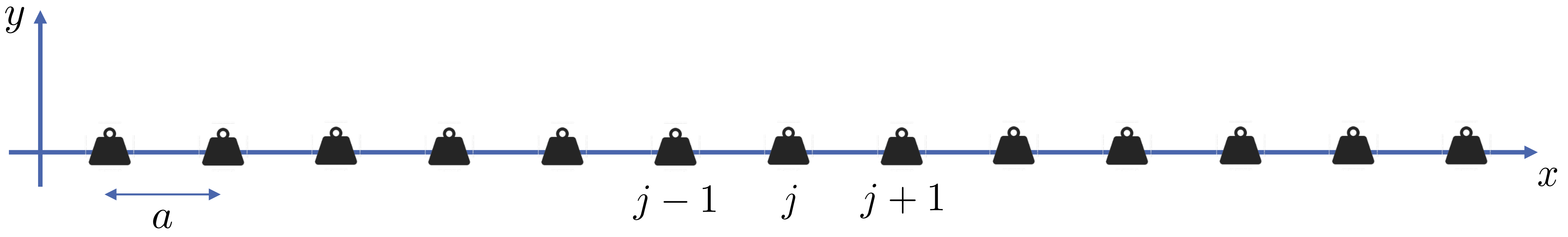


In a **classical** fluid, **vorticity** constitutes a **continuous** field.



In a superfluid, **vorticity** is quantized and the number of resulting vortices is **finite** and only depends on the initial **relative velocity** of the two fluids.

Toy-model: row of massive vortices



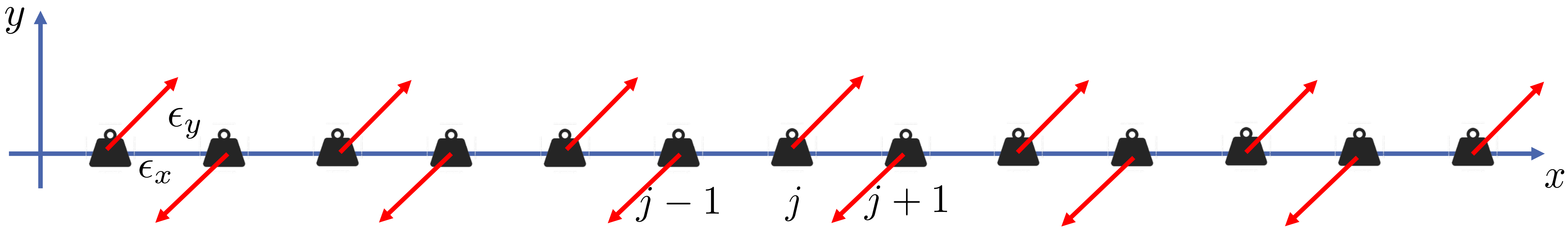
Equations of motion:

$$M_c \ddot{\boldsymbol{r}}_j = \kappa \dot{\boldsymbol{r}}_j \times (-m_a n_a \hat{z}) + \frac{m_a n_a}{2\pi} \kappa^2 \sum_{i=1}^{+\infty} \left(\frac{\boldsymbol{r}_j - \boldsymbol{r}_{j+i}}{|\boldsymbol{r}_j - \boldsymbol{r}_{j+i}|^2} + \frac{\boldsymbol{r}_j - \boldsymbol{r}_{j-i}}{|\boldsymbol{r}_j - \boldsymbol{r}_{j-i}|^2} \right)$$

Fixed point:

$$\boldsymbol{r}_j = (a\,j,\,0) \quad \forall\, t$$

Toy-model: row of massive vortices



In the most unstable mode, all the vortices are displaced from their equilibrium position according to:

$$(a(j \pm i), 0) \quad \rightarrow \quad \left(a(j \pm i) + (-1)^i \epsilon_x, (-1)^i \epsilon_y\right), \qquad i = 0, 1, \ldots, N_v$$

