The hadronic vacuum polarization contribution to the muon $g-2$ from $\mathrm{O}(a)$ improved Wilson quarks

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First LatticeNET workshop on challenges in Lattice field theory
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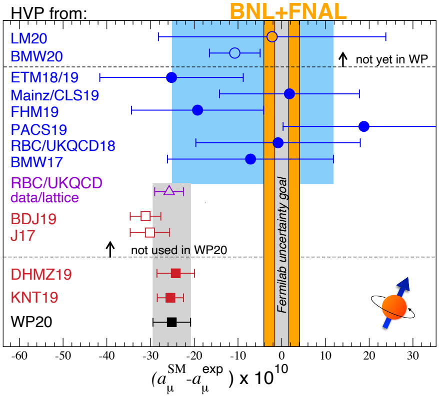


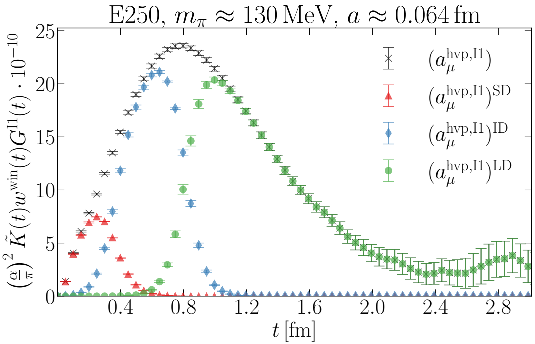
← Status for a_μ^{hvp} [2203.15810, Colangelo et al.]

- Prediction in [2002.12347, BMWc] deviates **significantly** from data-driven results.
- High-precision lattice calculations needed. Major challenges:
  - Cutoff effects at short distances t
  - Exponential deterioration of signal-to-noise ratio at large t (with traditional Monte Carlo methods)

- *Short term*: Focus on benchmark quantities to compare among collaborations. Time windows in the Time Momentum Representation [1801.07224, Blum et al.]
- *Long term*: Improve overall precision of a_μ^{hvp}.

Time-momentum representation [1107.4388, Bernecker and Meyer]:

$$(a_\mu^{\mathrm{hvp}})^i := \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dt \widetilde{K}(t)\, G(t) W^i(t;t_0;t_1)$$



- Current-current correlator:

  $G(t) = -\frac{a^3}{3} \sum_{k=1}^3 \sum_{\vec{x}} \langle j_k^{\mathrm{em}}(t,\vec{x})\, j_k^{\mathrm{em}}(0)\rangle$

- Time windows [1801.07224, Blum et al.]:

  $W^{\mathrm{SD}}(t;t_0;t_1) = [1 - \Theta(t,t_0,\Delta)]$

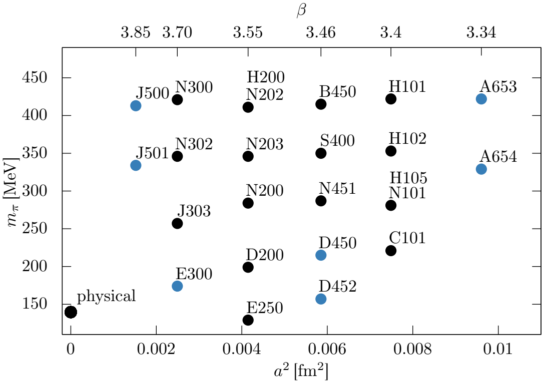
  $W^{\mathrm{ID}}(t;t_0;t_1) = [\Theta(t,t_0,\Delta) - \Theta(t,t_1,\Delta)]$

  $W^{\mathrm{LD}}(t;t_0;t_1) = \Theta(t,t_0,\Delta)$

  where

  $\Theta(t,t',\Delta) := \frac{1}{2}\left(1 + \tanh[(t-t')/\Delta]\right)$

  $t_0 = 0.4\,\mathrm{fm}, t_1 = 1.0\,\mathrm{fm}, \Delta = 0.15\,\mathrm{fm}.$



- $\mathrm{O}(a)$ improved Wilson-clover fermions.
- Six values of $a \in [0.039, 0.099]\,\mathrm{fm}$, a factor of 6.4 in a^2.
- Open boundary conditions in temporal direction.
- $m_\pi \in [129, 422]\,\mathrm{MeV}$
- $a\mathrm{Tr}[M_\mathrm{q}] = 2am_\mathrm{l} + am_\mathrm{s} = \mathrm{const.}$

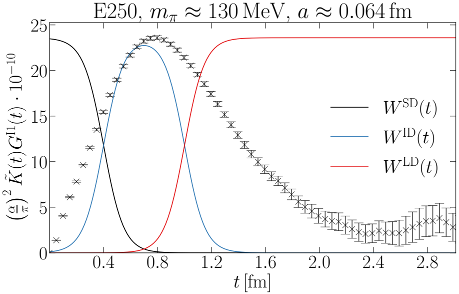
Scale: Either use $\sqrt{t_0^\mathrm{phys}} = 0.1443(15)\,\mathrm{fm}$ [2112.06696, Straßberger et al.] or express dimensionfull quantities in terms of af_π [1103.4818, Xu et al.][1904.03120, Gérardin et al.]

