Dark Matter Problems
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1. **Dark matter direct detection**
(problem originally formulated by David Cerdeño)

In Planetary Dependence of Melanoma (Zioutas and Valachovic, 2018), the authors claim to have observed a correlation between the occurrence of melanoma in the human skin and the annual modulation in the dark matter (DM) flux that arrives to the Earth.

- **Assume that the DM is a WIMP with mass of order 100 GeV. Calculate the number of expected DM recoils on the human skin over a whole human lifetime. Use the upper limit on the DM-nucleus scattering from recent experiments (LZ for example).**
- **Consider the possibility of lighter DM particles, so that the number density is larger, and the cross-section less tightly constrained by experiments. How large must the scattering cross section be so as to have a substantial number of interactions?**
- **Are there any sources of background that can give similar recoils (and that we are exposed to on a daily basis)?**
- **Based on all of the above, do you think that it is plausible that DM causes skin cancer?**

Very rough calculation (done on bus without a calculator...), should hopefully capture essentials, but likely contains errors.

Total number of events is given by

$$N_{\rm events} = n v N_{\rm T} \sigma T \,, \tag{1}$$

where n and v are the number density and speed of the WIMPs respectively, $N_{\rm T}$ is the number of target atoms σ the WIMP-nucleus scattering cross-section and T the total time.

$$\begin{aligned}
n &= \frac{\rho_\odot}{m_\chi} \approx \frac{0.3\,{\rm GeV\,cm^{-3}}}{100\,{\rm GeV}} = 3\times 10^{-3}\,{\rm cm^{-3}} = 3\times 10^{3}\,{\rm m^{-3}} \sim 10^{3}\,{\rm m^{-3}} \,, \\
v &\approx 200\,{\rm km\,s^{-1}} \sim 10^{5}\,{\rm m\,s^{-1}} \,, \\
f N_{\rm T} &\approx f \frac{m_{\rm person}}{m_{^{16}{\rm O}}} \approx f \frac{60\,{\rm kg}}{2.7\times 10^{-26}\,{\rm kg}} \sim f 10^{27} \,, \\
\sigma &\approx A^2 \sigma_{\rm p} = (16)^2 \times 10^{-47}\,{\rm cm^2} \sim (100)\times 10^{-51}\,{\rm cm^2} \sim 10^{-49}\,{\rm m^2} \,, \\
T &\approx 100\,{\rm yr} = 10^2 \times 3.1 \times 10^7\,{\rm s} \approx 10^9\,{\rm s} \,.
\end{aligned}$$

where $f \ll 1$ is the fraction of the total mass of a human in the form of skin.

Inserting these numbers into Eq. (1) gives $N_{\rm events} \sim f 10^{-5} \ll 1$!!

2. **Primordial Black Hole dark matter**

(As we'll see in lecture 3) Primordial Black Holes (PBHs) are black holes that may form in the early Universe. Following the LIGO-Virgo detection of gravitational waves from the mergers of $\sim 10\, M_{\odot}$ black hole binaries, PBHs have received a lot of attention, in particular as a DM candidate. In this question we'll work through the details of some simple calculations regarding the abundance of PBHs.

- **PBHs that form during radiation domination have mass roughly equal to the horizon mass, $M_{\rm H} = tc^3/G$, at the time they form. Show that the fraction of DM in the form of PBHs today, $f_{\rm PBH}$, is related to the initial abundance of PBHs, $\beta \equiv \rho_{\rm PBH}/\rho_{\rm rad}$ where $\rho_{\rm PBH}$ and $\rho_{\rm rad}$ are the densities of PBHs and radiation respectively at the time the PBHs form, by**

$$f_{\rm PBH} \sim 10^9 \left(\frac{M}{M_{\odot}} \right)^{-1/2} \beta \, . \tag{2}$$

  For this rough calculation you can, e.g., assume the universe is matter dominated from radiation-matter equality to the present day, and ignore the variation in the number of relativistic degrees of freedom.