The **linearized** equations of motion are:

$$M_c \ddot{\epsilon}_x = -\kappa m_a n_a \dot{\epsilon}_y - \frac{m_a n_a \pi \kappa^2}{4a^2} \epsilon_x$$

$$M_c \ddot{\epsilon}_y = \kappa m_a n_a \dot{\epsilon}_x + \frac{m_a n_a \pi \kappa^2}{4a^2} \epsilon_y$$

Toy-model: row of massive vortices

The linearized equations of motion are:

$$M_c \ddot{\epsilon}_x = -\kappa m_a n_a \dot{\epsilon}_y - \frac{m_a n_a \pi \kappa^2}{4a^2} \epsilon_x$$

Recoil force along x

$$M_c \ddot{\epsilon}_y = \kappa m_a n_a \dot{\epsilon}_x + \frac{m_a n_a \pi \kappa^2}{4a^2} \epsilon_y$$

Destabilizing force along y

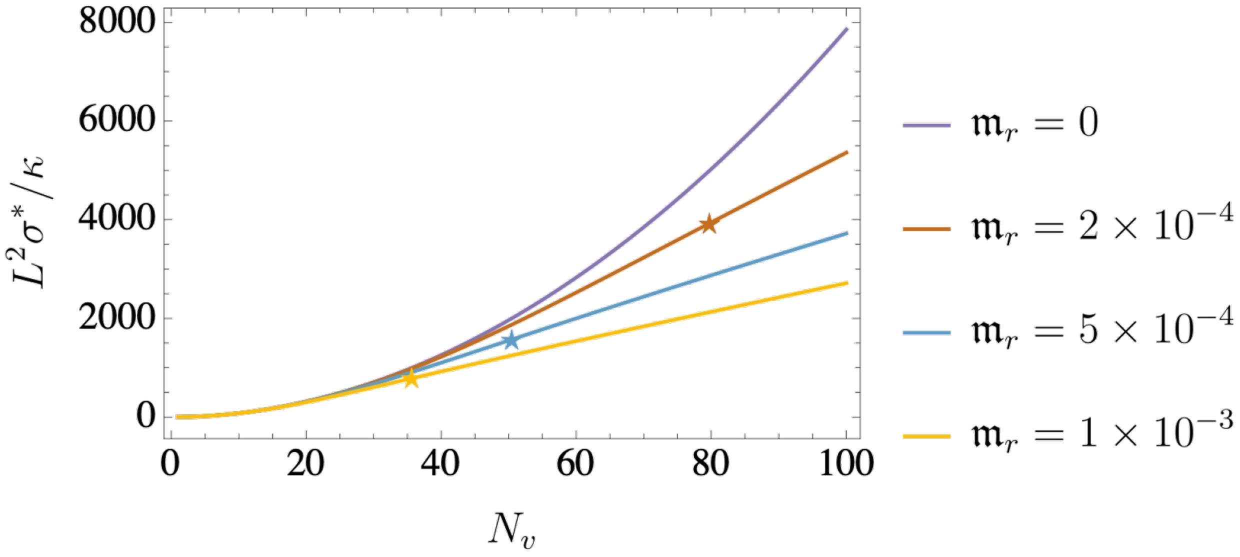
A straightforward analysis of these two coupled ODEs allows to compute the **maximum instability growth rate**:

$$\sigma^* = \frac{\kappa m_a n_a}{M_c \sqrt{2}} \sqrt{-1 + \sqrt{1 + \left(\frac{M_c \pi}{2a^2 m_a n_a} \right)^2}}$$

$$\epsilon_x,\ \epsilon_y \sim e^{\sigma^* t}$$

Toy-model: row of massive vortices

$$\sigma^* = \frac{\kappa m_a n_a}{M_c\sqrt{2}}\sqrt{-1+\sqrt{1+\left(\frac{M_c \pi}{2a^2 m_a n_a}\right)^2}}$$