$\rightarrow$ new $N_\mathrm{f} = 2+1$ result by RQCD: $\sqrt{t_0^\mathrm{phys}} = 0.1449^{(7)}_{(9)}\,\mathrm{fm}$ may be used in the future.

- We can determine the scale dependence via

$$\frac{\partial (a_\mu^{\mathrm{hvp}})^i}{\partial_\Lambda} = \left(\frac{\alpha}{\pi}\right)^2 \sum_0^\infty dt \left[\left(\frac{\partial}{\partial_\Lambda}\widetilde{K}(t)\right) W^i(t;t_0;t_1) + \widetilde{K}(t)\left(\frac{\partial}{\partial_\Lambda} W^i(t;t_0;t_1)\right)\right] G(t)$$

- Using a parametrization of the R-ratio, the Mainz group estimated $\frac{\Delta a_\mu^{\mathrm{hvp}}}{a_\mu^{\mathrm{hvp}} \Delta\Lambda} \approx 1.8$ [1705.01775, Della Morte et al.] $\rightarrow$ *Alberto:* what about the windows?



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- Computation via analytic formula or via $\partial(a_\mu^{\mathrm{hvp}})^i / \partial_\Lambda$ from the Γ-method.

- My estimates for $\frac{\Delta(a_\mu^{\mathrm{hvp}})^i}{(a_\mu^{\mathrm{hvp}})^i \Delta\Lambda}$ at m_π^{phys} (**no rigorous analysis!**):

$\delta a_\mu^{\mathrm{hvp}}$	$\delta(a_\mu^{\mathrm{hvp}})^{\mathrm{SD}}$	$\delta(a_\mu^{\mathrm{hvp}})^{\mathrm{ID}}$	$\delta(a_\mu^{\mathrm{hvp}})^{\mathrm{LD}}$
1.8	0.0	0.5	2.7

- Work in isospin decomposition of the electromagnetic current

$$j_\mu^{\mathrm{em}} = \tfrac{2}{3}\bar{u}\gamma_\mu u - \tfrac{1}{3}\bar{d}\gamma_\mu d - \tfrac{1}{3}\bar{s}\gamma_\mu s + \tfrac{2}{3}\bar{c}\gamma_\mu c + \ldots \quad = j_\mu^{I=1} + j_\mu^{I=0} + \tfrac{2}{3}\bar{c}\gamma_\mu c + \ldots, \,,$$

Isovector: $j_\mu^{I=1} = \tfrac{1}{2}(\bar{u}\gamma_\mu u - \bar{d}\gamma_\mu d)$, Isoscalar: $j_\mu^{I=0} = \tfrac{1}{6}(\bar{u}\gamma_\mu u + \bar{d}\gamma_\mu d - 2\bar{s}\gamma_\mu s)$

- Two discretizations of the vector current: local and conserved

$$J_\mu^{(\mathrm{L}),a}(x) = \overline{\psi}(x)\gamma_\mu \frac{\lambda^a}{2}\psi(x)\,,$$

$$J_\mu^{(\mathrm{C}),a}(x) = \frac{1}{2}\left(\overline{\psi}(x+a\hat{\mu})(1+\gamma_\mu)U_\mu^\dagger(x)\frac{\lambda^a}{2}\psi(x) - \overline{\psi}(x)(1-\gamma_\mu)U_\mu(x)\frac{\lambda^a}{2}\psi(x+a\hat{\mu})\right)$$

- Improved vector currents are given by

$$J_\mu^{(\alpha),a,\mathrm{I}}(x) = J_\mu^{(\alpha),a}(x) + a c_\mathrm{V}^{(\alpha)}(g_0)\,\tilde{\partial}_\nu \Sigma_{\mu\nu}^a(x)\,, \qquad \text{with} \quad \alpha \in \mathrm{L}, \mathrm{C}$$