  Using the variation of the matter and radiation densities with the scale factor we can relate the initial ('i', at the time of PBH formation) and present day ('0') PBH and radiation densities:

$$\begin{aligned} \rho_{\rm PBH,i} &= \rho_{\rm PBH,0} \left(\frac{a_0}{a_{\rm i}} \right)^3 , \\ \rho_{\rm rad,i} &= \rho_{\rm rad,0} \left(\frac{a_0}{a_{\rm i}} \right)^4 , \end{aligned}$$

  so that

$$\beta \equiv \frac{\rho_{\rm PBH,i}}{\rho_{\rm rad,i}} = \frac{\rho_{\rm PBH,0}}{\rho_{\rm rad,0}} \left(\frac{a_{\rm i}}{a_0} \right) ,$$

  We can break the ratio of scale factors into the product of ratios from the initial time to radiation-matter equality ('eq') and from equality to the present day:

$$\left(\frac{a_{\rm i}}{a_0} \right) = \left(\frac{a_{\rm i}}{a_{\rm eq}} \right) \left(\frac{a_{\rm eq}}{a_0} \right) ,$$

  Assuming that the Universe is matter dominated from equality until today

$$\left(\frac{a_{\rm eq}}{a_0} \right) = \left(\frac{t_{\rm eq}}{t_0} \right)^{2/3} = \left(\frac{2 \times 10^{12}\,{\rm s}}{4 \times 10^{17}\,{\rm s}} \right)^{2/3} = 2.8 \times 10^{-4} \, .$$

  The PBH mass, M, is roughly equal to the horizon mass which is proportional to t and the horizon mass at equality is equal to

$$M_{\rm eq} = \frac{c^3 t_{\rm eq}}{G} = \frac{(3 \times 10^8)^3 \times 2 \times 10^{12}}{6.7 \times 10^{-11}}\, {\rm kg} = 2.7 \times 10^{47}\, {\rm g} = 1.4 \times 10^{17} M_{\odot} \, ,$$

  so that

$$\left(\frac{a_{\rm i}}{a_{\rm eq}} \right) = \left(\frac{t_{\rm i}}{t_{\rm eq}} \right)^{1/2} = \left(\frac{M}{M_{\rm eq}} \right)^{1/2} = 4 \times 10^{-10} \left(\frac{M}{M_{\odot}} \right)^{1/2} ,$$

Finally, to calculate $f_{\rm PBH}$ we need to relate the radiation density today to the matter density today (using the values of the density parameters measured by Planck):

$$\rho_{\rm rad,0} = \rho_{\rm cdm,0} \left(\frac{\Omega_{\rm rad,0} h^2}{\Omega_{\rm cdm,0} h^2} \right) = \rho_{\rm cdm,0} \left(\frac{2.2 \times 10^{-5}}{0.12} \right) = 2.1 \times 10^{-4} \rho_{\rm cdm,0} \, .$$

Putting all of these pieces together we get

$$\begin{aligned} \beta &= \frac{\rho_{\rm PBH,0}}{\rho_{\rm rad,0}} \left(\frac{a_{\rm i}}{a_{\rm eq}} \right) \left(\frac{a_{\rm eq}}{a_0} \right) \, , \\ &= \frac{1}{2.1 \times 10^{-4}} \frac{\rho_{\rm PBH,0}}{\rho_{\rm cdm,0}} \times 4 \times 10^{-10} \left(\frac{M}{M_\odot} \right)^{1/2} \times 2.8 \times 10^{-4} \sim 10^{-9} \left(\frac{M}{M_\odot} \right)^{1/2} f_{\rm PBH} \, . \end{aligned}$$

- **The initial abundance of PBHs is related to the mass variance (the typical size of density perturbations) at horizon crossing, $\sigma(M)$, by**

$$\beta(M) \sim {\rm erfc} \left(\frac{\delta_{\rm c}}{\sqrt{2} \sigma(M)} \right) \tag{3}$$

where $\delta_{\rm c} \approx 0.5$ is the threshold for PBH formation.

Calculate the values of $\sigma(M)$ required for PBHs with mass M equal to
i) $1 M_\odot$**,**
ii) $10^{15}\,$g**,**
to make up all of the DM ($f_{\rm PBH} = 1$).

What are the implications of the dependence of β on $\sigma(M)$?

Using Eq. (2), $\beta(1 M_\odot) = 10^{-9}$ and

$$\beta(10^{15}\,{\rm g}) = 10^{-9} \left(\frac{10^{15}}{2 \times 10^{33}} \right)^{1/2} \sim 10^{-17} \, .$$

${\rm erfc}(4.3) = 1.2 \times 10^{-9}$ and ${\rm erfc}(6.0) = 2.2 \times 10^{-17}$ and hence $\sigma(1 M_\odot) \sim 0.09$ and $\sigma(10^{15}\,{\rm g}) \sim 0.06$.

$\beta(M)$ varies exponentially (in the mathematical sense of the word...) with $\sigma(M)$, so $\sigma(M)$ has to be fine tuned to obtain a specific value of $f_{\rm PBH}$. (The required values are also several orders of magnitude larger than the measured amplitude of the density perturbations on cosmological scales.)