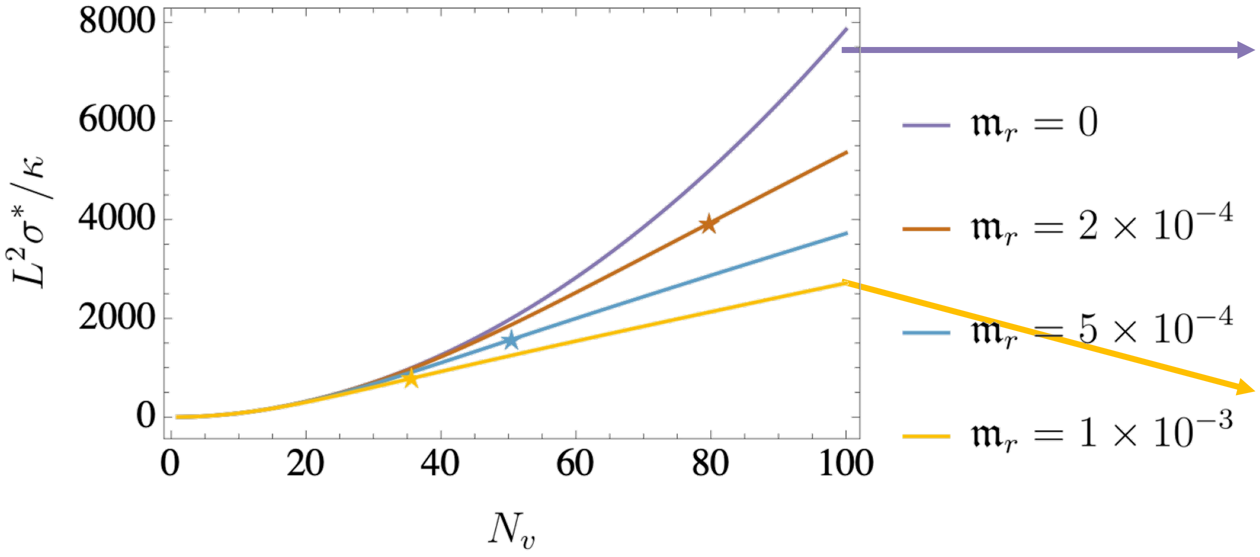
Core mass suppresses instability!

$L = N_v a$

$\mathfrak{m}_r = M_c/(m_a n_a L^2)$ is a dimensionless mass ratio.

Core mass affects the scaling

$$\sigma^* = \frac{\kappa m_a n_a}{M_c\sqrt{2}}\sqrt{-1+\sqrt{1+\left(\frac{M_c\pi}{2a^2 m_a n_a}\right)^2}}$$



In the massless limit:

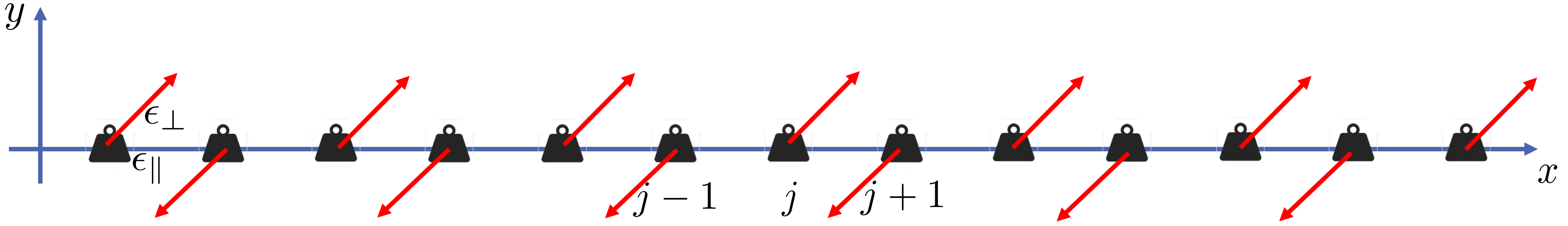
$$\sigma^* \propto N_v^2$$

[H. Aref, J. Fluid Mech. **290**, 167 (1995)]