- Renormalization and mass-dependent improvement of local currents via

$$J_\mu^{(\mathrm{L}),3,\mathrm{R}}(x) = Z_\mathrm{V}\left[1 + 3\bar{b}_\mathrm{V} a m_\mathrm{q}^\mathrm{av} + b_\mathrm{V} a m_{\mathrm{q},l}\right] J_\mu^{(\mathrm{L}),3,\mathrm{I}}(x)\,,$$

$$J_\mu^{(\mathrm{L}),8,\mathrm{R}}(x) = Z_\mathrm{V}\left[1 + 3\bar{b}_\mathrm{V} a m_\mathrm{q}^\mathrm{av} + \frac{b_\mathrm{V}}{3} a(m_{\mathrm{q},l} + 2m_{\mathrm{q},s})\right] J_\mu^{(\mathrm{L}),8,\mathrm{I}}(x)$$
$$+ Z_\mathrm{V}\left(\frac{1}{3} b_\mathrm{V} + f_\mathrm{V}\right)\frac{2}{\sqrt{3}} a(m_{\mathrm{q},l} - m_{\mathrm{q},s})\, J_\mu^{(\mathrm{L}),0,\mathrm{I}}(x)\,,$$

- Two independent non-perturbative determinations of $Z_\mathrm{V}, c_\mathrm{V}^L, c_\mathrm{V}^C, b_\mathrm{V}, \bar{b}_\mathrm{V}$:

Set 1: Large-volume, CLS ensembles [1811.08209, Gérardin et al.]

Set 2: Small volume, Schrödinger functional [2010.09539, ALPHA],[1805.07401, Fritzsch]

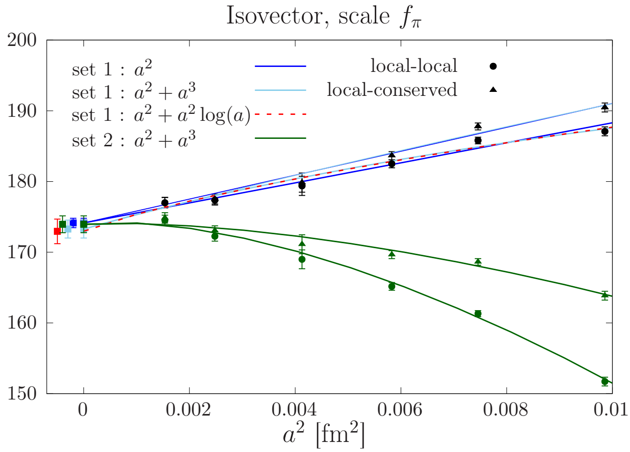
differ by higher order cutoff effects. f_V is of $\mathrm{O}(g_0^6)$ and unknown.

Finite-size effects

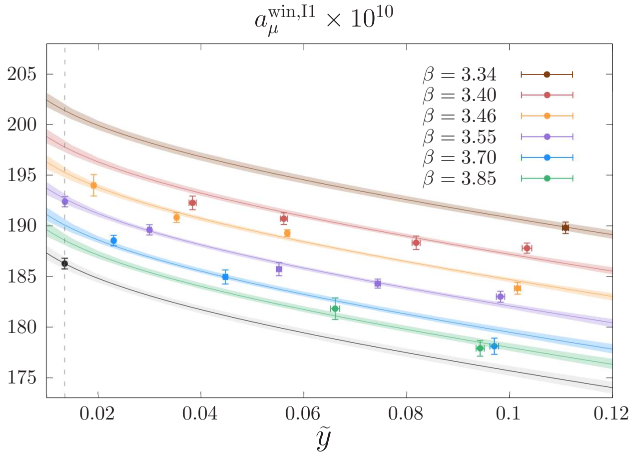
- Finite-size corrections applied to the isovector correlator.
- Correction for $t < \frac{(m_\pi L/4)^2}{m_\pi}$: Hansen-Patella method [1904.10010][2004.03935]
  - Expansion in the pion winding number.
  - Using monopole parametrization of the electromagnetic pion form factor.
- Large distances: MLL [1105.1892, Meyer] [hep-lat/0003023, Lellouch and Lüscher]:
  - Compute difference between finite and infinite-volume isovector correlator
  - Based on the time-like pion form factor.
  - Applied at large Euclidean distances $\rightarrow$ less relevant for short and intermediate distance windows.
- This is the only correction applied to the lattice data!
  Of similar size as statistical uncertainty for $a_\mu^{\text{win}} \equiv (a_\mu^{\text{hvp}})^{\text{ID}}$.

The intermediate-distance window

[2206.06582, Cè et al.]