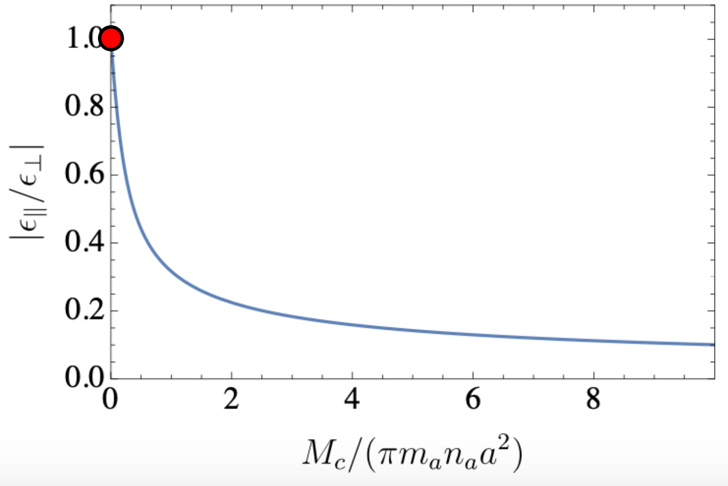
In the massive case:

$$\sigma^* \propto N_v$$

Longitudinal vs transversal instability

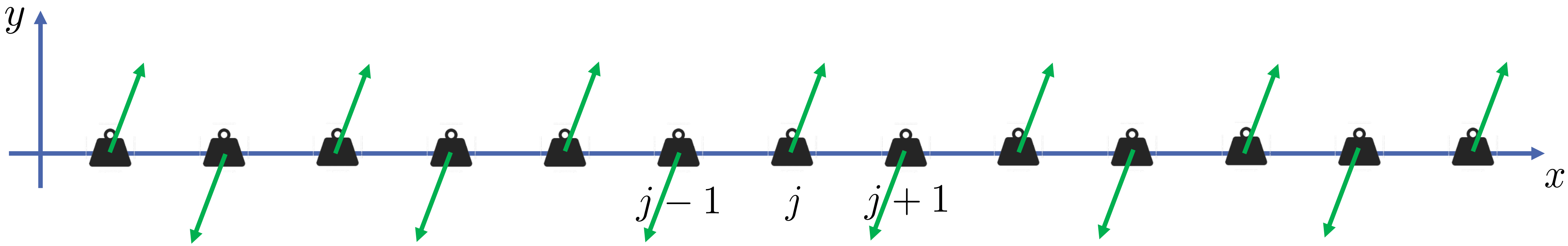


The eigenvector associated to σ^* provides info concerning the way a perturbation is amplified.

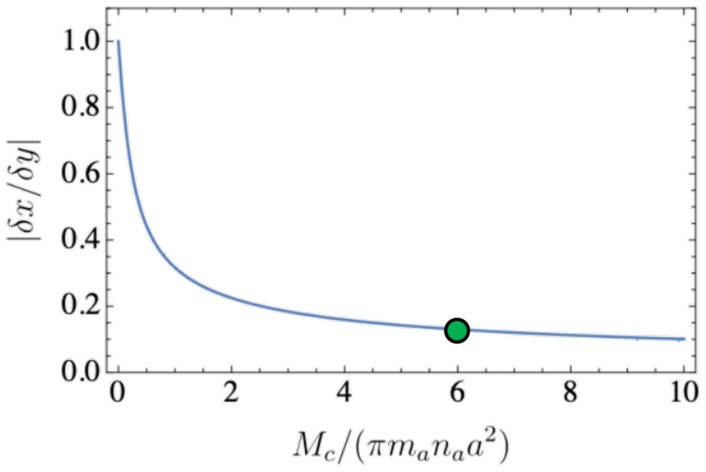


In the massless limit, the instability is both **longitudinal** and **transversal**, to the same extent.