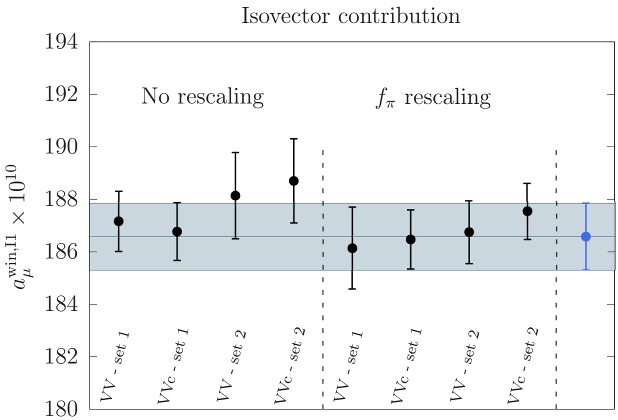


- Two sets of equally valid improvement coefficients.
- No cutoff effects of $\mathrm{O}(a^3)$ resolved for Set 1.
- Independent extrapolations compatible in the continuum $\rightarrow$ strong cross-check of our extrapolations.
- No sign of modification $a^2 \rightarrow (\alpha_s(1/a^2))^{\hat{\Gamma}} a^2$ [1912.08498, Husung et al.]

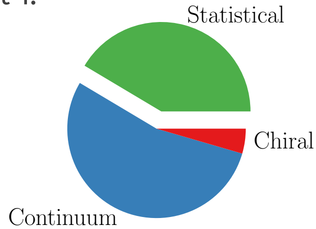


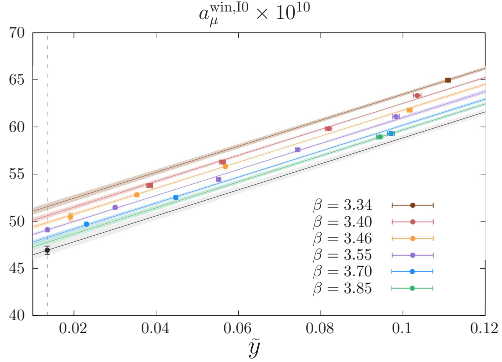
- f_π rescaling, local-local current and Set 1.
- Curvature in $\tilde{y} = \frac{m_\pi^2}{8\pi f_\pi^2}$ is needed to describe the data.
- Singular fit ansatz favored, also found in [2110.05493, Colangelo et al.]
- Variation in the chiral extrapolation does not change the result significantly.

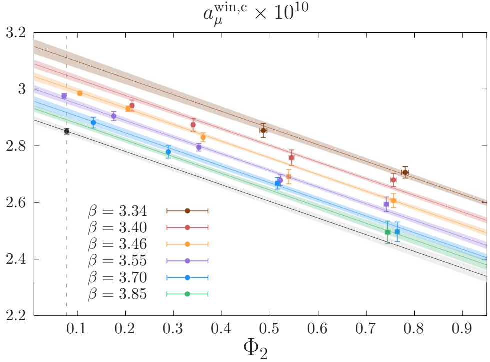
$$a_\mu^{\mathrm{win}}(\tilde{y}) = \gamma_1\ (\tilde{y} - \tilde{y}^{\mathrm{exp}}) + \gamma_2\ (f(\tilde{y}) - f(\tilde{y}^{\mathrm{exp}}))\ , \qquad f(\tilde{y}) \in \{0;\ \log(\tilde{y});\ \tilde{y}^2;\ 1/\tilde{y};\ \tilde{y}\log(\tilde{y})\}$$



- Eight combinations of discretization and improvement procedures.
- Model averages in each category to determine systematic uncertainty from choice of fit model. [2008.01069, Jay and Neil]
- Final result by combining L and C of Set 1.