Longitudinal vs transversal instability

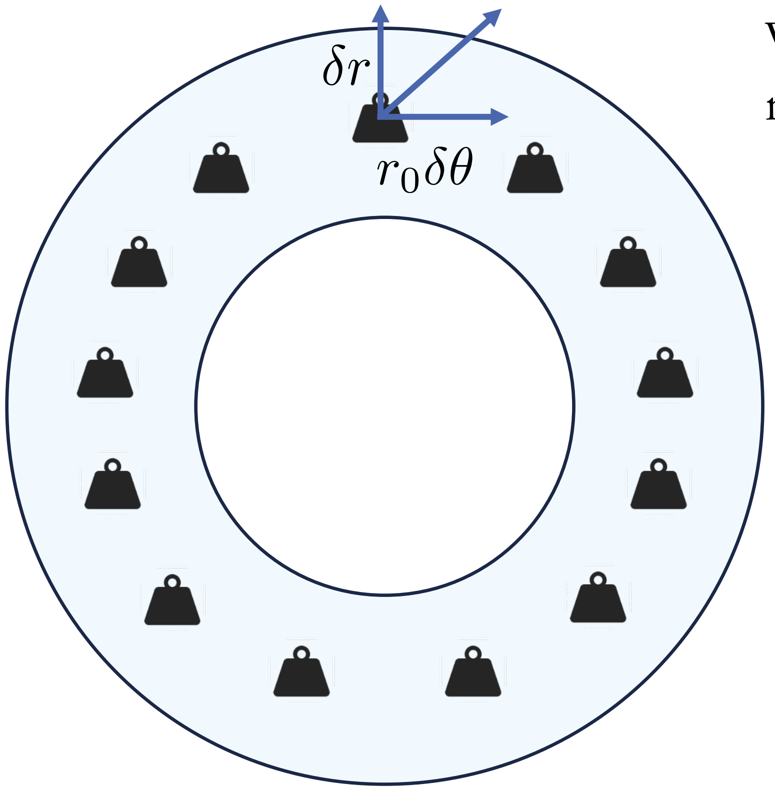


The eigenvector associated to σ^* provides info concerning the way a perturbation is amplified.

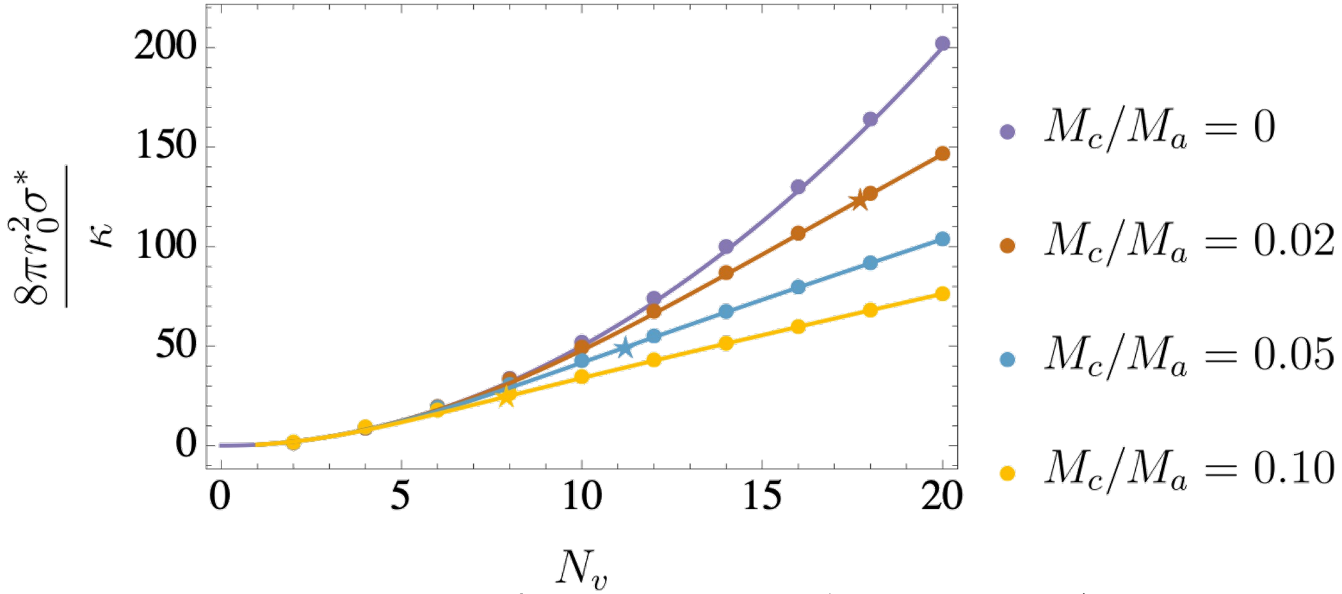


The presence of core mass makes the instability **more transverse** and less longitudinal.

Radial vs azimuthal instability



We generalized the result to circular arrays (i.e. necklaces) of massive vortices.

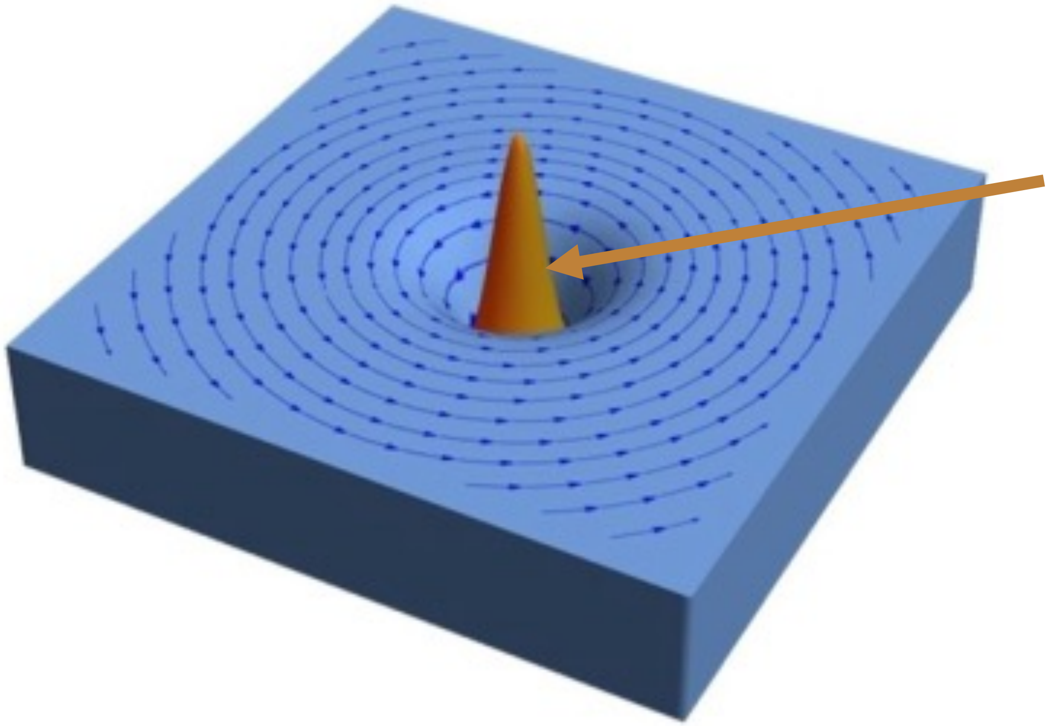


The presence of core mass makes the instability **slower and more radial.**

[M. Caldara, A. Richaud, M. Capone, P. Massignan arXiv:2403.11987]