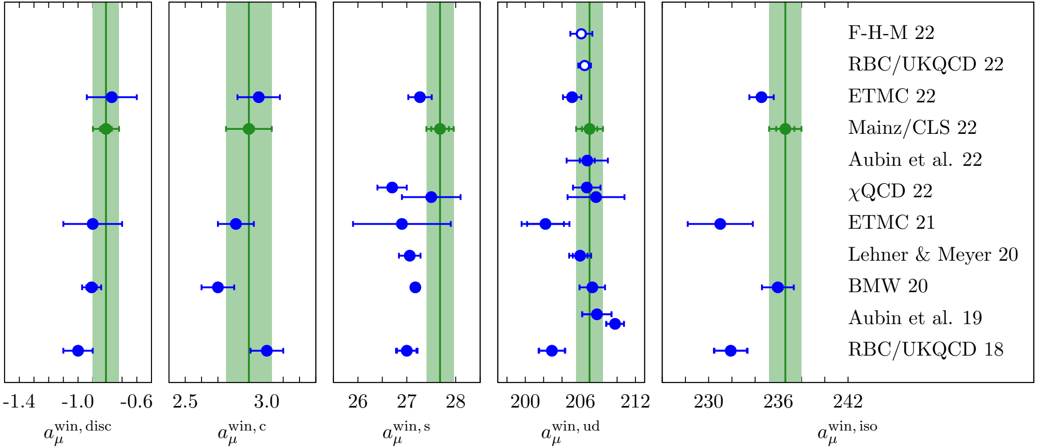






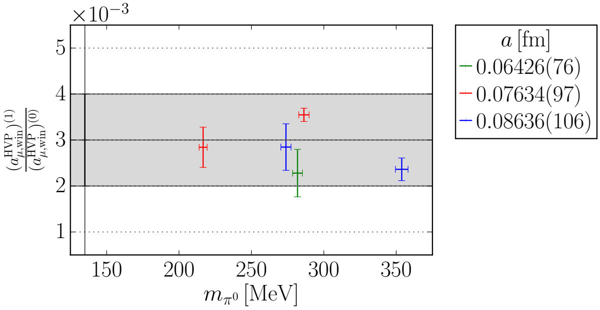
- Isoscalar contribution: Non-singular fit ansatz, $f(\tilde{y}) \in \{0;\ \tilde{y}^2;\ \tilde{y}\log(\tilde{y})\}$.
- Charm contribution:
  - Included in partially-quenched setup.
  - Effect of missing charm loops estimated to be $< 0.02\%$ for a_μ^{win}.
  - Mass-dependent renormalization scheme.

Comparison with lattice results for $a_\mu^{\mathrm{win,iso}}$



$$a_\mu^{\mathrm{win,iso}} = a_\mu^{\mathrm{win,I1}} + a_\mu^{\mathrm{win,I0}} + a_\mu^{\mathrm{win,c}} = (236.60 \pm 0.79_{\mathrm{stat}} \pm 1.13_{\mathrm{syst}} \pm 0.05_{\mathrm{Q}}) \times 10^{-10}$$

- Tensions within lattice results seem to be resolved.



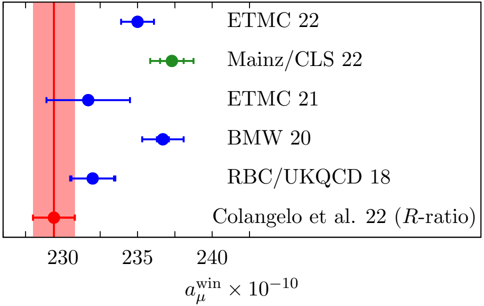
- $\mathrm{QED_L}$-action [0804.2044, Hayakawa and Uno] for IR regularisation, Coulomb gauge.
- Reweighting based on perturbative expansion [1303.4896, de Divitiis et al.] in $\Delta\varepsilon = \varepsilon - \varepsilon^{(0)} = (\Delta m_{\rm u}, \Delta m_{\rm d}, \Delta m_{\rm s}, \Delta\beta = 0, e^2)$

- Work in progress: Andreas Risch's contribution at the 2022 workshop of the TI.
- IB in scale setting [2112.08262, Segner et al.] and QED-FV effects to be considered.
- Uncertainty on relative correction $0.3(1)\%$ doubled in final result for $a_\mu^{\rm win}$.

Comparison with results for a_μ^{win}

- Isospin-breaking correction $+(0.70 \pm 0.47) \times 10^{-10}$ included:

$$a_\mu^{\text{win}} = (237.30 \pm 0.79_{\text{stat}} \pm 1.13_{\text{syst}} \pm 0.05_{\text{Q}} \pm 0.47_{\text{IB}}) \times 10^{-10}$$



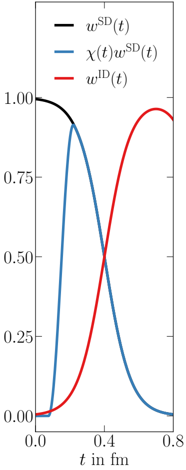
- 3.9σ tension with data-driven estimate in [2205.12963, Colangelo et al.].
- Genuine difference between lattice and data-driven results?

The short-distance window

The short-distance window

- Short-distance cutoff effects are dominant source of uncertainty in $(a_\mu^{\mathrm{hvp}})^{\mathrm{SD}}$.
- Log-enhanced cutoff effects are present at very short distances of the TMR integral [0807.1120, Della Morte et al.][2106.15293, Cè et al.] [Rainer Sommer's talk at Lattice22] $\rightarrow$ also present in a_μ^{hvp}!

- Tree-level improvement may help to reduce cutoff effects, as used for $(a_\mu^{\mathrm{hvp}})^{\mathrm{SD}}$ in [2206.15084, Alexandrou et al.].
- Use of perturbation theory at $\mathrm{O}(\alpha_s^4)$ at very short distances (already suggested in [1107.4388, Bernecker and Meyer]) removes logarithmic contribution.



- Regulate the short distance part [Rainer Sommer's talk at Lattice22]:

$$\int_0^\infty \mathrm{d}t F(t) = \int_0^\infty \mathrm{d}t[1-\chi(t)]\,F(t) + \lim_{a\to 0} a \sum_0^\infty \chi(t)F(t)\,.$$

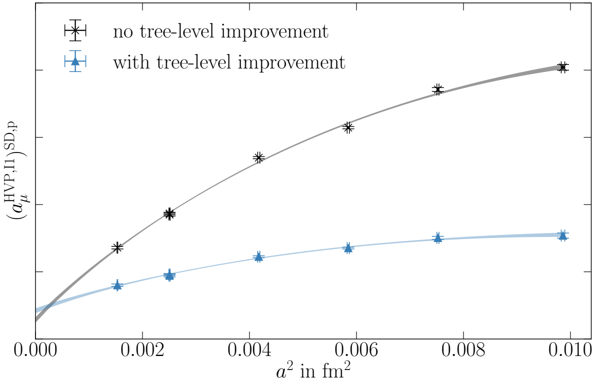
  Combine perturbation theory at short distances with the continuum lattice result.

- Choose sufficiently smooth and short ranged regulator, e.g.,

$$\chi(t) = \theta(t-u_0)\left(1-\cos\left[\frac{(t-u_0)\pi}{(2\delta)}\right]^2 \theta(u_0+\delta-t)\right)$$

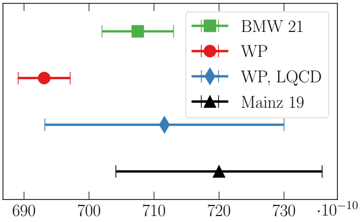
  with the Heaviside step function $\theta(t)$ and $u_0 = \delta = 0.075\,\mathrm{fm}$.

- Test regulators, improvement schemes and discretization prescriptions.
- Continuum extrapolation of regulated short-distance window $(a_\mu^{\mathrm{HVP,I1}})^{\mathrm{SD,p}}$ at the $\mathrm{SU}(3)_\mathrm{f}$ -symmetric point.

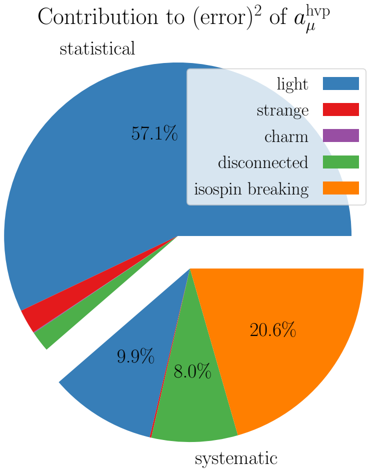


- No log-enhanced cutoff effects expected.
- Tree-level improvement:
  - Based on massless, free theory.
  - Reduces cutoff effects at $a = 0.1\,\mathrm{fm}$ from 18% to 6%.
- Different data sets offer insight in systematic uncertainties.

Noise reduction in the long-distance tail



- 2.2% uncertainty: Dominated by statistical uncertainties of light quark contribution.
- Variance reduction is needed to reach sub-percent precision (2.2% statistical uncertainty on physical mass ensemble).



- Employ Low Mode Averaging (LMA) [hep-lat/0106016, Neff et al.][hep-lat/0402002, Giusti et al.][hep-lat/0401011, DeGrand et al.][...] to reduce the variance of the isovector contribution.

- Split up the quark propagator ($Q = \gamma_5 D_{\mathrm{W}}$)

$$Q^{-1} = Q^{-1}(\mathbf{P}_{\mathrm{L}} + \mathbf{P}_{\mathrm{H}}) = \sum_{i=1}^{N_{\mathrm{L}}} \frac{1}{\lambda_i} v_i v_i^\dagger + Q^{-1}\mathbf{P}_{\mathrm{H}}$$

  in low and high mode contributions using the projectors

$$\mathbf{P}_{\mathrm{L}} = \sum_{i=1}^{N_{\mathrm{L}}} v_i v_i^\dagger\,, \qquad \mathbf{P}_{\mathrm{H}} = \mathbf{1} - \mathbf{P}_{\mathrm{L}}$$

  with the eigenmodes v_i and the (real) eigenvalues λ_i of Q.

- Even-odd preconditioning reduces memory by factor 2 [1004.2661, Blossier et al.].