CONCLUSIONS

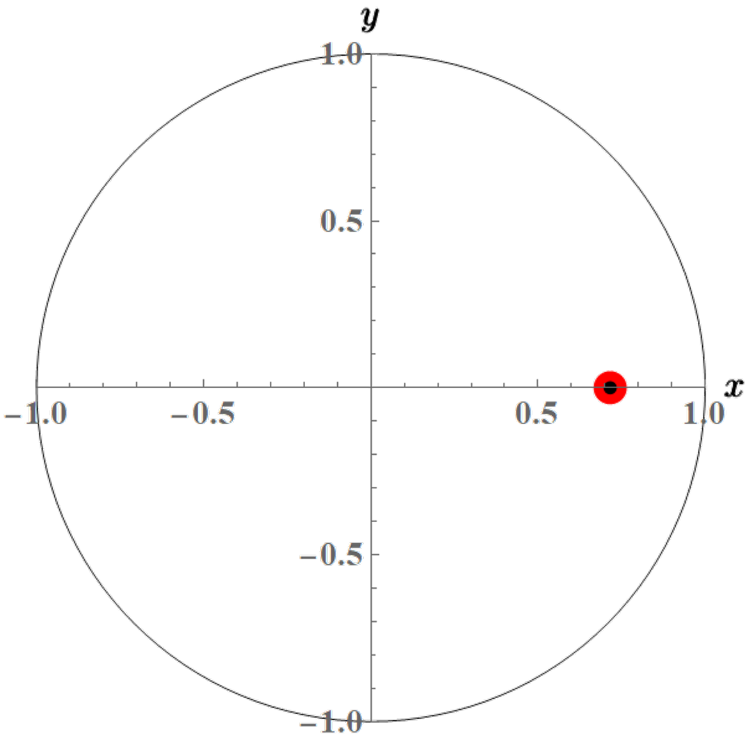
Most real superfluid vortices are massive



- **Tracer particles**
- **Thermal atoms**
- **Quasiparticle bound states**
- **Another (minority) BEC**

Superfluid vortices are often filled by massive cores (deliberately or accidentally!)

Massive Point Vortex Model



The massive point vortex model, unlike its massless counterpart, leads to second-order equations of motion:

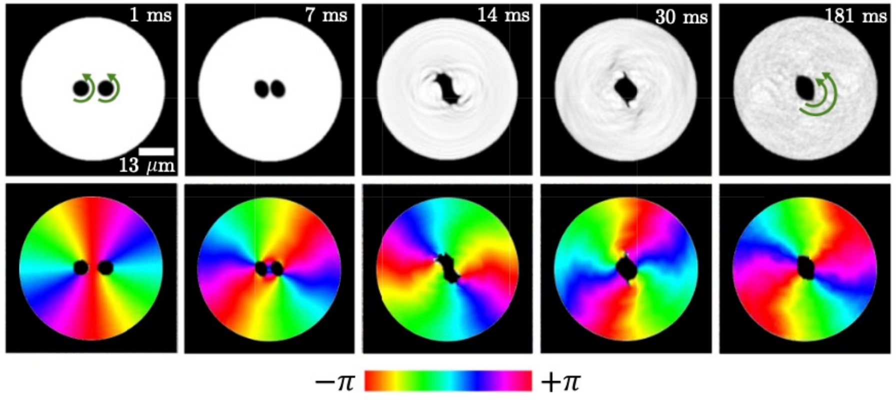
$$M_b \ddot{\boldsymbol{r}}_0 = 2 n_a \pi \hbar \left[\hat{z} \times \dot{\boldsymbol{r}}_0 + \frac{\hbar}{m_a} \frac{\boldsymbol{r}_0}{R^2 - r_0^2} \right]$$

The dynamical signature of vortex mass is represented by small-amplitude transverse oscillations.

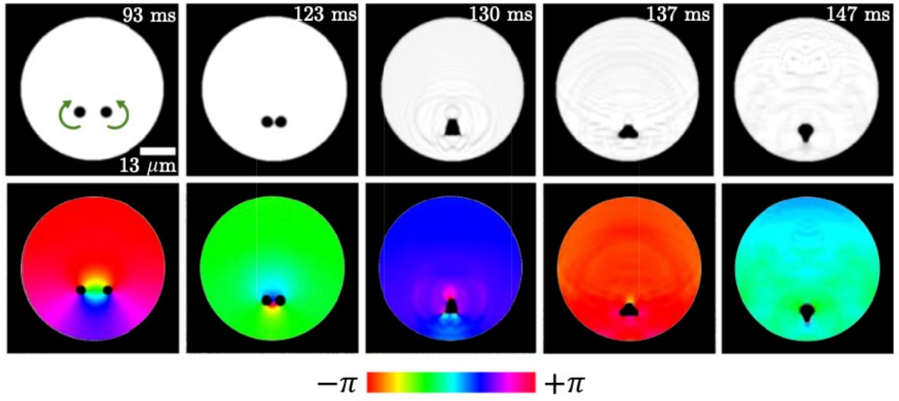
[A. Richaud, V. Penna, A. L. Fetter, Phys. Rev. A **103**, 023311 (2021)]