Low mode averaging: Connected two-point function

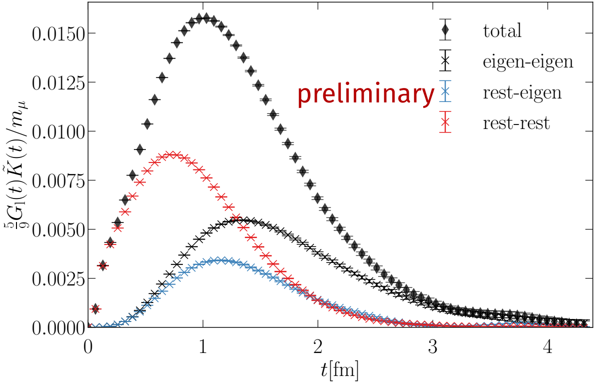
- The connected two-point function contains two quark propagators

$$C_{AB}(x_0, y_0) = \sum_{\mathbf{x},\mathbf{y}} \left\langle \mathrm{Tr}\left[\gamma_5 \Gamma_A Q^{-1}(x,y) \gamma_5 \Gamma_B Q^{-1}(y,x)\right] \right\rangle$$

  with γ_5 insertions because we use Q.
- We get four different terms ($t \equiv x_0 - y_0$),

$$C(t) = C^{ll}(t) + C^{hl}(t) + C^{lh}(t) + C^{hh}(t)\,,$$

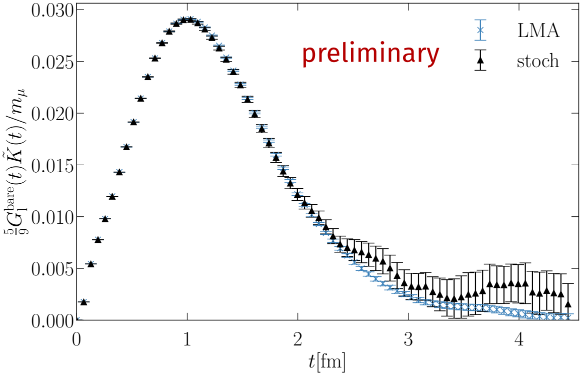
  each can be defined and computed separately:
  - Evaluate $C^{ll}(t)$ with full volume average.
  - Evaluate $C^{hl}(t) + C^{lh}(t)$ by inversion on eigenmodes (alternative: stochastic).
  - Evaluate C^{hh} stochastically.
- Use Truncated Solver Method [0910.3970, Bali et al.] to reduce cost of inversions.
- Exact deflation does not reduce the cost with Lüscher's deflated solver.



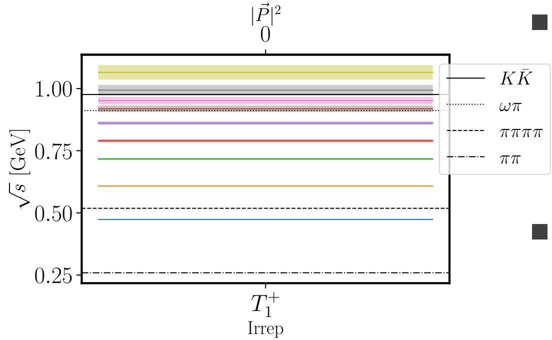
- Light-connected contribution to a_μ^{hvp} for $m_\pi \approx 129\,\mathrm{MeV}$ in a $12.4\,\mathrm{fm} \times (6.2\,\mathrm{fm})^3$ box at $a = 0.064\,\mathrm{fm}$.
- 800 eigenmodes of the even-odd preconditioned Dirac-Wilson operator $\gamma_5 \hat{D}$.

- All-to-all evaluation of low eigenmodes dominates correlator and its variance for $t > 1.5\,\mathrm{fm}$.

Variance reduction: Low Mode Averaging



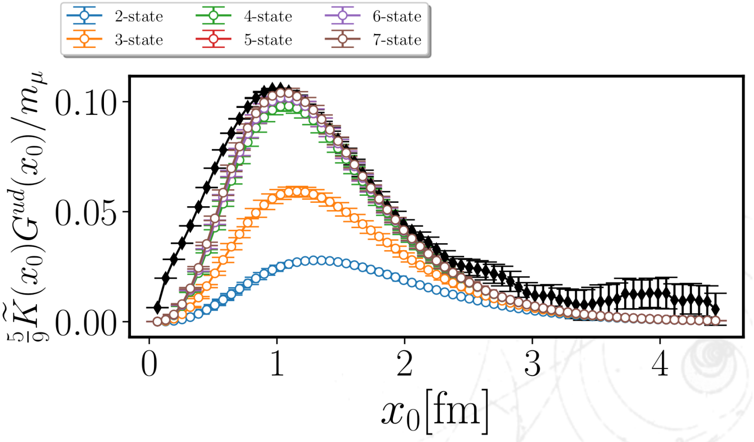
- Comparison of stochastic evaluation and LMA at similar statistics.
- Ongoing calculation: Sub-percent precision reached.
- LMA also applied on less challenging ensembles.



- Dedicated spectroscopy analysis of finite-volume energies and amplitudes to reconstruct the tail of the isovector correlation function [1808.05007, Andersen et al.][1904.03120, Gérardin et al.].
- Analysis at close-to-physical masses ongoing [2112.07385, Paul et al.], up to 9 energy levels resolved.
- Spectral decomposition of the vector correlator:

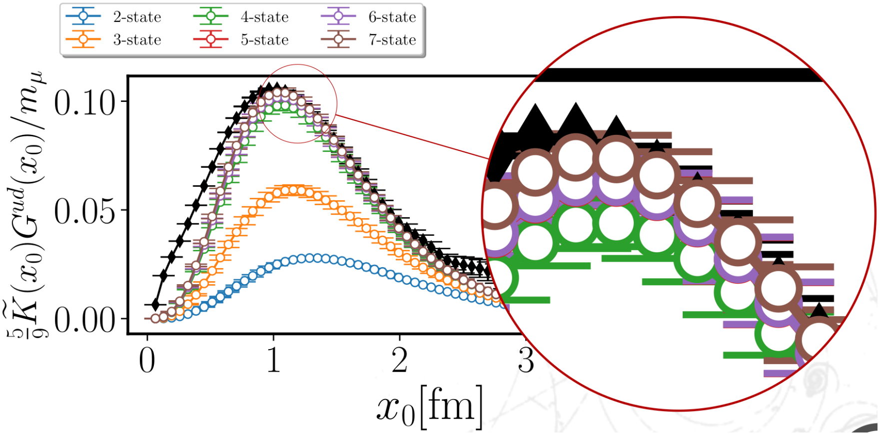
$$G_l(t) = \sum_n |A_n|^2 \mathrm{e}^{-E_n t}\,, \qquad E_n = 2\sqrt{m_\pi^2 + k^2}$$

- Computation of the pion transition form factor to correct for finite-size effects with less model dependence ongoing.



- Reconstruction of the light-connected TMR correlator at long distances at close-to-physical pion mass, presented in Srijit Paul's talk at Lattice22.

Variance reduction: Spectroscopy



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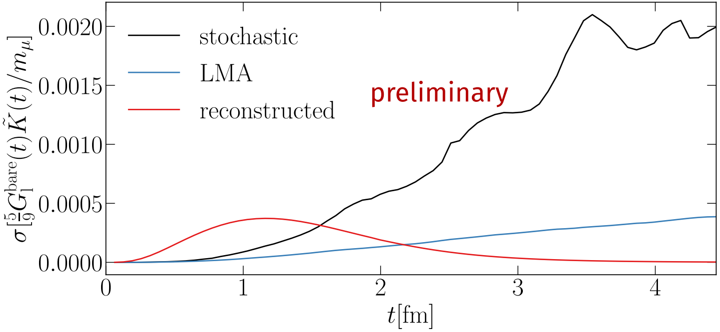

- Comparison of statistical uncertainties of the TMR correlator based on stochastic evaluation, reconstruction and LMA.
- Significantly less noise in LMA correlator.
- Exponential noise reduction in reconstructed correlator.

Intermediate window

- We observe tension with data-driven estimates for a_μ^{win}.
- Systematic effects from continuum extrapolation seem to be under control:
  - Non-perturbative $\mathrm{O}(a)$ improvement
  - 6 resolutions $< 0.1\,\mathrm{fm}$ with $a^2_{\mathrm{max}}/a^2_{\mathrm{min}} > 6$.
  - Two discretizations of the vector current, two sets of improvement procedures.
  - So-far no sign of logarithmic corrections to a^2 scaling.
- Uncertainties from chiral extrapolation and finite-volume correction are subleading.

Outlook

- Similar tension found for $\Delta\alpha_{\mathrm{had}}(Q^2)$ at low Q^2 [2203.08676, Cè et al.].
- Investigation of other windows might help to clarify the situation.
- Short-distance window:
  - Cutoff effects from short-distance singularities need proper treatment [0807.1120, Della Morte et al.][2106.15293, Cè et al.] [Rainer Sommer's talk].
  - Systematic uncertainties will dominate and need to be properly estimated.
- Sub-percent precision on a_μ^{hvp} needs reduction of our statistical uncertainties.
- Spectroscopy and variance reduction techniques will help to improve our calculation significantly close to physical pion mass.