Mass-driven vortex collisions

Inertial cores can be used to drive vortex collisions:



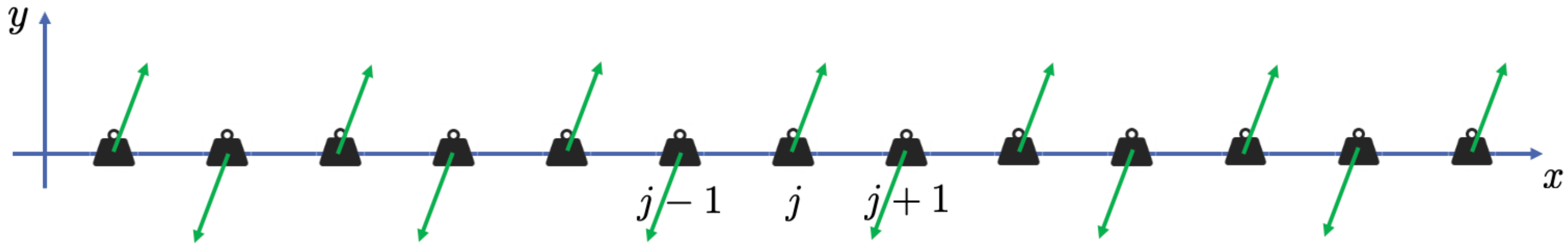
Vortex pair

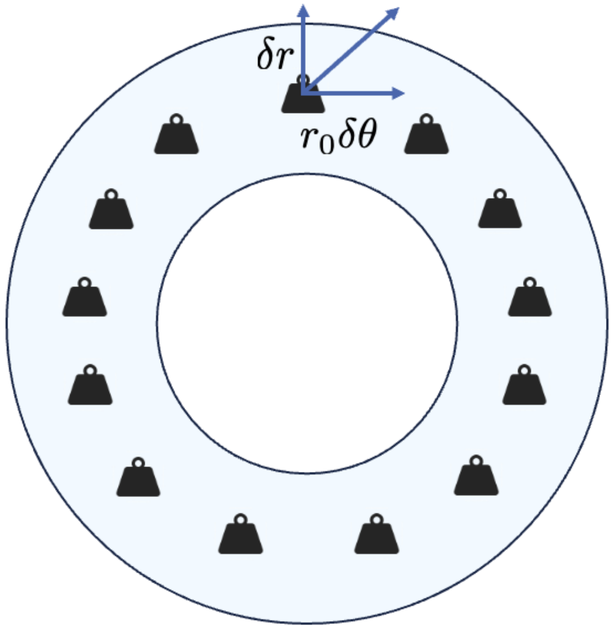


Vortex dipole

[A. Richaud, G. Lamporesi, M. Capone, A. Recati, Phys. Rev. **A** 107, 053317 (2023)]

Inertial effects on vortex-array instabilities





The presence of core mass makes the instability **slower and more transverse, and also alters the scaling law $\boldsymbol{\sigma}^*(\boldsymbol{N_v})$.**

[M. Caldara, A. Richaud, M. Capone, P. Massignan arXiv:2403.11987]

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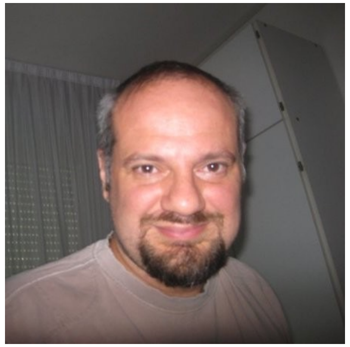

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Thanks for your attention!

QUESTIONS ?

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